Mathematical formalism for designing wide-field x-ray telescopes: mirror nodal positions and detector tilts

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We provide a mathematical formalism for optimizing the mirror nodal positions along the optical axis and the tilt of a commonly employed detector configuration at the focus of a x-ray telescope consisting of nested mirror shells with known mirror surface prescriptions. We adopt the spatial resolution averaged over the field-of-view as the figure of merit M. A more complete description appears in our paper in these proceedings.

1. Variance in ray position on a focal surface S

$$M = \left(\sigma_S^2\right)_M \equiv \frac{\int_0^{2\pi} d\phi \int_0^{\theta_{FOV}} d\theta\,\theta\, w_{FOV}(\theta,\phi)\,\sigma_S^2(\theta,\phi)}{\int_0^{2\pi} d\phi \int_0^{\theta_{FOV}} d\theta\,\theta\, w(\theta,\phi)}$$

$$\sigma_S^2(\theta,\phi) = \sigma_{S,1}^2(\theta,\phi) + \sigma_{S,2}^2(\theta,\phi)$$

where $w_{FOV}(\theta,\Phi)$ is a viewing angle weighting function and $\sigma_S^2(\theta, \Phi)$ is the spatial variance for a point source on the sky at polar and azimuthal off-axis angles (θ, Φ) determined on the surface S of the detectors. For J mirror shells with n_j multiply reflected rays through shell j, we have shown:

$$\sigma_{S,1}{}^2(\theta,\phi) = \left(\frac{N}{W}\right)\sum_{j=1}^{J}\left(\frac{w_j}{n_j}\right)\left(\frac{n_j-1}{N-1}\right)\sigma_{j,S}{}^2$$

$$\sigma_{j,S}^2 = \frac{n_j}{n_j-1}\left[\left(\left\langle x_j^2\right\rangle_S - \left\langle x_j\right\rangle_S^2\right) + \left(\left\langle y_j^2\right\rangle_S - \left\langle y_j\right\rangle_S^2\right) + \left(\left\langle z_j^2\right\rangle_S - \left\langle z_j\right\rangle_S^2\right)\right]$$

$$\sigma_{S,2}^2(\theta,\phi) = \left(\frac{N}{N-1}\right)\left[\sum_{j=1}^{J}\left(\frac{w_j}{W}\right)\left(1-\frac{w_j}{W}\right)q_{jj,S} - \sum_{i=1}^{J}\sum_{j\neq i}^{J}\left(\frac{w_i\,w_j}{W^2}\right)q_{ij,S}\right]$$

$$q_{ij,S} = \left\langle x_i\right\rangle_S \left\langle x_j\right\rangle_S + \left\langle y_i\right\rangle_S \left\langle y_j\right\rangle_S + \left\langle z_i\right\rangle_S \left\langle z_j\right\rangle_S$$

$$N = \sum_{j=1}^{J} n_j \qquad W = \sum_{j=1}^{J} w_j = \sum_{j=1}^{J}\sum_{k=1}^{n_j} w_{j,k}$$

When $w_{j,k} = 1$, then $w_j = n_j$ and $W = N$. The ensemble average of a quantity, say the ray x position on the surface S, for multiply reflected rays from shell J is given by:

$$\left\langle x_j\right\rangle_S = \frac{1}{w_j}\sum_{k=1}^{n_j} w_{j,k}\left(x_{j,k}\right)_S \qquad \left\langle x_j^2\right\rangle_S = \frac{1}{w_j}\sum_{k=1}^{n_j} w_{j,k}\left(x_{j,k}^2\right)_S$$

Where $w_{j,k}$ is the weight assigned to the k-th ray from the j-th mirror shell. In order to account for dependence on energy E, say for optics with two segments per mirror shell, the natural weight to use is the product of the reflectivities from the primary, R_P, and secondary, R_S, mirror surfaces:

$$w_{j,k} = R_P(\alpha_{P,j,k}, E) \times R_S(\alpha_{S,j,k}, E)$$

Here $\alpha_{P,j,k}$ and $\alpha_{S,j,k}$ are the primary and secondary graze angles for the k-th ray from the j-th mirror shell.

Important result: σ_S^2 is not a simple sum over the over the variances $\sigma_{j,S}^2$ of the individual shells.

3. The merit function

Making use of additional definitions given in our paper to make the notation more compact, we write:

$$\left(\sigma_{S_D}^2\right)_M = \left(A_0\right)_M + 2\left(D_0\right)_M \tan\vartheta + \left(F_0\right)_M \tan^2\vartheta + \sum_{j=1}^{J}\left[2\left(B_{j,0}\right)_M \delta z_j + \left(C_{j,0}\right)_M \delta z_j^2 + 2\left(E_{j,0}\right)_M \delta z_j \tan\vartheta\right]$$
$$- \sum_{i=1}^{J}\sum_{j\neq i}^{J}\left(\left[\left(B_{ij,0}\right)_M \delta z_i + \left(B_{ji,0}\right)_M \delta z_j\right] + \left(C_{ij,0}\right)_M \delta z_i\,\delta z_j + \left[\left(E_{ij,0}\right)_M \delta z_i + \left(E_{ji,0}\right)_M \delta z_j\right]\tan\vartheta\right)$$

The telescope/detector configuration is optimized by solving:

$$\frac{\partial\left(\sigma_{S_D}^2\right)}{\partial\delta z_k} = 2\left[\left(B'_{k,0}\right) + \left(E'_{k,0}\right)\tan\vartheta + \left(C_{k,0}\right)_M \delta z_k - \sum_{j\neq k}^{J}\left(C_{jk,0}\right)_M \delta z_j\right] = 0$$

$$\frac{\partial\left(\sigma_{S_D}^2\right)}{\partial\tan\vartheta} = 2\left[\left(D_0\right) + \left(F_0\right)\tan\vartheta + \sum_{k=1}^{J}\left(E'_{k,0}\right)_M \delta z_k\right] = 0$$

$$B'_{k,0} \equiv B_{k,0} - \sum_{i\neq k}^{J} B_{ki,0} \qquad E'_{k,0} \equiv E_{k,0} - \sum_{i\neq k}^{J} E_{ki,0} \qquad E'_{jk,0} \equiv E_{jk,0} - \sum_{i\neq(k,j)}^{J} E_{ji,0}$$

Solve for $\tan\varphi$:

$$\tan\vartheta = -\left[\left(D_0\right)_M + \sum_{k=1}^{J}\left(E'_{k,0}\right)_M \delta z_k\right]\Big/\left(F_0\right)_M$$

Substitute into equations for δz_k, and express in **linear algebra (matrix) form**:

$$\overline{\overline{\beta}} \bullet \vec{\delta z} = \vec{Y}$$

The column vectors **δz** and **Y** are given by

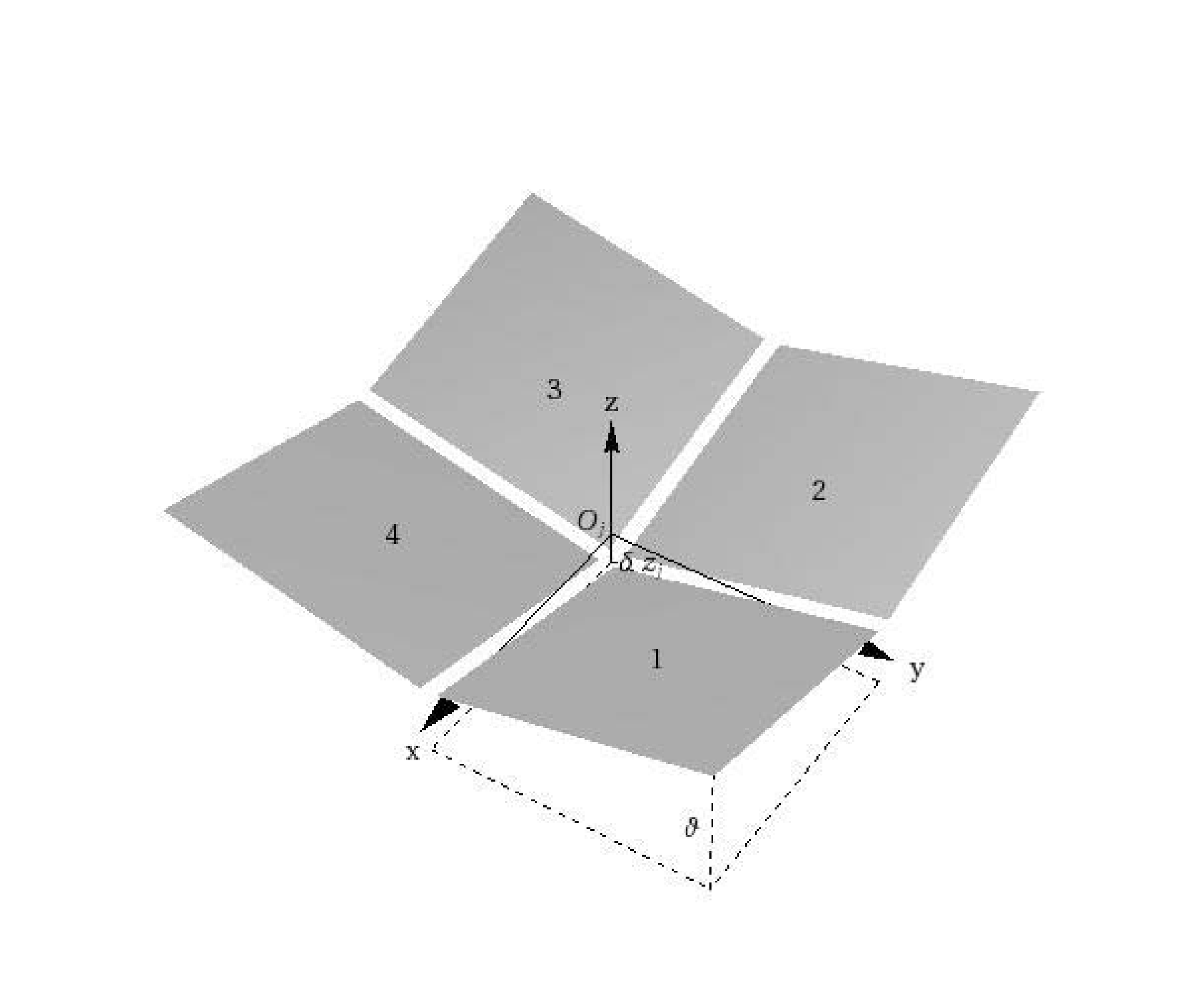
$$\vec{\delta z} = \left\{\delta z_k, (k=1,J)\right\} \qquad \vec{Y} = \left\{\left(D_0\right)_M \left(E'_{k,0}\right)_M - \left(F_0\right)_M \left(B'_{k,0}\right)_M, (k=1,J)\right\}$$

The element of the J × J matrix **β** in the k-th row and j-th column is given by:

$$\beta_{kj} = \delta_{kj}\left[\left(F_0\right)_M \left(C_{k,0}\right)_M - \left(E'_{k,0}\right)_M^2\right] - \left(1-\delta_{kj}\right)\left[\left(F_0\right)_M \left(C_{jk,0}\right)_M - \left(E'_{k,0}\right)_M \left(E'_{jk,0}\right)_M\right]$$

Here δ_{kj} is the Kronecker delta equal to 1 when k = j and 0 otherwise.

2. Application to an inverted pyramid of detectors



Consider a **pyramid of four tilted detectors**, with apex pointing away from the nested shells, and each occupying a quadrant with one corner on the diagonal intersecting the optical axis (see Figure). In this case, the focal surface $S \equiv S_D$ corresponds to the flat, but tilted, surfaces of the four detectors. We denote the tilt angle by φ. If shell j is displaced along the optical axis so that the apex of the inverted pyramid is a distance δz_j from the on-axis focus for that shell, then we have:

$$\sigma_{S,1}{}^2 = a_{j,0} + 2b_{j,0}\,\delta z_j + c_{j,0}\,\delta z_j^2 + 2d_{j,0}\tan\vartheta + 2e_{j,0}\,\delta z_j\tan\vartheta + f_{j,0}\tan^2\vartheta$$

In our coordinate system, $\delta z_j < 0$ if the apex is further from the shell mid-plane than its on-axis focus. The coefficients $a_{j,0}$ etc. are evaluated in the flat plane perpendicular to the optical axis and passing through the on-axis focus for shell j. Each coefficient is an ensemble average of the appropriate corresponding combination of the ray position and wave-vectors, and are given in our paper. To second order in tan φ and the δz, we also have:

$$q_{ij,S} = a_{ij,0} + \left(b_{ij,0}\,\delta z_i + b_{ji,0}\,\delta z_j\right) + c_{ij,0}\,\delta z_i\,\delta z_j$$
$$+ \left(d_{ij,0} + d_{ji,0}\right)\tan\vartheta + \left(e_{ij,0}\,\delta z_i + e_{ji,0}\,\delta z_j\right)\tan\vartheta + \left(\frac{1}{2}\right)\left(f_{ij,0} + f_{ji,0}\right)\tan^2\vartheta$$

Again our paper gives expressions for $a_{ij,0}$ etc. in terms of appropriate ensemble averages.

4. Configuration solutions

For a **single mirror shell** with j = J = 1, the solutions for δz_1 and tan φ reduce to:

$$\delta z_1 = \frac{\left(d_{1,0}\right)_M \left(e_{1,0}\right)_M - \left(b_{1,0}\right)_M \left(f_{1,0}\right)_M}{\left(c_{1,0}\right)_M \left(f_{1,0}\right)_M - \left(e_{1,0}\right)_M^2}$$

$$\tan\vartheta = \frac{\left(b_{1,0}\right)_M \left(e_{1,0}\right)_M - \left(c_{1,0}\right)_M \left(d_{1,0}\right)_M}{\left(c_{1,0}\right)_M \left(f_{1,0}\right)_M - \left(e_{1,0}\right)_M^2}$$

For a set of **J nested mirror shells**, we have J linear equations for the δz_k and an equation for tan φ in terms of the δz_k. We suggest that this system of linear equations ought to linearly independent in the sense that the determinant **|β| ≠ 0**. However, even If this condition is satisfied, current wide field telescope designs approach 100 closely nested mirror shells, so **numerical precision and convergence** my be issues for any computer implementation of the solution of these equations.

When **optimization of the prescriptions for the reflecting surfaces** of the mirror shells is desired, the above procedure becomes more complex. For example consider the case of so-called **polynomial x-ray optics** assuming two mirror segment surfaces and J mirror shells, for which M-1 higher order polynomial terms $p_{m,j,s}(z - z_{mid})^m$, with m = (2,M), j = (1,J) and s = (1,2) are added to a Wolter I prescription for the mirror segment radius squared. The merit function will now depend on the $p_{m,j,s}$ and derivates with respect to these new parameters set to zero and simultaneously solved for in addition to the δz_k and tan φ. Assuming the $p_{m,j,s}$ are small enough to permit linearization of the new conditions, the linear system of equations now consists of (M × J + 1) equations (including the equation for tan φ).