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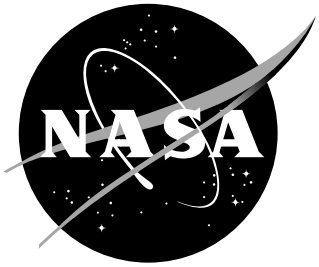

State-Based Implicit Coordination and Applications

Anthony J. Narkawicz and César A. Muñoz
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March 2011

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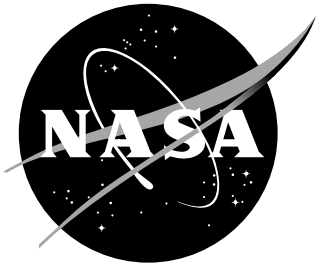

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March 2011

Acknowledgments

The authors would like to thank the members of the review committee presided by Ricky Butler and formed by Natalia Alexandrov, Jeffrey Maddalon, and David Wing for their insightful comments that helped to improve earlier versions of this paper.

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Abstract

In air traffic management, pairwise coordination is the ability to achieve separation requirements when conflicting aircraft simultaneously maneuver to solve a conflict. Resolution algorithms are implicitly coordinated if they provide coordinated resolution maneuvers to conflicting aircraft when only surveillance data, e.g., position and velocity vectors, is periodically broadcast by the aircraft. This paper proposes an abstract framework for reasoning about state-based implicit coordination. The framework consists of a formalized mathematical development that enables and simplifies the design and verification of implicitly coordinated state-based resolution algorithms. The use of the framework is illustrated with several examples of algorithms and formal proofs of their coordination properties. The work presented here supports the safety case for a distributed self-separation air traffic management concept where different aircraft may use different conflict resolution algorithms and be assured that separation will be maintained.

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1 Introduction

The next generation of air traffic management systems may enable a mode of operation where aircraft take a primary responsibility in the management of air traffic separation. This mode of operation, which is called *self-separation*, is supported by advances in hardware and software technologies. For example, global navigation satellite systems, such as Global Positioning System (GPS), will provide accurate surveillance information, which is then broadcast to traffic aircraft and ground elements by systems such as Automatic Dependent Surveillance-Broadcast (ADS-B). This information is then used by separation assurance systems, such as conflict detection and resolution algorithms (CD&R), to warn aircraft crew and air traffic controllers about traffic conflicts and to advise pilots on possible resolution maneuvers.

The conflict management function of the self-separation concept is a safety-critical component of the system. The safety case for that concept must guarantee that distributed separation assurance systems interact in a consistent way, i.e., aircraft do not fly into each other when they independently and simultaneously maneuver to solve a conflict using to their onboard CD&R logic. As discussed by Wing et al. [20], the conflict management function in a self-separation concept may rely on a multi-layered approach where one or more CD&R algorithms are used at different times by different aircraft. Providing a guarantee of safe interaction between these different algorithms requires verification that the distributed resolution maneuvers provided by the onboard systems are complementary. The characterization of complementary resolution maneuvers is not a trivial task. Different resolution algorithms have different safety goals. One algorithm, for example, may try to immediately recover separation, while another algorithm may try to iteratively improve the separation requirement. Even in the case where the safety goal is the same, algorithms that safely interact with themselves in a distributed environment do not necessarily interact with each other in a safe way.

In CD&R literature [10], the terms *cooperation* and *coordination* are often used to describe aircraft interaction when solving a conflict. Several approaches have been proposed to handle this interaction, for example by exchanging intent information [1,9,18,19], by a temporary delegation of responsibility for separation [7], or by geometric methods [2,4,5,8]. This paper provides a mathematical framework for understanding implicit coordination by geometric methods for state-based CD&R. *Geometric methods* refers to decision making rules that only depend on the geometry of the encounters, such as the rule in the Visual Flight Rules that states that when aircraft are approaching head-on, each aircraft shall alter their course to the right. *State-based CD&R* refers to the use of nominal aircraft trajectories that do not include intent information and are defined as linear projections of the current position and velocity of the aircraft for a given lookahead time. Finally, *implicit coordination* refers to the case where aircraft do not negotiate their resolution maneuvers but still take complimentary actions.

The implicit coordination concept considered in this paper only relies on periodically broadcast surveillance from traffic aircraft, for example via ADS-B. This concept is particularly suitable for distributed systems since it does not assume ex-

change of aircraft intention and does not rely on an explicit resolution negotiation between the aircraft. Furthermore, state-based separation assurance systems use simple models of the aircraft dynamics and the airspace geometry. These models yield analytical solutions that can be implemented very efficiently in software systems. In air/ground distributed air traffic management concepts, such as the self-separation concept, state-based CD&R often serves as backup for intent-based systems. Therefore, the overall safety of these concepts is ultimately dependent on state-based algorithms.

The proposed framework provides formal mathematical definitions of coordination and other fundamental concepts such as loss of separation, air traffic conflict, and pairwise resolution algorithm. The notion of coordination proposed here is relative to an abstract concept of a *safety property*, which characterizes safety goals that are intended to be maintained by a family of resolution algorithms. The framework also includes a set of theorems for reasoning about coordination for particular algorithms and particular safety properties. These theorems do not only enable the proof that a given algorithm is implicitly coordinated with itself, but more interestingly, they enable and simplify the proofs that the given algorithm is implicitly coordinated with other conflict resolution algorithms.

The mathematical development presented here can be used by developers of state-based separation assurance systems to design implicitly coordinated algorithms that are correct by construction and whose formal properties can be derived from the theorems provided by the framework. It could also be used by technical committees working on certification standards for distributed separation assurance systems. In the paper "A Criteria Standard for Conflict Resolution: A Vision for Guaranteeing the Safety of Self-Separation in NextGen" [12], the framework presented here is used to propose a standard for guaranteeing the safe interaction of state-based separation assurance algorithms. The proposed standard does not rely on a single mandated CD&R algorithm but rather it proposes a set of common criteria to be satisfied by algorithms that operate under a self-separation concept. That paper provides concrete examples of criteria for conflict resolution and loss of separation recovery algorithms. The fact that those criteria guarantee implicit coordination has been proved once and for all using the framework presented in this paper. Certifying that a particular algorithm complies the standard entails the verification that the algorithm satisfies the criteria, which is a relatively simple task that can also be accomplished by using the results in this framework.

This paper is logically structured in two parts. The first part, composed of sections 2 through 5, lays out the theoretical aspects of the framework. Section 2 concerns notation, basic definitions, and geometrical and physical assumptions. Section 3 presents the main theoretical contribution of this work: an abstract theory of state-based coordination. It is in this section that theorems providing sufficient and necessary conditions for proving coordination are stated and proved. Section 4 specifies safety properties for conflict-free, repulsive, and divergent resolution maneuvers. Section 5 provides a set of theorems for proving coordination of resolution algorithms for the safety properties presented in Section 4. The second part of the paper consists of sections 6 and 7. This part illustrates the theoretical concepts with practical examples. Section 6 formally specifies well-known CD&R algorithms.

Several coordination properties of these algorithms are formally proved in Section 7. Finally, the last section concludes this work.

2 Notation and Basic Definitions

This section presents the mathematical notation used in the paper and provides the basic definitions required to understand the concepts developed in the rest of the paper.

2.1 Notation

Vector variables are written in **boldface** and can be denoted by their components. For example, if $\mathbf{u}$ is a 2-dimensional vector, then $\mathbf{u}$ denotes the pair (u_x, u_y). The two-dimensional Euclidean norm of the vector $\mathbf{u}$ is denoted by[1]

$$\|\mathbf{u}\| \equiv \sqrt{u_x^2 + u_y^2},$$

and the dot product of the 2-dimensional vectors $\mathbf{u}$ and $\mathbf{w}$ is denoted

$$\mathbf{u} \cdot \mathbf{w} \equiv (u_x w_x + u_y w_y).$$

Furthermore, $\mathbf{0}$ denotes the zero vector, i.e.,

$$\mathbf{0} \equiv (0, 0),$$

and $\mathbf{u}^\perp$ denotes the right perpendicular vector to $\mathbf{u}$, i.e.,

$$\mathbf{u}^\perp \equiv (u_y, -u_x).$$

From these definitions, it can be easily proved that $\mathbf{u} \cdot \mathbf{u}^\perp = 0$.

The function `sign` maps real numbers to unit values in $\{-1, 1\}$ and is defined as follows.

$$\texttt{sign}(x) \equiv \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

The expression $\iota = \pm 1$ denotes the fact that an integer ι belongs to the set $\{-1, 1\}$. Moreover, the symbols $\neg$, $\implies$, $\iff$ denote logical negation, implication, and equivalence, respectively. In the context of this paper, a *predicate* is a Boolean function. For example, a predicate on vectors is a function that maps vectors into Boolean values.

The mathematical development presented in this paper has been specified and formally verified in the Prototype Verification System (PVS) [15]. PVS is a proof assistant that consists of a specification language, based on classical higher-order logic, and a mechanical theorem prover for this logic. The PVS specification language allows for the precise definition of mathematical objects such as *functions* and *relations*, and the precise statement of logical formulas such as *lemmas* and *theorems*. Proofs of logical formulas can be mechanically checked using the PVS

[1] The symbol $\equiv$ is used to introduce mathematical definitions.

theorem prover, which guarantees that every proof step is correct and that all possible cases of a proof are covered. All lemmas and theorems presented in this paper have been mechanically checked in PVS. For the sake of simplicity, only proof sketches of the main results are presented in the paper. The complete development, including all definitions and formal proofs, is available from http://shemesh.larc.nasa.gov/people/cam/ACCoRD.

The use of a formal language, e.g., in this case the specification language of PVS, enforces rigorous definitions of mathematical objects, where all dependencies are clearly specified. This level of rigor guarantees a very high confidence on the correctness of the results presented in this paper. However, this also makes the notation heavy and difficult to read for the non-expert reader. For this reason, the work presented here uses standard mathematical notation and does not assume that the reader is familiar with the syntax or semantics of the PVS language. In particular, some syntactical conventions are taken by the authors to make this development more accessible to the casual reader:

- The PVS specification language is *strongly typed*, i.e., all declarations are explicitly typed [16]. This feature guarantees that all PVS functions are total and well-defined. For instance, a mathematical formula that includes a division needs to make explicit the fact that the divisor is different from zero, otherwise the expression would be undefined. In PVS, these conditions are handled by a type system, which is enforced by the PVS type-checker. Since PVS type annotations tend to be verbose, formulas in this paper appear untyped. When necessary, the type domain of variables is made explicit in the context where the formula appears.

- PVS is based on higher-order logic, so it supports the definition of functions that return functions or that have functions as arguments. In this paper, arguments of a higher-order function are called *parameters*. For example, this paper uses the notions of *parametric predicate* and *parametric set*. A parametric predicate P on vectors, with parameters $\mathbf{s}, \mathbf{v}$, is a higher-order function that takes as arguments vectors $\mathbf{s}$ and $\mathbf{v}$, and returns a predicate on vectors. Similarly, a parametric set $\mathcal{A}$ of vectors, with parameters $\mathbf{s}, \mathbf{v}$, is a higher-order function that takes as arguments vectors $\mathbf{s}$ and $\mathbf{v}$, and returns a set of vectors. Sub- and super-indices will be used to denote parameters, e.g., $P_{\mathbf{s},\mathbf{v}}$ and $\mathcal{A}_{\mathbf{s},\mathbf{v}}$ are, respectively, the predicate and set resulting from the application of the parametric predicate P and parametric set A to $\mathbf{s}$ and $\mathbf{v}$.

- The PVS notation is declarative, i.e., there is not a notion of memory state as in a programming language. Algorithms are represented by functions. By convention, names of functions that are intended to have a logical meaning are written in *italics*. Functions that represent algorithms to be implemented in a programming language are written in `typewriter` font.

2.2 Assumptions and Basic Definitions

This paper considers pairwise resolution algorithms that return guidance maneuvers for aircraft. The terms *ownship* and *intruder* are used to distinguish between the

aircraft for which the resolution maneuver is computed, which corresponds to the ownship, and the traffic aircraft, which corresponds to the intruder. These designations are relative. Each aircraft will be from its point of view the ownship and the other aircraft will be the intruder. Without loss of generality, the development presented here takes the point of view of one of the aircraft, and that aircraft will be designated as the ownship.

The algorithms discussed here only use state-based information for the two aircraft, i.e., position and velocity vectors that are elements of a Euclidean space. Aircraft dynamics are represented by a simple kinematic model where points move at constant linear speed. For notational convenience, this paper mostly uses the Euclidean 2-dimensional geometry instead of the 3-dimensional one, but as shown in Section 5.4, all the results in this paper have been generalized to the Euclidean 3-dimensional airspace. The current state of the ownship and intruder aircraft are denoted by the following vectors.

$\mathbf{s}_o$	Initial position of the ownship aircraft
$\mathbf{v}_o$	Initial velocity of the ownship aircraft
$\mathbf{s}_i$	Initial position of the intruder aircraft
$\mathbf{v}_i$	Initial velocity of the intruder aircraft

It is assumed that the ground speeds of the ownship and intruder aircraft are not zero, i.e., $\|\mathbf{v}_o\| \neq 0$ and $\|\mathbf{v}_i\| \neq 0$. Therefore, $\mathbf{v}_o \neq \mathbf{0}$ and $\mathbf{v}_i \neq \mathbf{0}$.

In the airspace system, the separation requirement for two aircraft is specified by a minimum horizontal separation D (typically, D is 5 nautical miles). A loss of separation between two aircraft occurs when the distance between them is less than D.

Definition 1. *The ownship and intruder aircraft are in* loss of separation *if and only if it holds that*

$$\|\mathbf{s}_o - \mathbf{s}_i\| < D.$$

The separation requirement can be understood as an imaginary circle of diameter D around each aircraft, and a conflict between two aircraft as a future overlapping of these circles. In this paper, an alternative but equivalent view is considered where the intruder is surrounded by a circle, called the *protected zone*, of radius D. From this perspective, a conflict between the ownship and intruder aircraft is defined as the existence of a time $t \geq 0$ at which the ownship is in the interior of the intruder's protected zone. In conflict detection algorithms, it is also required that t is within a specified lookahead time. However, since this work concerns resolution algorithms, a lookahead time is not considered.

Definition 2. *The ownship and intruder aircraft are in* conflict *if and only if there exists $t \geq 0$ such that, at time t, separation is lost, i.e.,*

$$\|(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i)\| < D.$$

Since $(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i) = (\mathbf{s}_o - \mathbf{s}_i) + t\,(\mathbf{v}_o - \mathbf{v}_i)$, the mathematical expression that characterizes conflict can be defined on $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, i.e., the

relative position and velocity vectors, respectively, of the ownship with respect to the intruder. Therefore, conflict can be viewed as a predicate of two vectors $\mathbf{s}$ and $\mathbf{v}$ rather than a predicate of four vectors $\mathbf{s}_o$, $\mathbf{v}_o$, $\mathbf{s}_i$, and $\mathbf{v}_i$. This relative view simplifies the mathematical development presented in this paper. Thus, the predicate *Conflict* can be formally defined as follows.

$$\mathit{Conflict}(\mathbf{s}, \mathbf{v}) \equiv \exists\ t \geq 0 : \|\mathbf{s} + t\,\mathbf{v}\| < D. \tag{1}$$

In this paper, the relative position and velocity vectors, $\mathbf{s}$ and $\mathbf{v}$, will commonly be used in place of $\mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v}_o - \mathbf{v}_i$, respectively.

In a distributed airspace concept, a resolution algorithm can be defined as a function that computes one resolution maneuver for the aircraft that executes the algorithm, i.e., the ownship. In this paper, this definition is generalized such that a resolution algorithm returns a set of vectors, each of which represents a distinct resolution maneuver for the ownship. Since in PVS all functions are total, i.e., they are defined for all the elements of the domain, the generalization used in this paper has the advantage of encoding the case were no resolutions are available for the ownship as the empty set. That is, if there are no resolutions available, then an empty set is returned by the resolution algorithm.

Definition 3. *A* resolution algorithm *is a function* `cr` *that takes as arguments the current state of the ownship and intruder aircraft, e.g.,* $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i, \mathbf{v}_i$*, and returns a set of velocity vectors, where each of these vectors corresponds to a possible velocity maneuver for the ownship.*

The velocity maneuvers provided by a resolution algorithm are intended to maintain a safety objective between aircraft, which may be violated at the current state. Usually, the safety objective is that the aircraft are not in conflict. This paper uses an abstract concept of safety objectives, called *safety properties*. One characteristic of safety properties is that the ownship and the intruder agree on whether the given notion of safety is currently satisfied. That is, if from the perspective of one aircraft, the current state appears safe, then it should appear safe to the other aircraft as well. Section 4 will show that this restriction holds for typical safety objectives such as conflict-free, repulsion, and divergence.

Definition 4. *A* safety property *is a parametric predicate P on vectors, with parameters $\mathbf{s}, \mathbf{v}$, such that for all parameters $\mathbf{s}, \mathbf{v}$ and for all $\mathbf{v}'$,*

$$P_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff P_{-\mathbf{s},-\mathbf{v}}(-\mathbf{v}').$$

The vectors $\mathbf{s}$, $\mathbf{v}$, and $\mathbf{v}'$ in Definition 4 are intended to be relative vectors, where $\mathbf{s}$ and $\mathbf{v}$ are the current relative position and velocity of the aircraft.

Given a safety property P and current relative vectors $\mathbf{s}$ and $\mathbf{v}$, if $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ holds, then it is said that $\mathbf{v}'$ satisfies P. If a safety property P is satisfied when one of the aircraft maneuvers according to a resolution algorithm `cr`, while the other aircraft maintains its current velocity, then resolution algorithm `cr` is said to be independent for the safety property P or, more formally, *P-independent*.

Definition 5. *The resolution algorithm* $\mathtt{cr}$ *is* P-independent *if for all* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *and* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ *such that* $P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ *does not hold,* $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}_i)$ *holds for every vector* $\mathbf{v}'_o \in \mathtt{cr}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$.

Definition 5 can be read as "the resolution algorithm `cr` is independent for a safety property P if it computes velocity maneuvers for the ownship that restore P when P is not satisfied at the current state." Since vectors $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, and $\mathbf{v}'_o$ are universally quantified in this definition, if an algorithm `cr` is P-independent from the ownship's point of view, it is also P-independent from the intruder's point of view.

Even when two resolution algorithms $\mathtt{cr}_o$ and $\mathtt{cr}_i$ are both P-independent, it is still possible that the algorithms return maneuvers that do not satisfy the safety property when both aircraft simultaneously maneuver. The coordination property defined below ensures that the safety property is met when both aircraft simultaneously maneuver.

Definition 6. *A resolution algorithm* $\mathtt{cr}_o$ *is* P-coordinated *with a resolution algorithm* $\mathtt{cr}_i$ *if for all* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *and* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ *such that* $P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ *does not hold,* $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ *holds for all vectors* $\mathbf{v}'_o \in \mathtt{cr}_o(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ *and* $\mathbf{v}'_i \in \mathtt{cr}_i(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$.

Definition 6 can be read as "the resolution algorithm $\mathtt{cr}_o$ is coordinated with $\mathtt{cr}_i$ for a safety property P if they compute velocity maneuvers for the ownship and the intruder aircraft that restore the safety property P when P is not satisfied at the current state." This definition involves two algorithms simultaneously executed by two aircraft. From their own perspectives, each aircraft is the ownship while the other aircraft is the intruder. Since vectors $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, and $\mathbf{v}'_o$ are quantified universally, if an algorithm $\mathtt{cr}_o$ is P-coordinated with $\mathtt{cr}_i$, then it is also true that $\mathtt{cr}_i$ is P-coordinated with $\mathtt{cr}_o$.

It is noted that independence is a property held by one algorithm, while coordination is a property held by two algorithms. However, by abuse of notation, a resolution algorithm `cr` is said to be P-coordinated if it is P-coordinated with itself. This corresponds to the special case where the ownship and intruder aircraft are using the same algorithm.

3 A General Theory of Coordination

It is usually difficult to prove that resolution algorithms are P-independent or P-coordinated for a particular safety property P. Direct proofs of independence and coordination involve exhaustive case analyses that spell out of the control flow of the algorithms.

This section develops a mathematical theory that establishes sets of conditions, called criteria, that guarantee independence and coordination of resolution maneuvers for an abstract safety property. Using this theory, the proof that two algorithms $\mathtt{cr}_o$ and $\mathtt{cr}_i$ are independent and coordinated for a safety property P can be done in two steps:

1. Find a criterion that guarantees P-independence and P-coordination.

2. Prove the all resolution vectors computed by $\texttt{cr}_o$ and $\texttt{cr}_i$ satisfy the criterion.

This proof approach would be as difficult as a direct proof if it had to be done from scratch every time. However, the criterion constructed in the first step can often be defined in a general way so that it can be applied to a family of resolution algorithms. Section 5 gives several examples of criteria for different safety properties. The second step, i.e., the proof that the maneuvers computed by a resolution algorithm satisfy a criterion, still needs to be proved for every algorithm. But it can be argued that this proof is simpler than direct proofs of independence and coordination. Section 7 illustrates this technique with concrete resolution algorithms.

3.1 Independent and Coordinated Criteria

Definition 7. *A criterion is a parametric set of vectors $\mathcal{A}$, with parameters $\mathbf{s}, \mathbf{v}$.*

If $\mathcal{A}$ is a criterion, the set $\mathcal{A}_{\mathbf{s},\mathbf{v}}$ consists of vectors in the relative coordinate system, where the parameters $\mathbf{s}$ and $\mathbf{v}$ are, respectively, the current relative position and velocity of the aircraft. If all vectors in the set $\mathcal{A}_{\mathbf{s},\mathbf{v}}$ satisfy the safety property P, the criterion $\mathcal{A}$ is said to be independent for P or, more formally, *P-independent.*

Definition 8. *A criterion $\mathcal{A}$ is* P-independent *if for all for all parameters $\mathbf{s}, \mathbf{v}$ and for all $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$, $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ holds.*

A resolution algorithm satisfies a criterion when all the maneuvers computed by the algorithm are included in the criterion.

Definition 9. *A resolution algorithm $\texttt{cr}$ satisfies the criterion $\mathcal{A}$ if for all $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, $\mathbf{v}'_o \in \texttt{cr}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ implies $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$.*

The following theorem states that to prove that a resolution algorithm is P-independent, it is sufficient to prove that the algorithm satisfies a criterion that is P-independent. It is easily proved from the definitions.

Theorem 1. *If*

1. *the criterion $\mathcal{A}$ is P-independent and*
2. *the resolution algorithm $\texttt{cr}$ satisfies $\mathcal{A}$,*

then $\texttt{cr}$ is P-independent.

The concept of coordination for resolution algorithms can be generalized to a concept of coordination between criteria.

Definition 10. *A criterion $\mathcal{A}$ is P-coordinated with $\mathcal{B}$ if for all $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, and vectors $\mathbf{v}'_o, \mathbf{v}'_i$ such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds, $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{B}_{-\mathbf{s},-\mathbf{v}}$ imply that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ holds.*

By abuse of notation, it is said that a criterion $\mathcal{A}$ is P-coordinated if it is P-coordinated with itself.

The following theorem states that to prove that two resolution algorithms are coordinated for a safety property P, it is sufficient to show that the algorithms satisfy P-coordinated criteria.

Theorem 2. *If*

1. *the criterion $\mathcal{A}$ is P-coordinated with $\mathcal{B}$, and*

2. *the resolution algorithms $\mathtt{cr}_o$ and $\mathtt{cr}_i$ satisfy $\mathcal{A}$ and $\mathcal{B}$, respectively,*

then $\mathtt{cr}_o$ is P-coordinated with $\mathtt{cr}_i$.

3.2 A Theory of Criteria

Theorems 1 and 2 provide a way to prove that resolution algorithms are independent and coordinated for a safety property. At first glance, it seems that the problem of proving P-independence and P-coordination for resolution algorithms has been merely transformed into a problem of proving P-independence and P-coordination for set of the vectors in a criterion. However, the power of this approach is that criteria that satisfy P-independence and P-coordination can be defined in an abstract way, independently from specific safety properties or resolutions algorithms. This section presents basic conditions for the construction of these criteria. These conditions will be needed to prove the main theorems in Section 3.3, where the first conditions of theorems 1 and 2 are reduced to verifying simpler, geometric conditions.

Definition 11. *A set of vectors $\mathcal{S}$ is* closed under sum *if for all vectors $\mathbf{v}, \mathbf{u} \in \mathcal{S}$, $\mathbf{v} + \mathbf{u} \in \mathcal{S}$.*

It is often useful to know that for a given safety property P, the complement of P is closed under sum.

Definition 12. *A criterion $\mathcal{A}$ is sum independent for a safety property P if for all parameters $\mathbf{s}, \mathbf{v}$ such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds, for all vectors $\mathbf{u}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ and $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$, $P_{\mathbf{s},\mathbf{v}}(\mathbf{u}' + \mathbf{v}')$ holds.*

The relation between sum independence and coordination is the main focus of Section 3.3.

There is a notion of a set of vectors being independent of length, and this has slightly different definitions for criteria and safety properties.

Definition 13. *A criterion $\mathcal{A}$ is* independent of length *if for all vectors $\mathbf{s}, \mathbf{v}, \mathbf{v}'$, and for all positive real numbers r, $\mathcal{A}_{\mathbf{s},\mathbf{v}} = \mathcal{A}_{\mathbf{s},r\,\mathbf{v}}$.*

Definition 14. *A safety property P is* independent of length *if for all vectors $\mathbf{s}, \mathbf{v}, \mathbf{v}'$ and all positive real numbers r and p, $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff P_{\mathbf{s},r\,\mathbf{v}}(p\,\mathbf{v}')$.*

The notion of an open set is fundamental to the mathematical field of real analysis [17] and it is presented here, in the context of vector analysis, for completeness.

Definition 15. *A set $\mathcal{S}$ of vectors is* open *if for all $\mathbf{v} \in \mathcal{S}$, there exists a positive real number $\delta > 0$ such that for all vectors $\mathbf{u}$ with $\|\mathbf{u}\| < \delta$, $\mathbf{v} + \mathbf{u} \in \mathcal{S}$.*

For instance, if P is a safety property, then the set of vectors that do not satisfy P, denoted by $\neg P$, is open if for all vectors $\mathbf{s}, \mathbf{v}, \mathbf{v}'$ such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ holds, there exists $\delta > 0$ such that for all vectors $\mathbf{u}'$ with $\|\mathbf{u}'\| < \delta$, $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}')$.

The two criteria $\mathcal{A}$ and $\mathcal{B}$ in Definition 10 are usually defined by the same parametric set. In this case, the criterion is said to be *symmetric* if the two aircraft see the same set of vectors from their own perspectives.

Definition 16. *A criterion $\mathcal{A}$ is* symmetric *if for all parameters $\mathbf{s}, \mathbf{v}$ and for all vectors $\mathbf{v}'$, $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ if and only if $-\mathbf{v}' \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$.*

There is a dual concept to symmetry, called asymmetry, where the criteria $\mathcal{A}$ and $\mathcal{B}$ are defined by the same parametric set and the two aircraft see the same set of vectors except for a sign that encodes the perspective of the ownship. The formal definition of asymmetry requires a notion of *signed criterion* that takes an additional parameter $\varepsilon = \pm 1$. A signed criterion $\mathcal{A}^\varepsilon$ defines two criteria: $\mathcal{A}^{-1}$ and $\mathcal{A}^1$. The parameter ε refers to the side of the origin on which any trajectory from $\mathbf{s}$ along a vector in $\mathcal{A}^\varepsilon$ passes, from the perspective of the ownship. For example, in the Euclidean 2-dimensional airspace, ε may refer to the horizontal directions left and right. If a vertical dimension is considered, as in Section 5.4, the parameter ε may also refer to the vertical directions up and down.

Definition 17. *A signed criterion $\mathcal{A}^\varepsilon$ is* antisymmetric *if for all parameters $\mathbf{s}, \mathbf{v}, \varepsilon$ and for all vectors $\mathbf{v}'$, $\mathbf{v}' \in \mathcal{A}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ if and only if $-\mathbf{v}' \in \mathcal{A}^{-\varepsilon}_{-\mathbf{s},-\mathbf{v}}$.*

3.3 General Theorems

This section presents theorems that reduce the proof of criteria coordination to sum independence, which is a simpler geometric property.

Let $\mathcal{A}$ be a criterion and P be an arbitrary safety property. The results in this section use the following equality.

$$\mathbf{v}'_o - \mathbf{v}'_i = (\mathbf{v}'_o - \mathbf{v}_i) + (\mathbf{v}_o - \mathbf{v}'_i) - (\mathbf{v}_o - \mathbf{v}_i).$$

Theorem 3. *If*

1. *$\mathcal{A}$ is symmetric and sum independent for P, and*
2. *$\neg P$ is closed under sum,*

then $\mathcal{A}$ is P-coordinated.

Proof. Let $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, and $\mathbf{v}'_o, \mathbf{v}'_i$ be any vectors such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds. Suppose that both $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$. It suffices to prove that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ holds. Define $\mathbf{v}' = (\mathbf{v}'_o - \mathbf{v}_i)$ and $\mathbf{u}' = (\mathbf{v}_o - \mathbf{v}'_i)$. The goal is to prove that

$$P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}' - \mathbf{v}).$$

Assume that this is false. Since $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}' - \mathbf{v})$ and $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ both hold and $\neg P$ is closed under sum, it follows that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}')$ holds.

However, since $\mathcal{A}$ is symmetric and $-\mathbf{u}' \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$, it follows that $\mathbf{u}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$. Since $\mathcal{A}$ is sum independent for P, it follows that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}')$. This is a contradiction and therefore completes the proof. □

Theorem 4. *If*

1. *$\mathcal{A}$ is symmetric, independent of length, and P-coordinated, and*

2. *$\neg P$ is open and independent of length,*

then $\mathcal{A}$ is sum independent for P.

Proof. Suppose that $\mathbf{s}$, $\mathbf{v}$, $\mathbf{v}'$, and $\mathbf{u}'$ are vectors such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$, $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$, and $\mathbf{u}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$. It suffices to prove that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}')$ holds. Suppose that this is not true. Then $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}')$. Since $\neg P$ is open, there exists $\delta > 0$ such that if $\|\mathbf{w}\| < \delta$, then $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}' + \mathbf{w})$. In particular, $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v}' + \mathbf{u}' - c\,\mathbf{v})$, where $c = \frac{\delta}{2\|\mathbf{v}\|}$. Since $\neg P$ is independent of length, $\neg P_{\mathbf{s},c\,\mathbf{v}}(\mathbf{v}' + \mathbf{u}' - c\,\mathbf{v})$ holds.

Define vectors $\mathbf{v}_o$, $\mathbf{v}_i$, $\mathbf{v}'_o$ and $\mathbf{v}'_i$ as follows.

$$\begin{aligned}
\mathbf{v}_o &= \mathbf{0}, \\
\mathbf{v}_i &= -c\,\mathbf{v}, \\
\mathbf{v}'_o &= \mathbf{v}' - c\,\mathbf{v}, \\
\mathbf{v}'_i &= -\mathbf{u}'.
\end{aligned}$$

The following equations can easily be proved from these definitions.

$$\begin{aligned}
\mathbf{v}'_o - \mathbf{v}_i &= \mathbf{v}', \\
\mathbf{v}_o - \mathbf{v}'_i &= \mathbf{u}', \\
\mathbf{v}_o - \mathbf{v}_i &= c\,\mathbf{v}, \\
\mathbf{v}'_o - \mathbf{v}'_i &= \mathbf{v}' + \mathbf{u}' - c\,\mathbf{v}.
\end{aligned}$$

Thus, $\neg P_{\mathbf{s},c\,\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ holds. Since $\mathcal{A}$ is P-coordinated, it suffices to prove the following three properties.

1. $\neg P_{\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i}(\mathbf{v}_o - \mathbf{v}_i)$,

2. $\mathbf{v}'_o - \mathbf{v}_i \in \mathcal{A}_{\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i}$, and

3. $\mathbf{v}'_i - \mathbf{v}_o \in \mathcal{A}_{\mathbf{s}_i - \mathbf{s}_o, \mathbf{v}_i - \mathbf{v}_o}$.

The first of these properties is equivalent to $\neg P_{\mathbf{s},c\,\mathbf{v}}(c\,\mathbf{v})$, which follows from the facts that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds and that $\neg P$ is independent of length. The statement that $\mathbf{v}'_o - \mathbf{v}_i \in \mathcal{A}_{\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i}$ is equivalent to $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},c\,\mathbf{v}}$, which follows from the facts that $\mathbf{v}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ holds and that $\mathcal{A}$ is independent of length. Finally, the statement that $\mathbf{v}'_i - \mathbf{v}_o \in \mathcal{A}_{\mathbf{s}_i - \mathbf{s}_o, \mathbf{v}_i - \mathbf{v}_o}$ is equivalent to $-\mathbf{u}' \in \mathcal{A}_{-\mathbf{s},-c\,\mathbf{v}}$, which follows from the facts that $\mathbf{u}' \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ holds, that $\mathcal{A}$ is independent of length, and that $\mathcal{A}$ is symmetric. This completes the proof. □

Corollary 5 follows directly from Theorems 3 and 4.

Corollary 5 (Equivalence of Coordination and Sum Independence)**.** *Suppose that*

1. *$\mathcal{A}$ is symmetric, sum independent, and independent of length, and*

2. *$\neg P$ is open, closed under sum, and independent of length,*

then $\mathcal{A}$ is P-coordinated if and only if it is sum independent for P.

There are analogues of theorems 3 and 4 in the case where $\mathcal{A}^\varepsilon$ is antisymmetric. These are stated below, and the proofs are identical in form.

Theorem 6. *If*

1. *$\mathcal{A}^\varepsilon$ is antisymmetric and sum independent for P, and*

2. *$\neg P$ is closed under sum,*

then $\mathcal{A}^\varepsilon$ is P-coordinated with $\mathcal{A}^{-\varepsilon}$.

Theorem 7. *If*

1. *$\mathcal{A}^\varepsilon$ is antisymmetric, independent of length, and $\mathcal{A}^\varepsilon$ is P-coordinated with $\mathcal{A}^{-\varepsilon}$, and*

2. *$\neg P$ is open and independent of length,*

then $\mathcal{A}^\varepsilon$ is sum independent for P.

3.4 Derived and Composed Criteria

Criteria can be composed to form larger sets of vectors that preserve their coordination properties. This section presents two criteria combinators, called *derivation* and *composition*, and states theorems that provide sufficient conditions under which these combinators preserve coordination with respect to a safety property P.

Theorem 2 is used to prove that if $\mathtt{cr}_o$ and $\mathtt{cr}_i$ are resolution algorithms that satisfy the criterion $\mathcal{A}$, and if $\mathcal{A}$ is P-coordinated, then $\mathtt{cr}_o$ is P-coordinated with $\mathtt{cr}_i$. In some cases, the condition that $\mathtt{cr}_o$ satisfies $\mathcal{A}$ can be weakened in this statement. That is, given a criterion $\mathcal{A}$, it is often possible to construct a family of criteria from $\mathcal{A}$, called the *derived criteria* of $\mathcal{A}$, such that if $\mathtt{cr}_o$ satisfies one of the derived criteria and $\mathtt{cr}_i$ satisfies $\mathcal{A}$, the resolution algorithms $\mathtt{cr}_o$ and $\mathtt{cr}_i$ are still P-coordinated. The family of derived criteria of $\mathcal{A}$ is parameterized by a nonnegative number p.

Definition 18. *Let $\mathcal{A}$ be a criterion, the family of derived criteria of $\mathcal{A}$, denoted $Deriv^p(\mathcal{A})$, is defined as follows.*

$$Deriv^p(\mathcal{A})_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \mid (\mathbf{v}' - p\,\mathbf{v}) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}\}.$$

From this definition it is easy to see that $Deriv^0(\mathcal{A})_{\mathbf{s},\mathbf{v}} = \mathcal{A}$. Theorem 8 gives sufficient conditions under which the criterion $Deriv^p(\mathcal{A})$ is a weaker condition on the algorithm $\mathtt{cr}_o$ than the criterion $\mathcal{A}$.

Theorem 8. *If the criterion $\mathcal{A}$ is closed under sum and $-p\,\mathbf{v} \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$, then $\mathcal{A}$ is a subset of $Deriv^p(\mathcal{A})$, i.e.,*

$$\mathcal{A}_{\mathbf{s},\mathbf{v}} \subseteq Deriv^p(\mathcal{A})_{\mathbf{s},\mathbf{v}}.$$

The most important property of the derived criterion is coordination.

Theorem 9. *If $p + q = 1$ and the criterion $\mathcal{A}$ is symmetric and sum independent for a safety property P, then $Deriv^p(\mathcal{A})$ is P-coordinated with $Deriv^q(\mathcal{A})$.*

Proof. Let $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, and $\mathbf{v}'_o, \mathbf{v}'_i$ be any vectors such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds. Suppose that $(\mathbf{v}'_o - \mathbf{v}_i) \in Deriv^p(\mathcal{A})_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in Deriv^q(\mathcal{A})_{-\mathbf{s},-\mathbf{v}}$. It suffices to prove that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ holds. By the definition of $Deriv^q(\mathcal{A})$, $\mathbf{v}'_i - \mathbf{v}_o + q\,\mathbf{v} \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$. Since $\mathcal{A}$ is symmetric, $\mathbf{v}_o - \mathbf{v}'_i - q\,\mathbf{v} \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$. By the definition of $Deriv^p(\mathcal{A})$, $\mathbf{v}'_o - \mathbf{v}_i - p\,\mathbf{v} \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$. Since $\mathcal{A}$ is sum independent, it follows that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}_i - p\,\mathbf{v} + \mathbf{v}_o - \mathbf{v}'_i - q\,\mathbf{v})$ holds, and since $\mathbf{v}'_o - \mathbf{v}_i - p\,\mathbf{v} + \mathbf{v}_o - \mathbf{v}'_i - q\,\mathbf{v} = \mathbf{v}'_o - \mathbf{v}'_i$, the result follows. □

Corollary 10. *If the criterion $\mathcal{A}$ is symmetric and sum independent for P, then $\mathcal{A}$ is P-coordinated with $Deriv^1(\mathcal{A})$.*

It is important to note that if the criterion $\mathcal{A}_{\mathbf{s},\mathbf{v}}$ contains the vector $\mathbf{0}$, then the derived criterion $Deriv^p(\mathcal{A})_{\mathbf{s},\mathbf{v}}$ contains the relative velocity vector $p\,\mathbf{v}$. In particular, if $p = 1$ and $\mathcal{A}$ is symmetric and sum independent for P, then, by Corollary 10, an algorithm $\mathtt{cr}_o$ that always returns the current velocity vector is coordinated with an algorithm $\mathtt{cr}_i$ that satisfies the criterion $\mathcal{A}$.

Using the derivation combinator, a composition combinator is defined that takes two criteria $\mathcal{A}$ and $\mathcal{B}$ and composes them into one criterion $Comp(\mathcal{A}, \mathcal{B})$, called the composed criterion, that contains vectors from both criteria.

Definition 19. *The composed criterion of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows.*

$$Comp(\mathcal{A}, \mathcal{B}) \equiv \mathcal{A} \cup (\mathcal{B} \cap Deriv^1(\mathcal{A})). \tag{2}$$

The following theorem gives sufficient conditions under which the composed criterion preserves coordination for a safety property P.

Theorem 11. *Let $\mathcal{A}$ be sum independent for the safety property P and symmetric, such that $\neg P$ is closed under sum. If $\mathcal{B}$ is P-coordinated with $\mathcal{B}'$, then $Comp(\mathcal{A}, \mathcal{B})$ is P-coordinated with $Comp(\mathcal{A}, \mathcal{B}')$.*

Proof. Theorem 3 in Section 3.3 implies that $\mathcal{A}$ is coordinated. Let $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, and $\mathbf{v}'_o, \mathbf{v}'_i$ be any vectors such that $\neg P_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds. Suppose that $(\mathbf{v}'_o - \mathbf{v}_i) \in Comp(\mathcal{A}, \mathcal{B})_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in Comp(\mathcal{A}, \mathcal{B}')_{-\mathbf{s},-\mathbf{v}}$. It suffices to prove that $P_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}'_i)$ holds. There are four possibilities:

1. $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$.
2. $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{A}_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{B}'_{-\mathbf{s},-\mathbf{v}} \cap Deriv^1(\mathcal{A})_{-\mathbf{s},-\mathbf{v}}$.
3. $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{B}_{\mathbf{s},\mathbf{v}} \cap Deriv^1(\mathcal{A})_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{A}_{-\mathbf{s},-\mathbf{v}}$.
4. $(\mathbf{v}'_o - \mathbf{v}_i) \in \mathcal{B}_{\mathbf{s},\mathbf{v}} \cap Deriv^1(\mathcal{A})_{\mathbf{s},\mathbf{v}}$ and $(\mathbf{v}'_i - \mathbf{v}_o) \in \mathcal{B}'_{-\mathbf{s},-\mathbf{v}} \cap Deriv^1(\mathcal{A})_{-\mathbf{s},-\mathbf{v}}$.

The result follows in the first case from the fact that $\mathcal{A}$ is coordinated for P, in the second and third cases from the fact that $\mathcal{A}$ is coordinated with $Deriv^1(\mathcal{A})$ for P (Corollary 10), and in the final case from the fact that $\mathcal{B}$ and $\mathcal{B}'$ are coordinated for P. □

4 Safety Properties

As stated in Definition 4 in Section 2.2, a safety property is a predicate used by the ownship and intruder aircraft to agree on whether a particular state is safe. For example, if the ownship determines that a given relative resolution maneuver is conflict-free, the same resolution maneuver must be conflict-free when considered by the intruder aircraft.

The safety property of interest for conflict resolution algorithms is the absence of conflict between the ownship and the intruder aircraft. However, in some circumstances, that safety property cannot be immediately recovered, for example when the aircraft are already in loss of separation. In those cases, it may be useful to consider a stronger safety property such as *divergence*, i.e., the resolution maneuvers guarantee that the distance between the aircraft immediately increases, or a weaker safety property such as *repulsion*, i.e., the resolution maneuvers guarantee that the distance at time of closest approach increases.

In an expressive logic like the one provided by the verification system PVS, safety properties can be specified using universal or existential quantifiers. The statement of these definitions follow their natural logical description. However, quantifiers are not always implementable in an algorithmic way and, therefore, these natural definitions of safety properties cannot be mechanically checked by a computer. This section provides analytical definitions of safety properties that can be implemented by algorithms and that are equivalent to their intuitive logical description based on quantifiers.

4.1 Absence of Conflict

The safety property that determines whether a given state is conflict-free is defined by the predicate

$$\mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \equiv \neg \mathit{Conflict}(\mathbf{s},\mathbf{v}'). \tag{3}$$

The parametric predicate *ConflictFree* is a safety property according to Definition 4 in Section 2.2, i.e., it satisfies the following condition.

$$\mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff \mathit{ConflictFree}_{-\mathbf{s},-\mathbf{v}}(-\mathbf{v}').$$

Since Formula (3) involves a quantifier in the definition of *Conflict*, it is not practical for checking whether a given relative vector $\mathbf{v}'$ is conflict-free. This section offers an alternative, but equivalent, characterization of the safety property *ConflictFree*, where the quantifier has been eliminated.

If $\mathit{Conflict}(\mathbf{s},\mathbf{v})$ holds, then the half line $\mathbf{s} + t\,\mathbf{v}$, with $t \geq 0$, must intersect the circle of radius D at exactly two distinct times t. These times correspond to the solutions of the quadratic equation

$$\begin{aligned} 0 &= \|\mathbf{s} + t\,\mathbf{v}\|^2 - D^2 \\ &= \|\mathbf{v}\|^2\, t^2 + 2\,(\mathbf{s}\cdot\mathbf{v})\, t + (\|\mathbf{s}\|^2 - D). \end{aligned} \tag{4}$$

The discriminant of the polynomial on t in Formula (4) is given by $4\,\Delta(\mathbf{s},\mathbf{v})$, where

$$\Delta(\mathbf{s},\mathbf{v}) \equiv \|\mathbf{s}\cdot\mathbf{v}\|^2 - \|\mathbf{v}\|^2\,(\|\mathbf{s}\|^2 - D^2). \tag{5}$$

Equation (4) has solutions when the discriminant is nonnegative and it has exactly two distinct solutions when it is strictly positive, i.e., when $\Delta(\mathbf{s}, \mathbf{v}) > 0$. The roots of this quadratic equation are given by the following function, where $\iota = \pm 1$.

$$\Theta_D(\mathbf{s}, \mathbf{v}, \iota) \equiv \frac{-\mathbf{s} \cdot \mathbf{v} + \iota \sqrt{\|\mathbf{s} \cdot \mathbf{v}\|^2 - \|\mathbf{v}\|^2(\|\mathbf{s}\|^2 - D^2)}}{\|\mathbf{v}\|^2}. \tag{6}$$

If *Conflict*$(\mathbf{s}, \mathbf{v})$ holds, $\Theta_D(\mathbf{s}, \mathbf{v}, -1)$ is the time when the aircraft lose separation and $\Theta_D(\mathbf{s}, \mathbf{v}, 1)$ is the time when the aircraft recover separation. The following lemmas are proved by algebraic manipulations.

Lemma 12. *If* $\Delta(\mathbf{s}, \mathbf{v}) \geq 0$*, then* $\|\mathbf{s} + \Theta_D(\mathbf{s}, \mathbf{v}, \iota)\, \mathbf{v}\| = D$*, for* $\iota = \pm 1$*, and*

$$\Theta_D(\mathbf{s}, \mathbf{v}, -1) \leq \Theta_D(\mathbf{s}, \mathbf{v}, 1).$$

Furthermore, if $\Delta(\mathbf{s}, \mathbf{v}) > 0$*, then* $\Theta_D(\mathbf{s}, \mathbf{v}, -1) < \Theta_D(\mathbf{s}, \mathbf{v}, 1)$.

Lemma 13. *If* $\|\mathbf{s}\| \geq D$*, then* *Conflict*$(\mathbf{s}, \mathbf{v})$ *holds if and only if* $\mathbf{s} \cdot \mathbf{v} < 0$ *and* $\Delta(\mathbf{s}, \mathbf{v}) > 0$.

From Equation (3) and Lemma 13, the following equivalence holds.

$$\mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff \|\mathbf{s}\| \geq D \texttt{ and } (\mathbf{s} \cdot \mathbf{v}' \geq 0 \texttt{ or } \Delta(\mathbf{s}, \mathbf{v}') \leq 0). \tag{7}$$

Equation (7) provides and analytical way to check whether a relative vector $\mathbf{v}'$ results in a projected linear trajectory that is free of conflict.

4.2 Divergence

Another safety property that is useful, especially when the aircraft are already in loss of separation, is divergence. It is stronger than the notion of conflict in the case where the aircraft are currently horizontally separated. The ownship and the intruder aircraft are (horizontally) divergent if the distance between the aircraft is increasing, i.e.,

$$\mathit{Divergence}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \equiv \forall\, t > 0 : \|\mathbf{s} + t\, \mathbf{v}'\| > \|\mathbf{s}\|. \tag{8}$$

The parametric predicate *Divergence* is a safety property, i.e., it satisfies that for all vectors $\mathbf{v}'$,

$$\mathit{Divergence}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff \mathit{Divergence}_{-\mathbf{s},-\mathbf{v}}(-\mathbf{v}').$$

Equation (8) is not practical for checking divergence because of the universal quantification on t. However, it can be proved using basic algebra that divergence is equivalent to the dot product $\mathbf{s} \cdot \mathbf{v}$ being nonnegative. Thus, the following equivalence holds.

$$\mathit{Divergence}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff \mathbf{s} \cdot \mathbf{v}' \geq 0. \tag{9}$$

Equation (9) provides and analytical way to check whether a relative vector $\mathbf{v}'$ results in a projected linear trajectory that is divergent.

4.3 Repulsion

A resolution maneuver is repulsive if it increases the minimum future distance between the aircraft. Given the current relative velocity vector $\mathbf{v}_o - \mathbf{v}_i$, for the ownship aircraft with respect to the intruder, repulsion is a predicate on the relative velocity vector $\mathbf{v}'_o - \mathbf{v}_i$, where $\mathbf{v}'_o$ is a new velocity vector representing a maneuver for the ownship. It implies that the minimum distance achieved by the aircraft for positive time is greater if the new velocity vector $\mathbf{v}'_o$ is chosen by the ownship instead of the current vector $\mathbf{v}_o$. This is formalized as follows.

The time $\mathtt{tca}(\mathbf{s}, \mathbf{v})$, referred to as the *time of closest approach* for the vectors $\mathbf{s}$ and $\mathbf{v}$, is the time at which the aircraft achieve minimum horizontal separation. If the relative velocity vector $\mathbf{v}$ is zero, the distance between the aircraft remains constant at the value $\|\mathbf{s}\|$. In this case, the time of closest approach is defined to be 0. In the general case, the time of closest approach is defined as follows.

$$\mathtt{tca}(\mathbf{s}, \mathbf{v}) \equiv \begin{cases} 0, & \text{if } \mathbf{v} = \mathbf{0}, \\ -\frac{\mathbf{s}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}, & \text{otherwise.} \end{cases} \tag{10}$$

The following theorem, which is proved using elementary algebraic methods, states that `tca` indeed computes the time of closest approach between the aircraft.

Theorem 14. *For all real numbers t,*

$$\|\mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}\| \;\leq\; \|\mathbf{s} + t\,\mathbf{v}\|.$$

A stronger result can be proved when $\mathbf{v}$ is nonzero.

Theorem 15. *If $\mathbf{v} \neq \mathbf{0}$, then for all real numbers $t \neq \mathit{tca}(\mathbf{s}, \mathbf{v})$,*

$$\|\mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}\| \;<\; \|\mathbf{s} + t\,\mathbf{v}\|.$$

Using the function `tca`, repulsive resolution maneuvers are formally defined by the following predicate.

$$\begin{aligned} \mathit{Repulsion}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \equiv\; & \mathtt{tca}(\mathbf{s}, \mathbf{v}) > 0 \;\; \mathtt{and} \\ & (\mathtt{tca}(\mathbf{s}, \mathbf{v}') \leq 0 \;\; \mathtt{or} \\ & \|\mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}\| < \|\mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v}')\,\mathbf{v}'\|). \end{aligned} \tag{11}$$

Equation (11) provides an analytical way to check whether a relative vector $\mathbf{v}'$ results in a projected linear trajectory that is repulsive with respect to the original vector $\mathbf{v}$. The parametric predicate *Repulsion* is a safety property, i.e., it satisfies that for all vectors $\mathbf{v}'$,

$$\mathit{Repulsion}_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff \mathit{Repulsion}_{-\mathbf{s},-\mathbf{v}}(-\mathbf{v}').$$

5 Criteria

As stated in Definition 7 in Section 3.1, a criterion is a parametric set of vectors with parameters $\mathbf{s}$ and $\mathbf{v}$. This section presents concrete examples of criteria that are useful when proving coordination and independence for the safety properties *Divergence*, *ConflictFree*, and *Repulsion*. Theorems about these criteria are presented, including whether they are closed under sum, sum independent, independent of length, symmetric, and antisymmetric. One criterion that is particularly helpful when proving results about conflict resolution algorithms is the horizontal criterion, given by Definition 13 in Section 5.1. In Section 5.3, proofs of some fundamental properties of this criterion are presented, including a proof that it is maximal among signed, symmetric criteria that are independent of length and coordinated for *ConflictFree*. Finally, in Section 5.4, the notions of criteria, coordination, independence, and resolution algorithms are all extended to a three dimensional airspace. In that section, an antisymmetric criterion is defined for vertical maneuvers, and it is proved that this criterion is coordinated for the 3D version of the *ConflictFree* safety property. This new criterion is combined with the horizontal criterion using tools from Section 3.4 to form a new criterion that is coordinated and allows both vertical and horizontal resolution maneuvers.

5.1 Divergence, Horizontal, and Repulsion Criteria

Figure 1 illustrates the criterion $\mathcal{D}$, called *divergence criterion*, which is defined as follows.

$$\mathcal{D}_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \,|\, \mathbf{s} \cdot \mathbf{v}' \geq 0\}. \tag{12}$$

The following lemma states that the criterion $\mathcal{D}$ is independent for the safety property *Divergence* (Section 4.2) and, if $\|\mathbf{s}\|^2 \geq D$, it is also independent for the safety property *ConflictFree* (Section 4.1).

Lemma 16.

- *For all* $\mathbf{v}' \in \mathcal{D}_{\mathbf{s},\mathbf{v}}$, *Divergence*$_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ *holds.*
- *If* $\|\mathbf{s}\|^2 \geq D$, *then for all* $\mathbf{v}' \in \mathcal{D}_{\mathbf{s},\mathbf{v}}$, *ConflictFree*$_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ *holds.*

Proof. By Formula (9) in Section 4.2, $\mathcal{D}$ consists of all vectors that satisfy the predicate *Divergence*. The second part is a direct consequence of the definition of the predicate *Divergence*. □

Figure 2 illustrates the signed criterion $\mathcal{H}^{\varepsilon}$, called *horizontal criterion*, which is defined as follows.

$$\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \,|\, \|\mathbf{s}\| \geq D \text{ and } \mathbf{s} \cdot \mathbf{v}' \geq \frac{\varepsilon}{D}(\mathbf{s} \cdot \mathbf{v}'^{\perp})\sqrt{\|\mathbf{s}\|^2 - D^2}\}. \tag{13}$$

The signed criterion $\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ defines two criteria: $\mathcal{H}^{1}_{\mathbf{s},\mathbf{v}}$, which is shown in blue, and $\mathcal{H}^{-1}_{\mathbf{s},\mathbf{v}}$, which is shown in green. The sets $\mathcal{H}^{1}_{\mathbf{s},\mathbf{v}}$ and $\mathcal{H}^{-1}_{\mathbf{s},\mathbf{v}}$ are not disjoint. Indeed, vectors in the gray area are in both sets. The following lemmas state that the signed criterion $\mathcal{H}^{\varepsilon}$ is *ConflictFree* -independent.

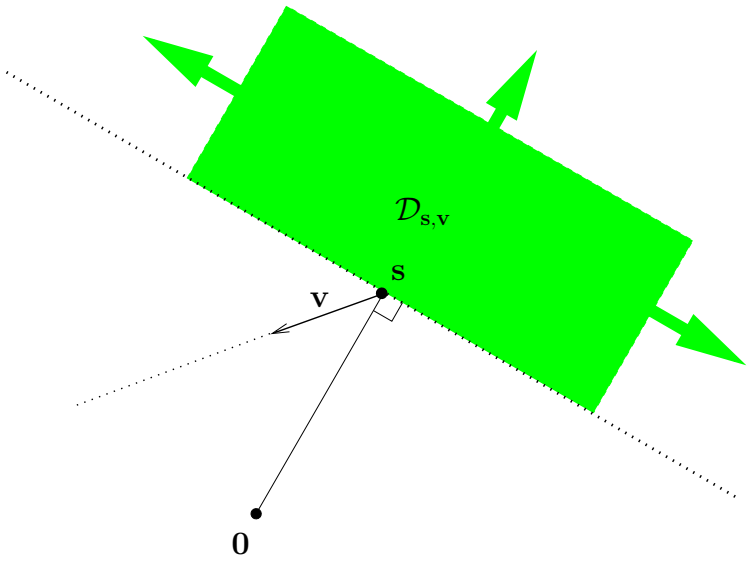

Figure 1. Divergence Criterion $\mathcal{D}$

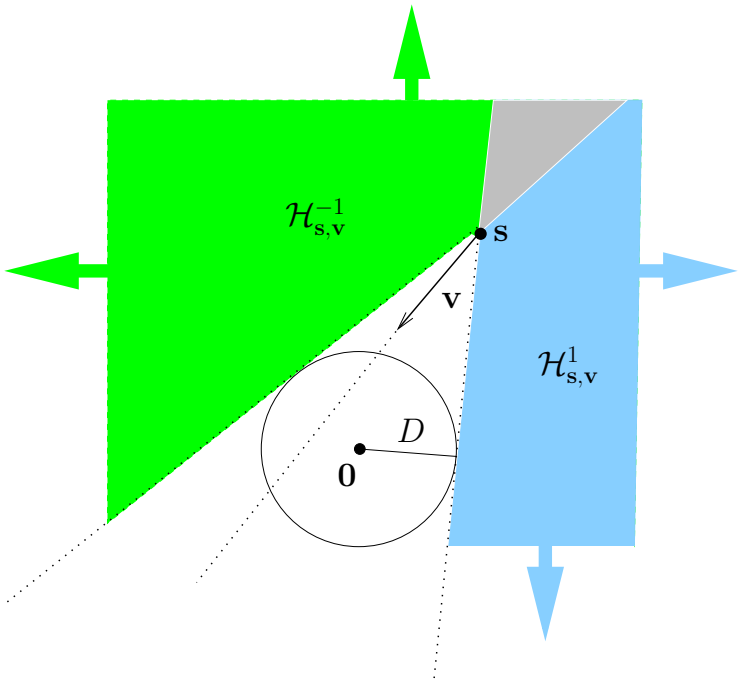

Figure 2. Horizontal Criterion $\mathcal{H}^{\varepsilon}$

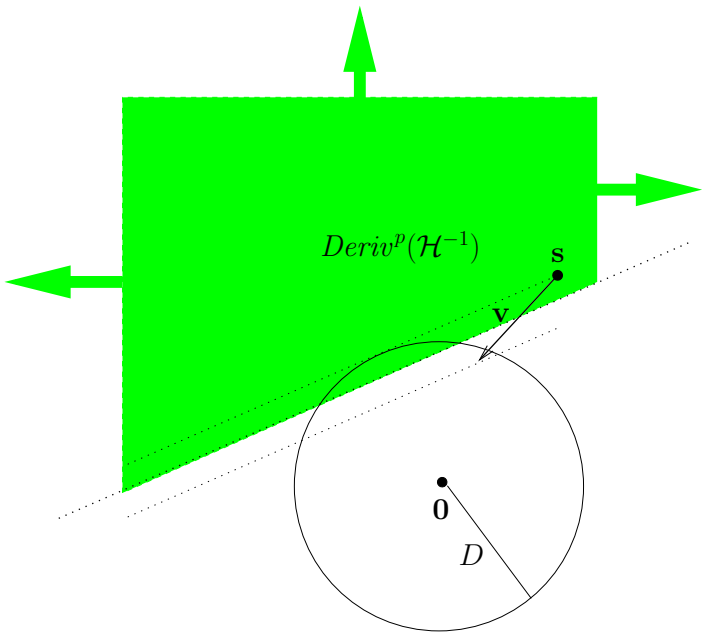

Figure 3. Derived Horizontal Criterion $Deriv^p(\mathcal{H}^{-1})$

Lemma 17. *For all* $\mathbf{v}' \in \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, $ConflictFree_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ *holds.*

Figure 3 illustrates the derived horizontal criterion $Deriv^p(\mathcal{H}^{-1})$, when $p \leq 1$. That criterion is a superset of $\mathcal{H}^{-1}$. However, in contrast to $\mathcal{H}^{-1}$, the derived criterion $Deriv^p(\mathcal{H}^{-1})$ is not independent for the safety property *ConflictFree* when $p > 0$. As shown by Figure 3, some vectors in $Deriv^p(\mathcal{H}^{-1})$ intersect the protected area around the intruder aircraft.

Figure 4 illustrates the signed criterion $\mathcal{R}^{\varepsilon}$, called *repulsion criterion*, which is defined as follows.

$$\mathcal{R}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \mid \varepsilon\, \mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0 \ \texttt{and}\ \mathbf{s} \cdot \mathbf{v} < 0 \ \texttt{and}\ \mathbf{s} \cdot \mathbf{v}' \leq 0 \ \texttt{and}\ \varepsilon\, \mathbf{v}' \cdot \mathbf{v}^{\perp} < 0\}. \quad (14)$$

The set $\mathcal{R}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ is always empty when $\varepsilon = \texttt{sign}(\mathbf{s} \cdot \mathbf{v}^{\perp})$. The following lemma gives sufficient conditions under which the signed criterion $\mathcal{R}^{\varepsilon}$ is *Repulsion*-independent (Section 4.3).

Lemma 18. *If* $Conflict(\mathbf{s}, \mathbf{v})$ *holds, then for all* $\mathbf{v}' \in \mathcal{R}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, $Repulsion_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ *holds.*

Proof. Suppose $Conflict(\mathbf{s}, \mathbf{v})$ holds and $\mathbf{v}' \in \mathcal{R}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, but that $Repulsion_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ does not hold. By the definition of the predicate *Repulsion*, $\texttt{tca}(\mathbf{s}, \mathbf{v})$ and $\texttt{tca}(\mathbf{s}, \mathbf{v}')$ are both positive. It suffices to prove that

$$\|\mathbf{s} + \texttt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v}\| < \|\mathbf{s} + \texttt{tca}(\mathbf{s}, \mathbf{v}')\, \mathbf{v}'\|.$$

It is a basic property of linear algebra that if $\mathbf{e}_1$ and $\mathbf{e}_2$ are any two nonzero orthogonal vectors, then any vector $\mathbf{w}$ can be written as a linear combination of $\mathbf{e}_1$

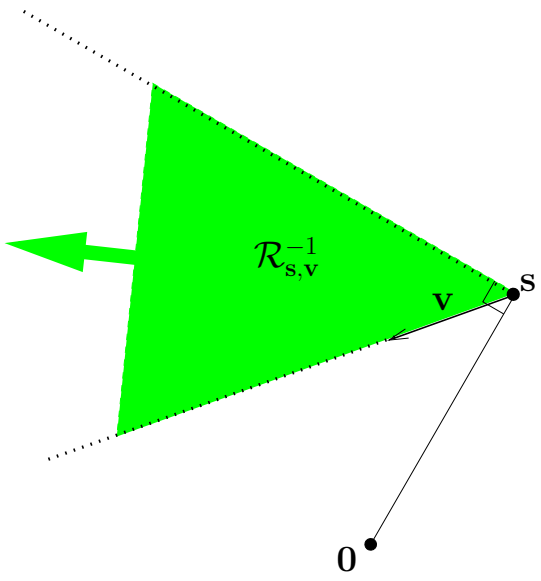

Figure 4. Repulsion Criterion $\mathcal{R}^{\varepsilon}$

and $\mathbf{e}_2$ as follows.

$$\mathbf{w} = \frac{\mathbf{w}\cdot\mathbf{e}_1}{\|\mathbf{e}_1\|^2}\,\mathbf{e}_1 + \frac{\mathbf{w}\cdot\mathbf{e}_2}{\|\mathbf{e}_2\|^2}\,\mathbf{e}_2.$$

Further, with such a decomposition, the squared norm $\|\mathbf{w}\|^2$ can be computed by the following equation.

$$\|\mathbf{w}\|^2 \;=\; \frac{(\mathbf{w}\cdot\mathbf{e}_1)^2}{\|\mathbf{e}_1\|^2} + \frac{(\mathbf{w}\cdot\mathbf{e}_2)^2}{\|\mathbf{e}_2\|^2}.$$

Let $\mathbf{s}_{\mathtt{tca}} = \mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v})\,\mathbf{v}$. It is easy to see from the definition of the function $\mathtt{tca}$ that $\mathbf{s}_{\mathtt{tca}}$ and $\mathbf{v}$ are orthogonal. Thus, the following equation is satisfied.

$$\mathbf{v}' = \frac{\mathbf{v}'\cdot\mathbf{s}_{\mathtt{tca}}}{\|\mathbf{s}_{\mathtt{tca}}\|^2}\,\mathbf{s}_{\mathtt{tca}} + \frac{\mathbf{v}'\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\,\mathbf{v}.$$

Similarly, $\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v})\,\mathbf{v}$ can be written as a linear combination of $\mathbf{v}$ and $\mathbf{v}^{\perp}$, and since it is perpendicular to $\mathbf{v}$ and $-\varepsilon\,\mathbf{s}\cdot\mathbf{v}^{\perp}$ is nonnegative, it follows that

$$\|\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v})\,\mathbf{v}\| = \frac{-1}{\|\mathbf{v}\|}\,\varepsilon\,\mathbf{s}\cdot\mathbf{v}^{\perp}.$$

If it can be proved that

$$\|\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}'\| \geq \frac{-1}{\|\mathbf{v}\|}\,(\varepsilon\,\mathbf{s}\cdot\mathbf{v}^{\perp} + \varepsilon\,\mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}'\cdot\mathbf{v}^{\perp}), \tag{15}$$

then the result will follow, because $\mathtt{tca}(\mathbf{s},\mathbf{v}') > 0$, and by hypothesis, $-\varepsilon\,\mathbf{v}'\cdot\mathbf{v}^{\perp} > 0$. Thus, it suffices to prove Equation (15). It follows from definitions that

$$\begin{aligned}
-\varepsilon\,\mathbf{s}\cdot\mathbf{v}^{\perp} - \varepsilon\,\mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}'\cdot\mathbf{v}^{\perp} &= (\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}')\cdot(-\varepsilon\,\mathbf{v}^{\perp}) \\
&\leq \|\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}'\|\,\|-\varepsilon\,\mathbf{v}^{\perp}\| \\
&= \|\mathbf{s} + \mathtt{tca}(\mathbf{s},\mathbf{v}')\,\mathbf{v}'\|\,\|\mathbf{v}\|,
\end{aligned}$$

where the inequality is given by the Cauchy-Schwartz inequality. The result follows from there. □

5.2 Coordination Properties of $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$

Proven facts about these criteria are presented below, including whether they are closed under sum, sum independent, independent of length, symmetric, and anti-symmetric. These results are used to deduce that the criteria $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$, defined in Section 5.1, are coordinated with themselves for the safety properties *Divergence*, *ConflictFree*, and *Repulsion*, respectively.

Lemma 19. *The criteria $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$, for $\varepsilon = \pm 1$, are all closed under sum.*

Lemma 20. *The complement of the predicates ConflictFree and Divergence are closed under sum.*

Lemma 21. *All of the following propositions hold.*

1. *The Criterion $\mathcal{D}$ is sum independent for Divergence.*
2. *The Signed criterion $\mathcal{H}^\varepsilon$ is sum independent for ConflictFree.*
3. *The Signed criterion $\mathcal{R}^\varepsilon$ is sum independent for Repulsion.*

Lemma 22. *The criteria $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$, for $\varepsilon = \pm 1$, are all independent of length.*

Lemma 23. *The safety properties Divergence, ConflictFree, Repulsion, and their complements, are all independent of length.*

Lemma 24. *The criteria $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$, for $\varepsilon = \pm 1$, are all symmetric.*

The results in Section 3.3, which relate coordination between criteria to these simpler geometric properties, are used to deduce coordination for the criteria defined in Section 5.1.

Theorem 25. *The criteria $\mathcal{D}$, $\mathcal{H}^\varepsilon$, and $\mathcal{R}^\varepsilon$ are coordinated with themselves for the safety properties Divergence, ConflictFree, and Repulsion, respectively.*

Proof. This follows from Corollary 5 in Section 3.3 and from lemmas 19 to 24. □

The horizontal criterion $\mathcal{H}^\varepsilon$ satisfies all the hypotheses of Theorem 9. Hence, even although in general the derived criterion $Deriv^p(\mathcal{H}^\varepsilon)$ is not independent or coordinated with itself for the safety property *ConflictFree*, it is always coordinated with $\mathcal{H}^\varepsilon$ for the safety property *ConflictFree*.

Lemma 26. *For $\varepsilon = \pm 1$ and $p \geq 0$, the derived criterion $Deriv^p(\mathcal{H}^\varepsilon)$ is ConflictFree-coordinated with $\mathcal{H}^\varepsilon$.*

The following theorem is a consequence of Lemma 26. Its usefulness is illustrated in Section 7.3 to prove that two distinct resolution algorithms are *ConflictFree*-coordinated.

Theorem 27. *Suppose that* $\mathtt{cr}_o$ *and* $\mathtt{cr}_i$ *are resolution algorithms for the ownship and the intruder aircraft, respectively, such that* $\mathtt{cr}_o$ *satisfies* $\mathcal{H}^\varepsilon$ *and that for all vectors* $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i, \mathbf{v}_i'$, $\mathbf{v}_i' \in \mathtt{cr}_i(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$ *implies* $\mathbf{v}_i' - \mathbf{v}_i \in \mathcal{H}^\varepsilon_{\mathbf{s}_i-\mathbf{s}_o, \mathbf{v}_i-\mathbf{v}_o}$. *Then* $\mathtt{cr}_o$ *and* $\mathtt{cr}_i$ *are coordinated for the safety property ConflictFree.*

Proof. By lemma 26, it suffices to prove that the algorithm $\mathtt{cr}_i$ satisfies the criterion $Deriv^1(\mathcal{H}^\varepsilon)$. That is, it suffices to prove that if $\mathbf{v}_i' \in \mathtt{cr}_i(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$, then $\mathbf{v}_i' - \mathbf{v}_o \in Deriv^1(\mathcal{H}^\varepsilon)_{\mathbf{s}_i-\mathbf{s}_o, \mathbf{v}_i-\mathbf{v}_o}$, or equivalently $(\mathbf{v}_i' - \mathbf{v}_o) - (\mathbf{v}_i - \mathbf{v}_o) \in \mathcal{H}^\varepsilon_{\mathbf{s}_i-\mathbf{s}_o, \mathbf{v}_i-\mathbf{v}_o}$. Since $(\mathbf{v}_i' - \mathbf{v}_o) - (\mathbf{v}_i - \mathbf{v}_o) = \mathbf{v}_i' - \mathbf{v}_i$, the result follows. □

5.3 Fundamental Properties of Horizontal Criterion

According to Lemma 17, the horizontal criterion $\mathcal{H}^\varepsilon$ is *ConflictFree*-independent and, according to Theorem 25, $\mathcal{H}^\varepsilon$ is also *ConflictFree*-coordinated. Since *ConflictFree* is the intended safety property of conflict resolution algorithms, the horizontal criterion is particularly important for designing CD&R algorithms and for verifying their coordination properties.

This section provides some fundamental results on the horizontal criterion. First, it is proved that any conflict resolution algorithm that computes relative velocity vectors that are tangent to the protected zone satisfies the horizontal criterion and, therefore, it is independent and coordinated for the safety property *ConflictFree*. Second, it is shown that the horizontal criterion is maximal among signed criteria that are symmetric, independent of length, and coordinated for *ConflictFree*.

5.3.1 Tangential Resolutions Satisfy Horizontal Criterion

A common approach to developing algorithms that resolve conflicts between the ownship and the intruder is to find a new velocity vector $\mathbf{v}_o'$ for the ownship such that the new relative velocity vector $\mathbf{v}' = \mathbf{v}_o' - \mathbf{v}_i$ has the property that the trajectory from $\mathbf{s}$ along $\mathbf{v}'$ is tangent to the circle of radius D around the origin. If the intruder does not maneuver, then the minimum separation between the aircraft is precisely D. Here, these kinds of resolutions are called *line solutions*.

In Figure 5, the vector $\mathbf{v}'$ is tangent to the right side of the circle. From this diagram, it is clear that $\mathbf{s} \cdot \mathbf{v}'^{\perp} = D\,\|\mathbf{v}'^{\perp}\| = D\,\|\mathbf{v}'\|$. Similarly, in the case where the trajectory from $\mathbf{s}$ along $\mathbf{v}'$ is tangent to the left side of the circle, the equation $-\mathbf{s} \cdot \mathbf{v}'^{\perp} = D\,\|\mathbf{v}'\|$ holds. In addition, since the trajectory from $\mathbf{s}$ along $\mathbf{v}'$ reaches the tangent point at a nonnegative time, and since this time is equal to $\mathtt{tca}(\mathbf{s}, \mathbf{v}')$, it follows from the definition of the function $\mathtt{tca}$ that $\mathbf{s} \cdot \mathbf{v}' \leq 0$.

This motivates the following definition of the predicate *LineSolution*, which determines whether a given trajectory, in the relative coordinate system, is tangent to the circle of radius D around the origin. The predicate depends on a unit value $\varepsilon = \pm 1$, with $\varepsilon = -1$ corresponding to a right tangent and $\varepsilon = 1$ to a left tangent.

$$\mathit{LineSolution}(\mathbf{s}, \mathbf{v}', \varepsilon) \equiv -\varepsilon\,\mathbf{s} \cdot \mathbf{v}'^{\perp} = D\|\mathbf{v}'\| \text{ and } \mathbf{s} \cdot \mathbf{v}' \leq 0. \tag{16}$$

This predicate holds for vectors $\mathbf{s}$ and $\mathbf{v}'$ precisely when the half line $\mathbf{s} + t\,\mathbf{v}'$, with $t \geq 0$, is tangent to the circle of radius D around the intruder in the relative coordinate system.

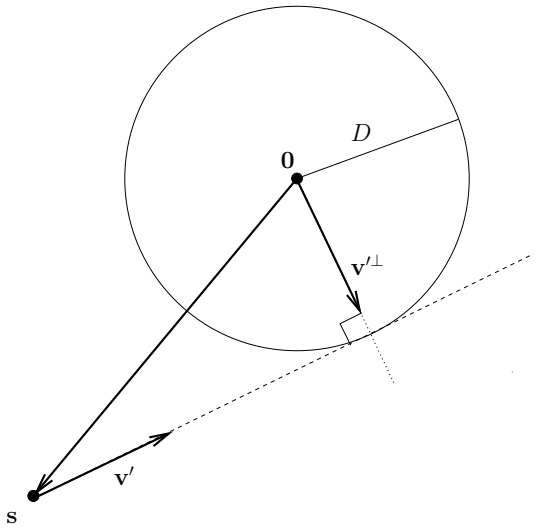

Figure 5. Relative Vector $\mathbf{v}'$ Tangent to the Circle

Lemma 28. *If LineSolution*$(\mathbf{s}, \mathbf{v}', \varepsilon)$ *holds, then*

$$\|\mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v}')\,\mathbf{v}'\| = D.$$

Since line solutions characterize tangent trajectories to the circle of radius D, they yield conflict free resolutions. This fact is a direct consequence of Theorem 14 and Lemma 28.

Theorem 29. *If LineSolution*$(\mathbf{s}, \mathbf{v}', \varepsilon)$ *holds, then* $\neg$*Conflict*$(\mathbf{s}, \mathbf{v}')$.

The function `tangent_line`, defined below, is used to compute vectors that satisfy the predicate *LineSolution*. It takes as arguments a relative position vector $\mathbf{s}$ such that $\|\mathbf{s}\| \geq D$ and a unit value $\varepsilon = \pm 1$. It returns a vector that is tangent to the protected zone.

$$\begin{array}{l}
\texttt{tangent_line}(\mathbf{s}, \varepsilon) \equiv \\
\quad \texttt{if } \|\mathbf{s}\| = D \texttt{ then} \\
\quad\quad \varepsilon\, \mathbf{s}^{\perp} \\
\quad \texttt{else} \\
\quad\quad \texttt{let } d = \|\mathbf{s}\|^2 \texttt{ in} \\
\quad\quad\quad (\dfrac{D^2}{d} - 1)\,\mathbf{s} + \dfrac{\varepsilon\, D\sqrt{d - D^2}}{d}\,\mathbf{s}^{\perp} \\
\quad \texttt{endif}
\end{array} \tag{17}$$

The proofs of the following lemmas rely on standard vector algebra.

Lemma 30. *If* $\|\mathbf{s}\| \geq D$ *and* $\varepsilon = \pm 1$, *then LineSolution*$(\mathbf{s}, \mathit{tangent\_line}(\mathbf{s}, \varepsilon), \varepsilon)$ *holds.*

Lemma 31. *If* $\|\mathbf{s}\| \geq D$*, then* $\mathit{LineSolution}(\mathbf{s}, \mathbf{v}', \varepsilon)$ *holds if and only if there exists* $k \geq 0$ *such that*

$$\mathbf{v}' = k \ \mathit{tangent\_line}(\mathbf{s}, \varepsilon).$$

Lemma 32 gives an alternative characterization of the horizontal criterion that uses the function `tangent_line`.

Lemma 32. *For all vectors* $\mathbf{s}$ *and* $\mathbf{v}$*, and* $\varepsilon = \pm 1$*,*

$$\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}} = \{\mathbf{v}' \mid \|\mathbf{s}\| \geq D \ \text{ and } \ \varepsilon\, \mathit{tangent\_line}(\mathbf{s}, \varepsilon) \cdot \mathbf{v}'^{\perp} \geq 0\}.$$

The following lemma states that vectors that satisfy the predicate *LineSolution* also satisfy the horizontal criterion. Therefore, since by Theorem 25, the horizontal criterion $\mathcal{H}^{\varepsilon}$ is *ConflictFree*-coordinated, resolution algorithms that compute line solutions are also *ConflictFree*-coordinated. This result is stated by Theorem 34.

Lemma 33. *If* $\|\mathbf{s}\| \geq D$ *and* $\mathit{LineSolution}(\mathbf{s}, \mathbf{u}, \varepsilon)$ *holds, then* $\mathbf{u} \in \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$*.*

Proof. By Lemma 31, if $\mathit{LineSolution}(\mathbf{s}, \mathbf{u}, \varepsilon)$ holds, then there exists $k \geq 0$ such that $\mathbf{u} = k\,\mathtt{tangent\_line}(\mathbf{s}, \varepsilon)$. Thus, $\varepsilon\,\mathtt{tangent\_line}(\mathbf{s}, \varepsilon) \cdot \mathbf{u}^{\perp} = 0$, and the result follows directly from Lemma 32. □

Theorem 34. *Suppose that* $\mathtt{cr}_o$ *and* $\mathtt{cr}_i$ *are resolution algorithms for the ownship and the intruder, respectively, and that for all vectors* $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i, \mathbf{v}'_o, \mathbf{v}'_i$*,*

1. $\mathbf{v}'_o \in \mathtt{cr}_o(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ *implies that* $\mathit{LineSolution}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$ *holds, and*
2. $\mathbf{v}'_i \in \mathtt{cr}_i(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$ *implies that* $\mathit{LineSolution}(-\mathbf{s}, \mathbf{v}'_i - \mathbf{v}_o, \varepsilon)$ *holds.*

Then $\mathtt{cr}_o$ *and* $\mathtt{cr}_i$ *are coordinated for the safety property ConflictFree.*

Proof. By Theorem 2 in Section 3.1 and Theorem 25, it suffices to prove that $\mathtt{cr}_o$ and $\mathtt{cr}_i$ each satisfy the criterion $\mathcal{H}^{\varepsilon}$. This follows immediately from Lemma 33. □

5.3.2 Maximality of Horizontal Criterion

Theorem 35 (Maximality of $\mathcal{H}^{\varepsilon}$)**.** *Suppose that* $\mathcal{A}^{\varepsilon}$ *is a symmetric signed criterion that is independent of length and contains the horizontal criterion* $\mathcal{H}^{\varepsilon}$*. If* $\mathcal{A}^{\varepsilon}$ *is coordinated for ConflictFree and* $\neg \mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ *holds, then* $\mathcal{A}^{\varepsilon}_{\mathbf{s},\mathbf{v}} = \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$*.*

Proof. Note that Theorem 4 implies that $\mathcal{A}^{\varepsilon}$ is sum independent. By contradiction, suppose that there are vectors $\mathbf{s}$, $\mathbf{v}$, and $\mathbf{v}'$ such that $\mathbf{v}' \in \mathcal{A}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, $\mathbf{v}' \notin \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, and $\neg \mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v})$ holds. By Lemma 32,

$$\varepsilon\,\mathtt{tangent\_line}(\mathbf{s}, \mathbf{v}) \cdot \mathbf{v}'^{\perp} < 0.$$

By using the Cauchy-Schwartz inequality, it is shown that there exists $\delta > 0$ such that for every vector $\mathbf{w}$ with $\|\mathbf{w}\| < \delta$, $\varepsilon\,\mathtt{tangent\_line}(\mathbf{s}, \mathbf{v}) \cdot (\mathbf{v}' + \mathbf{w})^{\perp} < 0$. For any such vector $\mathbf{w}$, $\mathbf{v}' + \mathbf{w} \notin \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$.

Choose any positive real number c such that $\|c\,\mathbf{s}\| < \delta$. Therefore, $\mathbf{v}' + c\,\mathbf{s} \notin \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$. By Lemma 32, if $\mathbf{u}$ is any vector that is not an element of the set $\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, then the

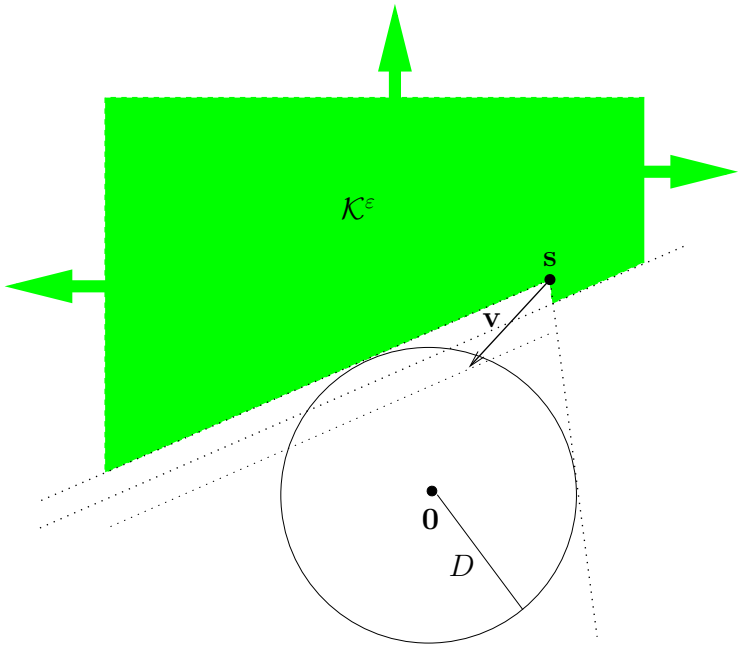

Figure 6. Criterion $\mathcal{K}^{\varepsilon}$

negative vector $-\mathbf{u}$ is an element of this set. Applying this to the vector $\mathbf{v}' + c\,\mathbf{s}$, it follows that $-\mathbf{v}' - c\,\mathbf{s}$ is an element of $\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ and therefore an element of $\mathcal{A}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ as well.

Since $-\mathbf{v}' - c\,\mathbf{s}$ and $\mathbf{v}'$ are both elements of $\mathcal{A}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, and since $\mathcal{A}$ is sum independent, it follows that their sum, which is equal to $-c\,\mathbf{s}$, satisfies $\mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(-c\,\mathbf{s})$. This is a contradiction since $\mathbf{s} + \frac{1}{c}\,(-c\,\mathbf{s}) = \mathbf{0}$. □

The theorem above states that the horizontal criterion is maximal among symmetric, coordinated, signed criteria that are independent of length. This result is false if the hypothesis of length-independence is removed, even among criteria that are *ConflictFree*-independent. In particular, the derived criterion $\mathit{Deriv}^{\frac{1}{2}}(\mathcal{H}^{\varepsilon})$ contains the horizontal criterion $\mathcal{H}^{\varepsilon}$. This derived criterion is coordinated with itself by Theorem 9 in Section 3.4. Define a new signed criterion $\mathcal{K}^{\varepsilon}$ as follows.

$$\mathcal{K}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \equiv \mathit{Deriv}^{\frac{1}{2}}(\mathcal{H}^{\varepsilon})_{\mathbf{s},\mathbf{v}} \;\cap\; (\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \cup \mathcal{H}^{-\varepsilon}_{\mathbf{s},\mathbf{v}}). \tag{18}$$

This criterion is shown graphically in Figure 6.

It follows from definitions that the criterion $\mathcal{K}^{\varepsilon}$ is symmetric. By Lemma 17 , $\mathcal{H}^{\varepsilon}$ and $\mathcal{H}^{-\varepsilon}$ are *ConflictFree*-independent. Since $\mathcal{K}^{\varepsilon}$ is contained in the union of these two sets, $\mathcal{K}^{\varepsilon}$ is *ConflictFree*-independent as well. However, since $\mathit{Deriv}^{\frac{1}{2}}(\mathcal{H}^{\varepsilon})$ is not independent of length, neither is the signed criterion $\mathcal{K}^{\varepsilon}$. If $\neg \mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ holds, then $\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \subset \mathcal{K}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ and this inclusion is proper. Thus, the horizontal criterion $\mathcal{H}^{\varepsilon}$ is not maximal among symmetric, *ConflictFree*-coordinated, signed criteria.

5.4 Three-Dimensional Criteria

For notational convenience, the mathematical framework presented in this paper is illustrated in the Euclidean 2-dimensional airspace. However, all definitions such as those of safety property, conflict resolution algorithm, independence, coordination, criterion, etc., and their properties, naturally extend to the Euclidean 3-dimensional geometry. The formal development in PVS discussed in this paper is both 2-dimensional and 3-dimensional.

As opposed to the rest of this paper, all vectors in this section, e.g., the position and velocity vectors of the aircraft $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i$, are assumed to be 3-dimensional. Furthermore, $\mathbf{u}_{(x,y)}$ and $\mathbf{u}_z$ denote the 2-dimensional and vertical projections of $\mathbf{u}$, respectively. When r is a real number, the notation $\mathbf{u}$ `with` $[z \leftarrow r]$ denotes the 3-dimensional vector $(\mathbf{u}_x, \mathbf{u}_y, r)$.

In the Euclidean 3-dimensional airspace, the separation requirement for two aircraft is specified by a minimum horizontal separation D and a minimum vertical separation H, which is typically 1000 feet. In the relative 3-dimensional coordinate system, the separation requirement is represented by a cylinder of radius D and half-height H around the intruder aircraft. A loss of separation between two aircraft occurs when the ownship enters this cylinder, i.e., when the following inequalities hold

$$\begin{aligned} \|(\mathbf{s}_o - \mathbf{s}_i)_{(x,y)}\| &< D, \\ |(\mathbf{s}_o - \mathbf{s}_i)_z| &< H. \end{aligned}$$

A conflict in the 3-dimensional airspace is defined as a projected loss of separation and is formally defined by the predicate $3D\,Conflict$.

$$3D\,Conflict(\mathbf{s}, \mathbf{v}) \equiv \exists\ t \geq 0 : \|(\mathbf{s} + t\,\mathbf{v})_{(x,y)}\| < D \ \texttt{and}\ |(\mathbf{s} + t\,\mathbf{v})_z| < H,$$

where $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$. Therefore, the 3-dimensional predicate $3D\,ConflictFree$ is defined as follows.

$$3D\,ConflictFree_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \equiv \neg 3D\,Conflict(\mathbf{s}, \mathbf{v}'). \tag{19}$$

The following lemma relates two-dimensional and three-dimensional conflicts.

Lemma 36. *A 3-dimensional conflict implies a 2-dimensional one, i.e.,*

$$3D\,Conflict(\mathbf{s}, \mathbf{v}) \implies Conflict(\mathbf{s}_{(x,y)}, \mathbf{v}_{(x,y)}).$$

Moreover, 3-dimensional ConflictFree is implied by the 2-dimensional one, i.e.,

$$ConflictFree_{\mathbf{s}_{(x,y)},\mathbf{v}_{(x,y)}}(\mathbf{v}'_{(x,y)}) \implies 3D\,ConflictFree_{\mathbf{s},\mathbf{v}}(\mathbf{v}').$$

The parametric predicate $3D\,ConflictFree$ is a safety property, i.e., it satisfies that for all 3-dimensional vectors $\mathbf{v}'$,

$$3D\,ConflictFree_{\mathbf{s},\mathbf{v}}(\mathbf{v}') \iff 3D\,ConflictFree_{-\mathbf{s},-\mathbf{v}}(-\mathbf{v}').$$

Furthermore, the complement of $3D\,ConflictFree$, i.e., the set of vectors that satisfy $\neg 3D\,ConflictFree$, is closed under sum.

The natural 3-dimensional extension of the horizontal criterion $\mathcal{H}^{\varepsilon}$, called $3D\mathcal{H}^{\varepsilon}$, is defined as follows.

$$3D\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \mid \mathbf{v}'_{(x,y)} \in \mathcal{H}^{\varepsilon}_{\mathbf{s}_{(x,y)},\mathbf{v}_{(x,y)}}\}$$

As stated by the following theorem, the criterion $3D\mathcal{H}^{\varepsilon}$ satisfies all the properties of its 2-dimensional counterpart $\mathcal{H}^{\varepsilon}$.

Theorem 37. *The three-dimensional signed criterion $3D\mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ is independent for the safety property $3D\,ConflictFree$. Furthermore, it is symmetric and sum independent for $3D\,ConflictFree$.*

The following theorem follows from Theorem 3 in Section 3.3, Theorem 37, and properties of the predicate $3D\,ConflictFree$.

Theorem 38. *The signed criterion $3D\,ConflictFree$ is coordinated with itself for $3D\,ConflictFree$.*

A more interesting example of a 3-dimensional signed criterion is the vertical criterion $\mathcal{V}^{\varepsilon}$, which is defined as follows.

$$\begin{aligned}
\mathcal{V}^{\varepsilon}_{\mathbf{s},\mathbf{v}} \equiv \{\mathbf{v}' \mid\ & (\|\mathbf{v}_{(x,y)}\| = 0 \ \texttt{and}\ \varepsilon\,\mathbf{v}'_z \geq 0 \ \texttt{and}\ \varepsilon\,\mathbf{s}_z \geq H) \\
& \quad \texttt{or} \\
& (\texttt{let}\ \iota = \texttt{if}\ |\mathbf{s}_z| \geq H\ \texttt{then}\ \varepsilon\,\texttt{sign}(\mathbf{s}_z)\ \texttt{else}\ -1\ \texttt{endif}\ \texttt{in} \\
& \Delta(\mathbf{s}_{(x,y)}, \mathbf{v}_{(x,y)}) > 0 \ \texttt{and}\ \Theta_D(\mathbf{s}_{(x,y)}, \mathbf{v}_{(x,y)}, \iota) > 0 \ \texttt{and} \\
& \texttt{let}\ \mathbf{p} = (\mathbf{s} + \Theta_D(\mathbf{s}_{(x,y)}, \mathbf{v}_{(x,y)}, \iota)\,\mathbf{v})\ \texttt{with}\ [z \leftarrow \varepsilon\,H]\ \texttt{in} \\
& \mathit{IntersectsHalfPlane}(\mathbf{s}, \mathbf{v}', \mathbf{p}, \varepsilon))\},
\end{aligned}$$

where

$$\begin{aligned}
\mathit{IntersectsHalfPlane}(\mathbf{s}, \mathbf{v}', \mathbf{p}, \varepsilon) \equiv\ & \mathbf{v}' \cdot \mathbf{p} \neq 0 \ \texttt{and} \\
& \texttt{let}\ t = \frac{D^2 - \mathbf{s} \cdot \mathbf{p}}{\mathbf{v}' \cdot \mathbf{p}}\ \texttt{in} \\
& t \geq 0 \ \texttt{and} \\
& \varepsilon\,(\mathbf{s}_z + t\,\mathbf{v}'_z) \geq \varepsilon\,\mathbf{p}_z,
\end{aligned}$$

Intuitively, the set $\mathcal{V}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$ consists of vectors $\mathbf{v}'$ that solve a predicted conflict by maintaining vertical separation when the aircraft are not horizontally separated. In the vertical criterion $\mathcal{V}^{\varepsilon}$, the unit value ε represents the two possible regions for vertical resolution: up, when $\varepsilon = 1$, and down, when $\varepsilon = -1$. Figure 7 illustrates $\mathcal{V}^{\varepsilon}$ when $\varepsilon = 1$. Reference [12] provides a detailed description of this criterion.

Lemma 39. *The signed criterion $\mathcal{V}^{\varepsilon}$ is independent for $3D\,ConflictFree$, i.e., for all $\mathbf{v}' \in \mathcal{V}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$, $3D\,ConflictFree_{\mathbf{s},\mathbf{v}}(\mathbf{v}')$ holds.*

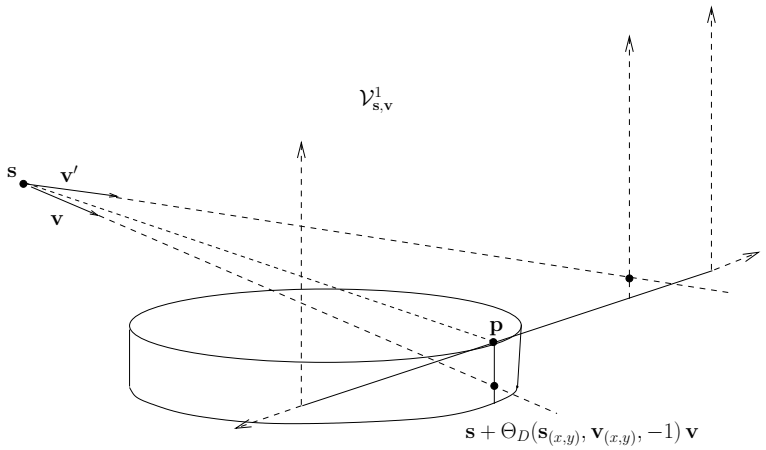

Figure 7. Vertical Criterion $\mathcal{V}^1$

The vertical criterion $\mathcal{V}^\varepsilon$ is an example of an antisymmetric criterion. Furthermore, it is sum independent for the safety property *3D ConflictFree*. Therefore, by Theorem 6 in Section 3.3, the following coordination property holds.

Theorem 40. *The vertical criterion $\mathcal{V}^\varepsilon$ is coordinated with $\mathcal{V}^{-\varepsilon}$ for the safety property 3D ConflictFree.*

In contrast to the horizontal criterion where both aircraft have to use the same horizontal direction, i.e., both left or both right, to solve a predicted conflict, the vertical criterion requires that the aircraft use opposite vertical directions, i.e., up/down or down/up, to solve the conflict.

Theorems 38 and 40 guarantee that the horizontal and vertical criteria are each coordinated with themselves for *3D ConflictFree*. However, those theorems do not guarantee that the criteria are coordinated with each other. In general, it does not hold that two algorithms $\texttt{cr}_o$ and $\texttt{cr}_i$ that satisfy, respectively, the horizontal criterion $3D\mathcal{H}^\varepsilon$ and the vertical criterion $\mathcal{V}^\varepsilon$ are coordinated for *3D ConflictFree*. Theorem 11 in Section 3.4 provides a simple way to combine different criteria, which are coordinated with themselves for a safety property P. The composed criterion contains vectors in both criteria and is coordinated for P. The following theorem provides a criterion $\mathcal{C}^{\varepsilon_h,\varepsilon_v}$, parametric by two unit values ε_h and ε_v, that is coordinated for *3D ConflictFree*.

Theorem 41. *Let $\mathcal{C}^{\varepsilon_h,\varepsilon_v}$ be the criterion defined as the following set of 3-dimensional vectors.*

$$\mathcal{C}^{\varepsilon_h,\varepsilon_v}_{\mathbf{s},\mathbf{v}} \equiv \mathit{Comp}(3D\mathcal{H}^\varepsilon, \mathcal{V}^\varepsilon)_{\mathbf{s},\mathbf{v}}.$$

The criterion $\mathcal{C}^{\varepsilon_h,\varepsilon_v}$ is coordinated with $\mathcal{C}^{\varepsilon_h,-\varepsilon_v}$ for the safety property 3D ConflictFree.

Proof. The results follows directly from theorems 11 (Section 3.4), 37, 40, and the fact that the complement of the predicate *3D ConflictFree* is closed under sum. □

Finally, Theorem 41 gives sufficient conditions to show that two, possibly different, 3-dimensional conflict resolution algorithms are coordinated.

Theorem 42. *Let* $\texttt{cr}_o$ *and* $\texttt{cr}_i$ *be 3-dimensional conflict resolution algorithms. If* $\texttt{cr}_o$ *satisfies* $\mathcal{C}^{\varepsilon_h,\varepsilon_v}$ *and* $\texttt{cr}_i$ *satisfies* $\mathcal{C}^{\varepsilon_h,-\varepsilon_v}$*, then* $\texttt{cr}_o$ *and* $\texttt{cr}_i$ *are coordinated for* $3D\,ConflictFree$*.*

Proof. The result follows from Theorem 2 in Section 3.1 and Theorem 41. □

The criterion $\mathcal{C}^{\varepsilon_h,\varepsilon_v}$ illustrates the usefulness of composing two criteria using *Comp*. The composed criteria is coordinated for $3D\,ConflictFree$. Since it is 3-dimensional, it allows for both horizontal and vertical maneuvers. This criterion is at the basis of a standard for guaranteeing implicit coordination of 3-dimensional conflict resolution algorithms [12] in a distributed self-separation airspace concept.

6 Conflict Resolution Algorithms

As defined in Section 2, an algorithm that returns guidance maneuvers that attempt to restore a safety property is called a resolution algorithm. When the safety property is *ConflictFree*, the algorithm is called a *conflict* resolution algorithm.

This section provides several examples of conflict resolution algorithms and states some basic properties of these algorithms. In particular, the following algorithms are described: the Modified Voltage Potential algorithm [8] developed at the National Aerospace Laboratory (NLR) in the Netherlands, an algorithm for track angle maneuvers developed at NASA Langley Research Center as part of the Airborne Coordinated Conflict Resolution and Detection (ACCoRD) framework [13], and the Geometric Optimization algorithm [2] for track angle maneuvers developed at NASA Ames Research Center. The main results in this section are that the Modified Voltage Potential algorithm is not *ConflictFree*-independent and that the track-angle algorithms of ACCoRD and Geometric Optimization are *ConflictFree*-independent.

6.1 Modified Voltage Potential

The Modified Voltage Potential [8] algorithm `MVP` is a conflict resolution algorithm developed by NLR in the Netherlands. It takes as inputs the current state vectors of the aircraft, i.e., $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i$, and returns either an empty set if a resolution is not found or a singleton set $\{\mathbf{v}'_o\}$, where $\mathbf{v}'_o$ is a new velocity vector for the ownship.

The algorithm relies on the function `tca`, defined in Section 4.3, which gives the time of closest approach between the aircraft. The first step of the `MVP` algorithm is to compute the time $\texttt{tca}(\mathbf{s}, \mathbf{v})$, where $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$. Then, the algorithm computes the relative position of the two aircraft at the time $\texttt{tca}(\mathbf{s}, \mathbf{v})$, when the aircraft achieve minimum separation. The relative position at this time is $\mathbf{s} + \texttt{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}$, which is denoted $\mathbf{s}_{\texttt{tca}}$ and graphically shown in Figure 8. Here, it is assumed that the aircraft are currently in conflict, although this is not required for the definition of the algorithm `MVP`.

In Figure 8, the point $\mathbf{s}_{\texttt{tca}}$ is shown at the end of the dotted line. It is perpendicular to the vector $\mathbf{v}$, i.e., $\mathbf{s}_{\texttt{tca}} \cdot \mathbf{v} = 0$. If $\mathbf{s}_{\texttt{tca}} \neq \mathbf{0}$, then the algorithm `MVP` returns a

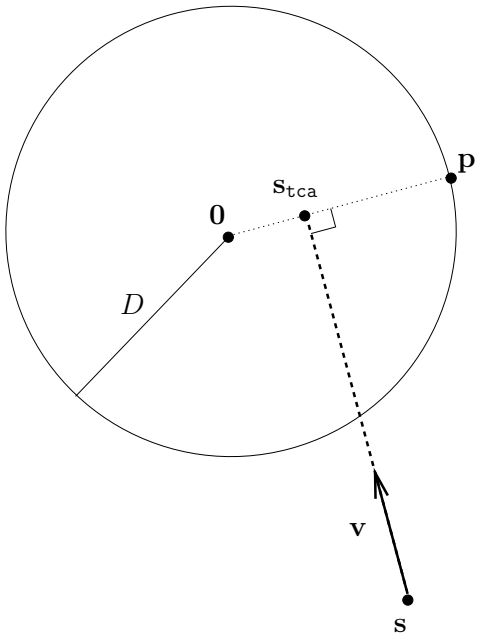

Figure 8. Relative Position $\mathbf{s}_{\mathtt{tca}}$ at Time of Closest Approach

velocity vector $\mathbf{v}'_o$ for the ownship such that the relative vector $\mathbf{v}' = \mathbf{v}'_o - \mathbf{v}_i$ satisfies

$$\mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v}' = \mathbf{p},$$

where $\mathbf{p}$ denotes $\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}}$. This is illustrated in Figure 9.

Thus, if $\mathtt{tca}(\mathbf{s}, \mathbf{v}) > 0$, then $\mathbf{v}'_o$ can be calculated as follows.

$$\begin{aligned}
\mathbf{v}'_o &= \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\, (\mathbf{p} - \mathbf{s}) + \mathbf{v}_i \\
&= \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\, (\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}} - \mathbf{s}) + \mathbf{v}_i \\
&= \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\, (\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}} - (\mathbf{s}_{\mathtt{tca}} - \mathtt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v})) + \mathbf{v}_i \\
&= \left(\frac{D - \|\mathbf{s}_{\mathtt{tca}}\|}{\mathtt{tca}(\mathbf{s}, \mathbf{v})\, \|\mathbf{s}_{\mathtt{tca}}\|} \right) \mathbf{s}_{\mathtt{tca}} + \mathbf{v} + \mathbf{v}_i \\
&= \left(\frac{D - \|\mathbf{s}_{\mathtt{tca}}\|}{\mathtt{tca}(\mathbf{s}, \mathbf{v})\, \|\mathbf{s}_{\mathtt{tca}}\|} \right) \mathbf{s}_{\mathtt{tca}} + \mathbf{v}_o.
\end{aligned} \tag{20}$$

Equation (20) motivates the following definition of the algorithm `MVP`, which returns the empty set if the time $\mathtt{tca}(\mathbf{s}, \mathbf{v})$ is not positive or the vector $\mathbf{s}_{\mathtt{tca}}$ is equal to $\mathbf{0}$.

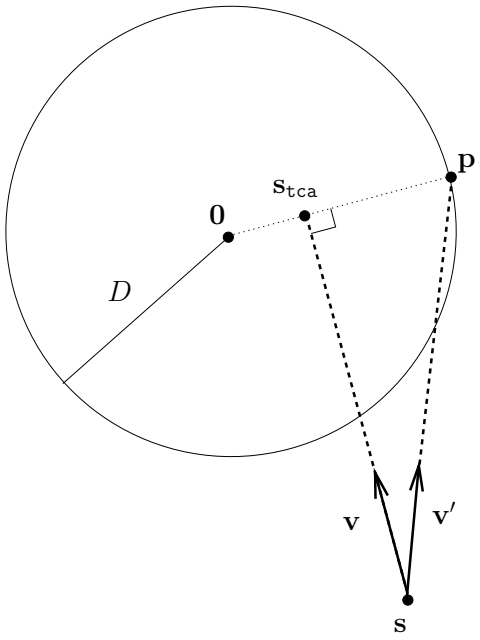

Figure 9. MVP's Relative Vector $\mathbf{v}' = \mathbf{v}'_o - \mathbf{v}_i$

$$
\begin{array}{l}
\mathtt{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i) \equiv \\
\quad \mathtt{let} \\
\quad\quad \mathbf{s} \quad = \mathbf{s}_o - \mathbf{s}_i, \\
\quad\quad \mathbf{v} \quad = \mathbf{v}_o - \mathbf{v}_i, \\
\quad\quad \mathbf{s}_{\mathtt{tca}} = \mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v} \\
\quad \mathtt{in} \\
\quad \mathtt{if}\ \mathtt{tca}(\mathbf{s}, \mathbf{v}) > 0\ \mathtt{and}\ \mathbf{s}_{\mathtt{tca}} \neq \mathbf{0}\ \mathtt{then} \\
\quad\quad \mathtt{let}\ \ \mathbf{v}'_o = \left(\dfrac{D - \|\mathbf{s}_{\mathtt{tca}}\|}{\mathtt{tca}(\mathbf{s}, \mathbf{v})\, \|\mathbf{s}_{\mathtt{tca}}\|} \right) \mathbf{s}_{\mathtt{tca}} + \mathbf{v}_o\ \ \mathtt{in} \\
\quad\quad\quad \{\mathbf{v}'_o\} \\
\quad \mathtt{else} \\
\quad\quad \emptyset \\
\quad \mathtt{endif}
\end{array}
\tag{21}
$$

The next lemma follows directly from Equation (20) and states that the resolution maneuver provided by MVP achieves a distance D at the original time of closest approach $\mathtt{tca}(\mathbf{s}, \mathbf{v})$. Unfortunately, as shown by Theorem 44, this result does not imply that MVP achieves a distance D at the time of closest approach for the resolution maneuver, i.e., $\mathtt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i)$.

Lemma 43. *For all* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *and* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$*, if* $\mathbf{v}'_o \in \mathit{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ *then*

$$\mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v})\,(\mathbf{v}'_o - \mathbf{v}_i) = \frac{D}{\|\mathbf{s}_{\mathit{tca}}\|}\,\mathbf{s}_{\mathit{tca}},$$

where $\mathbf{s}_{\mathit{tca}} = \mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}$.

The Modified Voltage Potential algorithm `MVP` is not independent for *Conflict-Free*. In fact, a stronger result can be shown: if `MVP` returns a vector, then this vector is always in conflict.

Theorem 44. *Let* $\mathbf{s}$ *be the relative vector* $\mathbf{s}_o - \mathbf{s}_i$*. If* $\|\mathbf{s}\| > D$*, Conflict*$(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i)$ *holds, and* $\mathbf{v}'_o \in \mathit{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$*, then Conflict*$(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i)$ *holds.*

Proof. The formal proof is based on algebraic reasoning. A geometric, intuitive proof of this result is given here. To understand the reasoning behind this theorem, consider the diagram in Figure 9, which shows the geometric interpretation of the vector $\mathbf{v}'_o$ computed by $\mathtt{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$. The triangle formed by the segments $\overline{\mathbf{s}, \mathbf{s}_{\mathtt{tca}}}$, $\overline{\mathbf{s}_{\mathtt{tca}}, \mathbf{p}}$, and $\overline{\mathbf{p}, \mathbf{s}}$ is a right triangle. Since the sum of the interior angles of any triangle is π, it follows that the interior angle formed by the segments $\overline{\mathbf{s}_{\mathtt{tca}}, \mathbf{p}}$ and $\overline{\mathbf{p}, \mathbf{s}}$ is strictly less than $\frac{\pi}{2}$. Thus, the trajectory from $\mathbf{s}$ along the relative vector $\mathbf{v}'_o - \mathbf{v}_i$ is not tangent to the circle. By Lemma 43, this trajectory does touch the circle at the point $\mathbf{p}$. It follows that this trajectory must touch the circle at two distinct places, and it therefore passes through the interior of the circle. □

6.2 ACCoRD's Track Angle Resolution

ACCoRD is a mathematical framework for the design and formal verification of state-based separation assurance algorithms [13]. The framework is written in PVS and includes conflict resolution algorithms for track angle, ground speed, combined track angle and ground speed, and vertical speed maneuvers. This paper only considers the algorithm for track angle maneuvers, which will be denoted $\mathtt{ACCoRDtrack}^{\varepsilon}$, where ε is a unit value ± 1. The main theorem in this section states that for $\varepsilon = \pm 1$, $\mathtt{ACCoRDtrack}^{\varepsilon}$ is *ConflictFree*-independent.

The algorithm $\mathtt{ACCoRDtrack}^{\varepsilon}$, where $\varepsilon = \pm 1$, has as arguments the vectors $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, and $\mathbf{v}_i$. It returns a set of at most two vectors where each one of these vectors, say $\mathbf{v}'_o$, is a new velocity vector for the ownship such that $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$, i.e., $\mathbf{v}'_o$ represents a track angle maneuver for the ownship. Furthermore, the relative velocity vector $\mathbf{v}' = \mathbf{v}'_o - \mathbf{v}_i$ is tangent to the circle on the side corresponding to the unit value ε, i.e., it satisfies *LineSolution*$(\mathbf{s}, \mathbf{v}', \varepsilon)$ as defined in Section 5.3.1.

If $\mathbf{v}'_o$ is a track angle maneuver for the ownship and *LineSolution*$(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$ holds, then it follows from Lemma 31 that there is some $k \geq 0$ such that

$$\|\mathbf{v}_o\|^2 = \|k\ \mathtt{tangent\_line}(\mathbf{s}, \varepsilon) + \mathbf{v}_i\|^2. \tag{22}$$

Equation (22) has the form $\|\mathbf{v}_o\|^2 = \|k\ \mathbf{u} + \mathbf{v}_i\|^2$, for a given vector $\mathbf{u}$. It is possible to define a function that solves equations of this form for real numbers k. It follows from the equation $\|\mathbf{v}_o\|^2 = \|k\ \mathbf{u} + \mathbf{v}_i\|^2$ that

$$\begin{aligned} 0 &= \|k\ \mathbf{u} + \mathbf{v}_i\|^2 - \|\mathbf{v}_o\|^2 \\ &= \|\mathbf{u}\|^2 k^2 + (2\,\mathbf{v}_i \cdot \mathbf{u})k + (\|\mathbf{v}_i\|^2 - \|\mathbf{v}_o\|^2). \end{aligned} \tag{23}$$

This is a quadratic equation in k, which has at most two distinct solutions. Each one of these solutions yields a resolution vector $\mathbf{v}'_o$ for the ownship. The solutions to Equation (23) are given by $\frac{-b+\iota\sqrt{b^2-4ac}}{2a}$, where $\iota = \pm 1$, and

$$\begin{aligned} a &= \|\mathbf{u}\|^2, \\ b &= 2\,\mathbf{v}_i \cdot \mathbf{u}, \\ c &= \|\mathbf{v}_i\|^2 - \|\mathbf{v}_o\|^2. \end{aligned}$$

Thus, if $b^2 - 4\,a\,c \geq 0$ and $k = \frac{-b+\iota\sqrt{b^2-4\,a\,c}}{2a} > 0$, then the vector $\mathbf{v}'_o$ defined by $\mathbf{v}'_o = k + \mathbf{v}_i$ satisfies both $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$ and $\mathit{LineSolution}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$. This motivates the definition of the function `track_only_line`, which returns a real number.

$$\begin{array}{l} \texttt{track\_only\_line}(\mathbf{u}, \mathbf{v}_o, \mathbf{v}_i, \iota) \equiv \\ \quad \texttt{let} \\ \quad\quad a = \|\mathbf{u}\|^2, \\ \quad\quad b = 2\,\mathbf{v}_i \cdot \mathbf{u}, \\ \quad\quad c = \|\mathbf{v}_i\|^2 - \|\mathbf{v}_o\|^2 \\ \quad \texttt{in} \\ \quad\quad \texttt{if}\ \ b^2 - 4ac \geq 0\ \ \texttt{then} \\ \quad\quad\quad \dfrac{-b + \iota\sqrt{b^2 - 4ac}}{2a} \\ \quad\quad \texttt{else} \\ \quad\quad\quad 0 \\ \quad\quad \texttt{endif} \end{array} \tag{24}$$

The next lemma states that the algorithm `track_only_line` computes solutions for k to the equation $\mathbf{v}'_o = k\ \mathbf{u} + \mathbf{v}_i$, where $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$.

Lemma 45. *If $\mathbf{u} \neq \mathbf{0}$, then $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$ and $k\ \mathbf{u} = \mathbf{v}'_o - \mathbf{v}_i$ if and only if*

$$k = \mathit{track\_only\_line}(\mathbf{u}, \mathbf{v}_o, \mathbf{v}_i, \iota),$$

for some $\iota = \pm 1$.

Using `track_only_line`, the algorithm `ACCoRDtrack`$^{\varepsilon}$, which computes track angle maneuvers $\mathbf{v}'_o$ for the ownship that satisfy $\mathit{LineSolution}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$, for $\varepsilon =$

± 1, can be defined as follows.

$$
\begin{array}{l}
\texttt{ACCoRDtrack}^{\varepsilon}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i) \equiv \\
\quad \texttt{let} \\
\quad\quad \mathbf{s} \;= \mathbf{s}_o - \mathbf{s}_i, \\
\quad\quad \mathbf{u} \;= \texttt{tangent\_line}(\mathbf{s}, \varepsilon), \\
\quad\quad k_1 = \texttt{track\_only\_line}(\mathbf{u}, \mathbf{v}_o, \mathbf{v}_i, 1), \\
\quad\quad k_2 = \texttt{track\_only\_line}(\mathbf{u}, \mathbf{v}_o, \mathbf{v}_i, -1), \\
\quad \texttt{in} \\
\quad\quad \texttt{if } k_1 \geq 0 \texttt{ then } \{k_1\,\mathbf{u} + \mathbf{v}_i\} \texttt{ else } \emptyset \texttt{ endif} \\
\quad\quad \cup \\
\quad\quad \texttt{if } k_2 \geq 0 \texttt{ then } \{k_2\,\mathbf{u} + \mathbf{v}_i\} \texttt{ else } \emptyset \texttt{ endif}
\end{array}
\tag{25}
$$

Lemma 46 states that $\texttt{ACCoRDtrack}^{\varepsilon}$ resolutions are correct and complete for line solutions that are track angle maneuvers.

Lemma 46. *Let* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *such that* $\|\mathbf{s}\| \geq D$. *For all* $\varepsilon = \pm 1$, $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$ *and* *LineSolution*$(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$ *holds if and only if*

$$\mathbf{v}'_o \in \textit{ACCoRDtrack}^{\varepsilon}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i).$$

The next theorem states that $\texttt{ACCoRDtrack}^{\varepsilon}$ is *ConflictFree*-independent.

Theorem 47 ($\texttt{ACCoRDtrack}^{\varepsilon}$ Independence)**.** *For all vectors* $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i, \mathbf{v}'_o$ *and* $\varepsilon = \pm 1$, *if*

$$\mathbf{v}'_o \in \textit{ACCoRDtrack}^{\varepsilon}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$$

and *Conflict*$(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$ *holds, then it holds that*

$$\textit{ConflictFree}_{\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i}(\mathbf{v}'_o - \mathbf{v}_i).$$

Proof. By Theorem 29 and Lemma 46. □

6.3 Geometric Optimization's Track Angle Resolution

The geometric optimization approach to state-based conflict resolution [2] consists of algorithms for track angle, ground speed, and combined track angle and ground speed maneuvers. This paper only considers the track angle algorithm, which will be denoted $\texttt{GOtrack}^{\varepsilon}_{f}$, where ε is a unit value ± 1 and $f \leq 1$ is a nonnegative real number. In the case where $f = 1$, the algorithm returns a maneuver vector for the ownship such that if the intruder does not maneuver, then the resulting relative velocity vector is tangent to the circle of radius D around the origin. The unit value ε corresponds to the side of the circle, from the perspective of the ownship, on which this relative vector is tangent, with $\varepsilon = -1$ corresponding to a right tangent and $\varepsilon = 1$ to a left tangent.

The algorithm takes as inputs the current state vectors of the aircraft, i.e., vectors $\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i$. It returns a set of at most two vectors where each one of these

vectors, say $\mathbf{v}'_o$, is a new velocity vector for the ownship such that $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$, i.e., $\mathbf{v}'_o$ represents a track angle maneuver for the ownship.

In the construction that follows, it will be implicit that the aircraft are currently in conflict. If this is not the case, then the algorithm returns the empty set. The main theorem in this section states that $\texttt{GOtrack}^{\varepsilon}_{f}$ is *ConflictFree*-independent when the parameter f is equal to 1.

In order to specify the algorithm, some basic notation and trigonometric functions are needed. The first of these is the function `track`, which computes the *track angle* of a vector, relative to true North. It is defined for $\mathbf{u} \neq \mathbf{0}$ as follows.

$$\texttt{track}(\mathbf{u}) \equiv \texttt{atan2}(\mathbf{u}_y, \mathbf{u}_x). \tag{26}$$

Here, $\texttt{atan2}(\mathbf{u}_y, \mathbf{u}_x)$ is the angle α that satisfies the equation $\mathbf{u} = (\sin(\alpha), \cos(\alpha))$. By convention, $\texttt{track}(\mathbf{0}) = 0$.

The second function that is needed in the definition of the algorithm takes an angle α, which is any real number, and returns another angle, trigonometrically equivalent to α, which lies in the interval $[0, 2\pi)$. It is defined by the following equation.

$$\texttt{to2pi}(\alpha) \equiv \alpha - 2\pi \cdot \texttt{floor}\left(\frac{\alpha}{2\pi}\right). \tag{27}$$

It is easy to see that if $\alpha \in [0, 2\pi)$, then $\texttt{to2pi}(\alpha) = \alpha$.

Next, the algorithm $\texttt{GOtrack}^{\varepsilon}_{f}$ relies on the function `angleto`, which returns the angle from one angle to another. The angle returned by this function lies in the interval $[-\pi, \pi)$. This function is defined as follows.

$$\texttt{angleto}(\alpha, \beta) \equiv \texttt{to2pi}((\beta - \alpha) + \pi) - \pi. \tag{28}$$

Here, α and β are any real numbers.

Lemma 48. *The angle* $\alpha + \texttt{angleto}(\alpha, \beta)$ *is* trigonometrically equivalent *to* β *in the sense that* $(\alpha + \texttt{angleto}(\alpha, \beta)) - \beta$ *is an integer multiple of* 2π.

There are various ways to define the function `angleto` so that it has the desired properties. One such property, which has been proved for the definition of `angleto` given in Equation (28), is given by the following lemma.

Lemma 49. *If* $|\alpha - \beta| < \pi$, *then* $\texttt{angleto}(\alpha, \beta) = \beta - \alpha$.

The first step in the algorithm $\texttt{GOtrack}^{\varepsilon}_{f}$ is to compute the angle change α needed in the relative velocity vector $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ in order to achieve a tangent to the circle of radius D around the origin. The side of the circle on which the tangent occurs is determined by ε. The angle α is illustrated in Figure 10 for $\varepsilon = -1$ (a right tangent), and its value is given in Formula (29).

If $f \leq 1$, then any velocity vector $\mathbf{v}'_o$ returned by the algorithm results in a new relative velocity vector $\mathbf{v}' = \mathbf{v}'_o - \mathbf{v}_i$ that lies between the current velocity vector $\mathbf{v}$ and a tangent vector to the circle on the side corresponding to ε. That is, the angle from $\mathbf{v}$ to the vector $\mathbf{v}'$ is equal to $f\,\alpha$. The function χ^{*}_{rel}, defined below, computes the track angle β of $\mathbf{v}'$. This is illustrated in Figure 11.

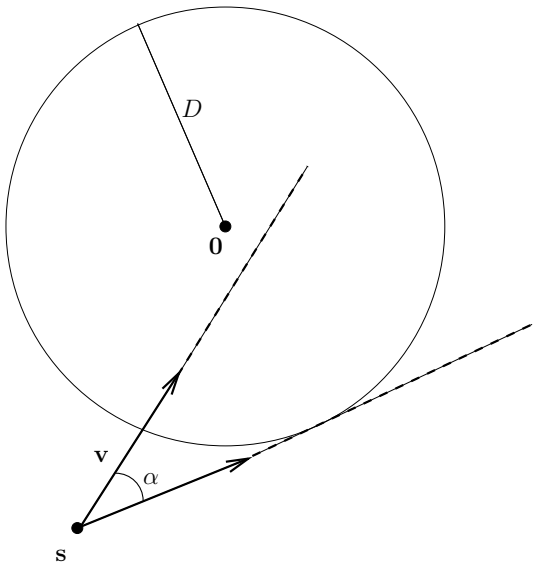

Figure 10. Angle Change α Needed for a Tangent

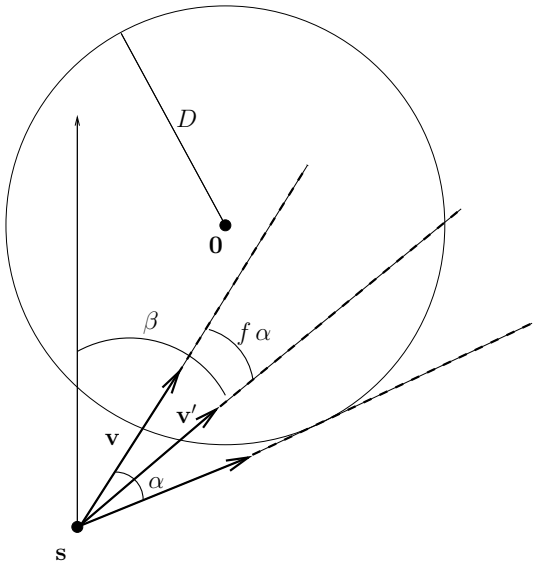

Figure 11. Track Angle β of *GOtrack*'s $\mathbf{v}' = \mathbf{v}'_o - \mathbf{v}_i$

$$\chi^*_{rel}(\mathbf{s},\mathbf{v},f,\varepsilon) \equiv \texttt{to2pi}(\texttt{track}(\mathbf{v}) + f\,\alpha), \tag{29}$$

where α denotes the angle $\texttt{angleto}(\texttt{track}(\mathbf{v}), \texttt{track}(-\mathbf{s}) - \varepsilon \operatorname{asin}(\frac{D}{\|\mathbf{s}\|}))$.

The next lemma shows that if $f = 1$, then χ^*_{rel} computes the track angle of a vector that is tangent to the circle, on the side corresponding to the unit value ε.

Lemma 50. *If* $k > 0$, $\beta = \chi^*_{rel}(\mathbf{s},\mathbf{v},1,\varepsilon)$, *and* $\mathbf{u} = k \cdot (\sin(\beta),\cos(\beta))$, *then* *LineSolution*$(\mathbf{s},\mathbf{u},\varepsilon)$ *holds.*

The algorithm $\texttt{GOtrack}^{\varepsilon}_{f}$ is defined using the function χ^*_{rel}. Any vector $\mathbf{v}'_o$ returned by the algorithm is a track angle maneuver for the ownship, i.e., $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$. Furthermore, if $f \leq 1$, then the relative velocity vector $\mathbf{v}'_o - \mathbf{v}_i$ satisfies

$$\texttt{track}(\mathbf{v}'_o - \mathbf{v}_i) = \chi^*_{rel}(\mathbf{s},\mathbf{v}_o - \mathbf{v}_i, f, \varepsilon).$$

The algorithm $\texttt{GOtrack}^{\varepsilon}_{f}$ is defined as follows.

$$\begin{array}{l}
\texttt{GOtrack}^{\varepsilon}_{f}(\mathbf{s}_o,\mathbf{s}_i,\mathbf{v}_o,\mathbf{v}_i) \equiv \\
\quad \texttt{let} \\
\quad\quad \mathbf{s} = \mathbf{s}_o - \mathbf{s}_i, \\
\quad\quad \mathbf{v} = \mathbf{v}_o - \mathbf{v}_i, \\
\quad\quad \beta = \chi^*_{rel}(\mathbf{s},\mathbf{v},f,\varepsilon) \\
\quad \texttt{in} \\
\quad\quad \texttt{if } \mathbf{v} = \mathbf{0} \texttt{ or } \dfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_o\|}\,|\sin(\beta - \texttt{track}(\mathbf{v}_i))| > 1 \texttt{ then} \\
\quad\quad\quad \emptyset \\
\quad\quad \texttt{else} \\
\quad\quad\quad \texttt{let} \\
\quad\quad\quad\quad e = \dfrac{\|\mathbf{v}_i\|}{\|\mathbf{v}_o\|}\,|\sin(\beta - \texttt{track}(\mathbf{v}_i))|, \\
\quad\quad\quad\quad \theta_1 = \beta - \operatorname{asin}(e), \\
\quad\quad\quad\quad \mathbf{v}'_{o1} = \|\mathbf{v}_o\|\,(\sin(\theta_1),\cos(\theta_2)), \\
\quad\quad\quad\quad \theta_2 = \beta - \texttt{sign}(\operatorname{asin}(e))\pi + \operatorname{asin}(e), \\
\quad\quad\quad\quad \mathbf{v}'_{o2} = \|\mathbf{v}_o\|\,(\sin(\theta_2),\cos(\theta_2)) \\
\quad\quad\quad \texttt{in} \\
\quad\quad\quad\quad \texttt{if } \mathbf{s}\cdot(\mathbf{v}'_{o1} - \mathbf{v}_i) \geq 0 \texttt{ then } \{\mathbf{v}'_{o1}\} \texttt{ else } \emptyset \texttt{ endif} \\
\quad\quad\quad\quad \cup \\
\quad\quad\quad\quad \texttt{if } \mathbf{s}\cdot(\mathbf{v}'_{o2} - \mathbf{v}_i) \geq 0 \texttt{ then } \{\mathbf{v}'_{o2}\} \texttt{ else } \emptyset \texttt{ endif} \\
\quad\quad\quad \texttt{endif}
\end{array} \tag{30}$$

The following lemma states that $\texttt{GOtrack}^{\varepsilon}_{f}$ returns the ownship's current velocity vector when $f = 0$.

Lemma 51. *If* $\mathbf{s}\cdot(\mathbf{v}_o - \mathbf{v}_i) < 0$ *and* $\mathbf{v}'_o \in \mathit{GOtrack}^{\varepsilon}_{0}(\mathbf{s}_o,\mathbf{s}_i,\mathbf{v}_o,\mathbf{v}_i)$, *then* $\mathbf{v}'_o = \mathbf{v}_o$.

An important property of the algorithm $\texttt{GOtrack}_f^\varepsilon$ is given by Theorem 53. The proof of that theorem relies on the following lemma.

Lemma 52. *If* $\mathbf{v}'_o \in \mathit{GOtrack}_f^\varepsilon(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ *and* $\beta = \chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, f, \varepsilon)$*, then*

$$\|\mathbf{v}_i\| \sin(\beta - \mathit{track}(\mathbf{v}_i)) = \|\mathbf{v}_o\| \sin(\beta - \mathit{track}(\mathbf{v}'_o)).$$

Proof. By hypothesis, the algorithm returns a nonempty set. Therefore, $\mathbf{v} \neq \mathbf{0}$, $0 \leq e \leq 1$, and $\mathbf{s} \cdot (\mathbf{v}'_o - \mathbf{v}_i) < 0$, where

$$\begin{aligned} e &= \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_o\|} \left|\sin(\beta - \texttt{track}(\mathbf{v}_i))\right|, \\ \theta &= \beta - \texttt{sign}(\operatorname{asin}(e))\frac{\pi}{2}(1+\iota) + \iota \operatorname{asin}(e), \\ \mathbf{v}'_o &= \|\mathbf{v}_o\| \left(\sin(\theta), \cos(\theta)\right), \end{aligned}$$

for some $\iota = \pm 1$.

Since θ is trigonometrically equivalent to $\texttt{track}(\mathbf{v}'_o)$, it suffices to prove that $e = \sin(\chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, f, \varepsilon) - \theta)$. Expanding the definition of θ and cancelling equal terms reduces the proof to the verification of the following equality.

$$e = \sin(\texttt{sign}(\operatorname{asin}(e))\frac{\pi}{2}(1+\iota) - \iota \operatorname{asin}(e)).$$

If $\iota = -1$, then this equation becomes $e = \sin(\operatorname{asin}(e))$, which is trivial. Alternatively, if $\iota = 1$, then $\texttt{sign}(\operatorname{asin}(e))\frac{\pi}{2}(1+\iota)$ is equal to $\texttt{sign}(\operatorname{asin}(e))\pi$, which is either $-\pi$ or π. Thus, it suffices to prove that

$$e = \sin(\pm\pi - \operatorname{asin}(e)),$$

which is also trivial. □

Theorem 53. *If* $\mathbf{s} \cdot (\mathbf{v}_o - \mathbf{v}_i) < 0$ *and* $\mathbf{v}'_o \in \mathit{GOtrack}_f^\varepsilon(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$*, then*

1. $\|\mathbf{v}'_o\| = \|\mathbf{v}_o\|$*, and*
2. $\mathit{track}(\mathbf{v}'_o - \mathbf{v}_i) = \chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, f, \varepsilon)$.

Proof. The proof is sketched here, with the special cases and minor details are omitted for brevity. As in the proof of Lemma 52, $\mathbf{v} \neq \mathbf{0}$, $0 \leq e \leq 1$, and $\mathbf{s} \cdot (\mathbf{v}'_o - \mathbf{v}_i) < 0$, where

$$\begin{aligned} \beta &= \chi^*_{rel}(\mathbf{s}, \mathbf{v}, f, \varepsilon), \\ e &= \frac{\|\mathbf{v}_i\|}{\|\mathbf{v}_o\|} \left|\sin(\beta - \texttt{track}(\mathbf{v}_i))\right|, \\ \theta &= \beta - \texttt{sign}(\operatorname{asin}(e))\frac{\pi}{2}(1+\iota) + \iota \operatorname{asin}(e), \\ \mathbf{v}'_o &= \|\mathbf{v}_o\| \left(\sin(\theta), \cos(\theta)\right), \end{aligned}$$

for some $\iota = \pm 1$. The first part of the theorem is proved by simple algebraic and trigonometric manipulations:

$$\|\mathbf{v}'_o\| = \|\mathbf{v}_o\| \|(\sin(\theta), \cos(\theta))\| = \|\mathbf{v}_o\| \sqrt{(\sin^2(\theta) + \cos^2(\theta)} = \|\mathbf{v}_o\|.$$

The second part of the theorem follows from Lemma 52:

$$\|\mathbf{v}_i\| \sin(\beta - \texttt{track}(\mathbf{v}_i)) = |\mathbf{v}_o\| \sin(\beta - \texttt{track}(\mathbf{v}'_o)).$$

Since θ is trigonometrically equivalent to $\texttt{track}(\mathbf{v}'_o)$, by the subtraction property of the sine function,

$$\begin{aligned}\|\mathbf{v}_i\| (\sin(\beta)\cos(\texttt{track}(\mathbf{v}_i)) - \sin(\texttt{track}(\mathbf{v}_i))\cos(\beta)) = \\ \|\mathbf{v}_o\| (\sin(\beta)\cos(\theta) - \sin(\theta)\cos(\beta)).\end{aligned}$$

Therefore,

$$\begin{aligned}\tan(\beta) &= \frac{\|\mathbf{v}_o\|\sin(\theta) - \|\mathbf{v}_i\|\sin(\texttt{track}(\mathbf{v}_i))}{\|\mathbf{v}_o\|\cos(\theta) - \|\mathbf{v}_i\|\cos(\texttt{track}(\mathbf{v}_i))} \\ &= \frac{v'_{ox} - v_{ix}}{v'_{oy} - v_{iy}} \\ &= \tan(\texttt{track}(\mathbf{v}'_o - \mathbf{v}_i)).\end{aligned}$$

It is easy to prove that if $\mathbf{w}$ and $\mathbf{u}$ are any nonzero vectors where $\mathbf{w}_y \neq 0$, $\mathbf{u}_y \neq 0$, $\mathbf{w}_x/\mathbf{w}_y = \mathbf{u}_x/\mathbf{u}_y$, $\mathbf{s}\cdot\mathbf{w} < 0$, and $\mathbf{s}\cdot\mathbf{u} < 0$, then $\texttt{track}(\mathbf{w}) = \texttt{track}(\mathbf{u})$. Applying this to the vectors $(\sin(\beta), \cos(\beta))$ and $\mathbf{v}'_o - \mathbf{v}_i$ gives the desired result. □

The lemma below follows directly from Lemma 50 and Theorem 53.

Lemma 54. *If* $\mathit{Conflict}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i)$ *holds and* $\mathbf{v}'_o \in \mathit{GOtrack}_1^\varepsilon(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$*, then* $\|\mathbf{v}_o\| = \|\mathbf{v}'_o\|$ *and* $\mathit{LineSolution}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$ *holds.*

The next theorem states that $\texttt{GOtrack}_1^\varepsilon$ is *ConflictFree*-independent.

Theorem 55 ($\texttt{GOtrack}_1^\varepsilon$ Independence)**.** *For all vectors* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, *and* $\mathbf{v}'_o$*, and for all* $\varepsilon = \pm 1$*, if*

$$\mathbf{v}'_o \in \mathit{GOtrack}_1^\varepsilon(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$$

and $\mathit{Conflict}(\mathbf{s}, \mathbf{v})$ *holds, then it holds that*

$$\mathit{ConflictFree}_{\mathbf{s},\mathbf{v}}(\mathbf{v}'_o - \mathbf{v}_i).$$

Proof. By Theorem 29 and Lemma 54. □

6.4 Numerical Example

This section compares the resolution maneuvers computed by the algorithms presented before for a concrete scenario. The algorithms `MVP` and $\texttt{GOtrack}_f^\varepsilon$ are implemented in Python. Java and C++ implementations of ACCoRD's CD&R algorithms are available from http://shemesh.larc.nasa.gov/people/cam/ACCoRD. All the implementations use the floating point arithmetic provided by their respective languages.

cr	$\mathbf{v}_o' \in \texttt{cr}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$	$\mathbf{v}_i' \in \texttt{cr}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$
MVP	$(484.50, -127.47)$	$(265.49, 77.47)$
ACCoRDtrack$^{\varepsilon}$	$(483.99, -135.10)$	$(225.13, 108.69)$
GOtrack$^{\varepsilon}_{1}$	$(483.99, -135.10)$	$(225.13, 108.69)$
GOtrack$^{\varepsilon}_{\frac{1}{2}}$	$(494.00, -91.98)$	$(-154.34, 196.67)$

Table 1. Maneuvers From Different Resolution Algorithms

Assume that D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, and $\mathbf{v}_i$ are given as follows, where distances are in nautical miles (nmi) and speeds are in knots, i.e., nautical miles per hour.

$$\begin{aligned} D &= 5, \\ \mathbf{s}_o &= (0, 0), \\ \mathbf{s}_i &= (10, 0), \\ \mathbf{v}_o &= (500, -50), \\ \mathbf{v}_i &= (250, 0). \end{aligned}$$

Let $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ be the relative position and velocity vectors, respectively. In this case,

$$\begin{aligned} \mathbf{s} &= (-10, 0), \\ \mathbf{v} &= (250, -50). \end{aligned}$$

The time of closest approach between the aircraft is given by $\texttt{tca}(\mathbf{s}, \mathbf{v}) = \frac{1}{26}$ hours, which is about 138.5 seconds, and the distance at time of closest approach is about 1.96 miles. Since the minimum safe separation D is 5 nautical miles, the aircraft are in conflict.

Table 1 shows resolution vectors computed by the conflict resolution algorithms MVP, ACCoRDtrack$^{\varepsilon}$, and GOtrack$^{\varepsilon}_{f}$, where $\varepsilon = -1$, for the given values from the ownship's and intruders' perspectives. In the case of GOtrack$^{\varepsilon}_{f}$, the values $f = 1$ and $f = \frac{1}{2}$ are considered. The results have been rounded to 2 decimal places. According to Lemma 46 and Lemma 54, all maneuvers computed by GOtrack$^{\varepsilon}_{1}$ are also computed by ACCoRDtrack$^{\varepsilon}$. This property is illustrated in the example by the fact that ACCoRDtrack$^{\varepsilon}$ and GOtrack$^{\varepsilon}_{1}$ compute the same track angle resolution maneuvers.

Tables 2 and 3 show the time of closest approach and distance of closest approach for the resolutions in Table 1 for the independent and coordinated cases, respectively. The results have been rounded to 2 decimal places. In the independent case, the distance at time of closest approach is the same from the ownship's and intruder's perspective. Table 2 shows that in the given scenario, MVP and GOtrack$^{\varepsilon}_{\frac{1}{2}}$ do not achieve separation when only one of the aircraft maneuvers. The fact that MVP does not achieve separation is a numerical illustration of Theorem 44, which states that resolution maneuvers computed MVP are always in conflict. Table 2 also shows that in the independent case, for this scenario, both ACCoRDtrack$^{\varepsilon}$ and GOtrack$^{\varepsilon}_{1}$

cr	$\texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i)$	$\texttt{tca}(-\mathbf{s}, \mathbf{v}'_i - \mathbf{v}_o)$	Distance at tca
MVP	118.50 s	118.50	4.77 nmi
$\texttt{ACCoRDtrack}^{\varepsilon}$	115.39 s	98.23 s	5.00 nmi
$\texttt{GOtrack}^{\varepsilon}_{1}$	115.39 s	98.23 s	5.00 nmi
$\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$	129.18 s	48.17 s	3.53 nmi

Table 2. Time and Distance of Closest Approach (Independent Case)

cr	$\tau = \texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i)$	Distance at τ
MVP	87.63 s	6.83 nmi
$\texttt{ACCoRDtrack}^{\varepsilon}$	73.70 s	6.85 nmi
$\texttt{GOtrack}^{\varepsilon}_{1}$	73.70 s	6.85 nmi
$\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$	46.34 s	4.07 nmi

Table 3. Time and Distance of Closest Approach (Coordinated Case)

achieve a minimum separation of 5 nautical miles. The fact that $\texttt{ACCoRDtrack}^{\varepsilon}$ and $\texttt{GOtrack}^{\varepsilon}_{1}$ achieve exactly the required minimum separation is a numerical illustration of Lemma 46 and Lemma 54, which state, respectively, that $\texttt{ACCoRDtrack}^{\varepsilon}$ and $\texttt{GOtrack}^{\varepsilon}_{1}$ compute solutions that are tangent to the relative protected zone. Table 3 shows that in this particular scenario the resolutions computed by MVP, $\texttt{ACCoRDtrack}^{\varepsilon}$, and $\texttt{GOtrack}^{\varepsilon}_{1}$ are coordinated. However, the minimum separation achieved when both aircraft simultaneously maneuver according to $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ is less than the required minimum separation of 5 nautical miles.

This example suggests that $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ is neither *ConflictFree*-independent nor *ConflictFree*-coordinated. These apparent counterexamples to independence and coordination need to be formally verified. It may be that the figures in the tables are imprecise due to the effect of rounding errors in the floating point arithmetic used by the programming languages where the algorithms were implemented. The scenario also suggests that the conflict resolution algorithms MVP, $\texttt{ACCoRDtrack}^{\varepsilon}$, and $\texttt{GOtrack}^{\varepsilon}_{1}$ are *ConflictFree*-coordinated with themselves. However, this numerical example cannot be considered a proof of coordination. In addition to possible floating point error imprecisions, the existence of one scenario where coordination holds cannot be generalized to all possible scenarios. The next section provides formal, incontrovertible proofs of the coordination properties of MVP, $\texttt{GOtrack}^{\varepsilon}_{f}$, and $\texttt{ACCoRDtrack}^{\varepsilon}$.

7 Formal Properties of MVP, $\texttt{GOtrack}^{\varepsilon}_{f}$, and $\texttt{ACCoRDtrack}^{\varepsilon}$

This section presents formal proofs of several results regarding coordination of MVP, $\texttt{GOtrack}^{\varepsilon}_{f}$, and $\texttt{ACCoRDtrack}^{\varepsilon}$. In particular, it is shown that MVP is coordinated for *Repulsion* and that $\texttt{GOtrack}^{\varepsilon}_{1}$ and $\texttt{ACCoRDtrack}^{\varepsilon}$ are *ConflictFree*-coordinated

with themselves, with each other, and with MVP. Formal proofs of the facts that MVP and $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ are not *ConflictFree*-coordinated with themselves are also presented. Furthermore, numerical evidence is provided to support the claim that they are not *ConflictFree*-coordinated with each other.

7.1 MVP is Coordinated for *Repulsion*

The proof that the Modified Voltage Potential algorithm MVP is coordinated for *Repulsion* illustrates the use of the repulsion criterion $\mathcal{R}^{\varepsilon}$ defined in Section 5. The coordination result for MVP follows from the fact that it satisfies $\mathcal{R}^{\varepsilon}$.

Lemma 56. *Let* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ *and suppose that* $\mathit{Conflict}(\mathbf{s}, \mathbf{v})$ *holds,* $\mathbf{s} + \mathit{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v} \neq \mathbf{0}$*, and* $\mathbf{v}'_o \in \mathit{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$*. Then*

$$\mathbf{v}'_o - \mathbf{v}_i \in \mathcal{R}^{\varepsilon}_{\mathbf{s},\mathbf{v}},$$

where $\varepsilon = -\mathit{sign}(\mathbf{s} \cdot \mathbf{v}^{\perp})$*.*

Proof. It can be proved from the definition of MVP that

$$\mathbf{v}'_o - \mathbf{v}_i \;=\; \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,\mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right),$$

where $\mathbf{s}_{\mathtt{tca}} = \mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\,\mathbf{v}$. It is easy to see that $\mathtt{tca}(\mathbf{s}, \mathbf{v})$ must be positive. Thus, by the definition of $\mathcal{R}^{\varepsilon}$, it suffices to prove the following four conditions.

1. $\varepsilon\,\mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$.
2. $\mathbf{s} \cdot \mathbf{v} < 0$.
3. $\mathbf{s} \cdot \left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,\mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right) \leq 0$.
4. $\varepsilon\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,\mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right) \cdot \mathbf{v}^{\perp} \leq 0$.

The first condition follows directly from the fact that $\varepsilon = -\mathtt{sign}(\mathbf{s} \cdot \mathbf{v}^{\perp})$, and the second follows from $\mathit{Conflict}(\mathbf{s}, \mathbf{v})$. The third condition follows from the Cauchy-Schwartz inequality:

$$\begin{aligned}\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,(\mathbf{s} \cdot \mathbf{s}_{\mathtt{tca}}) &\leq \frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,\|\mathbf{s}\|\,\|\mathbf{s}_{\mathtt{tca}}\| \\ &= D\|\mathbf{s}\| \\ &\leq \mathbf{s} \cdot \mathbf{s}.\end{aligned}$$

The fourth condition follows from the facts that $\mathbf{s}_{\mathtt{tca}} \cdot \mathbf{v}^{\perp} = \mathbf{s} \cdot \mathbf{v}^{\perp}$ and $\varepsilon\,\mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$:

$$\begin{aligned}\varepsilon\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\,\mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right) \cdot \mathbf{v}^{\perp} &= \varepsilon\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|} - 1\right)(\mathbf{s} \cdot \mathbf{v}^{\perp}) \\ &\leq 0.\end{aligned} \tag{31}$$

The final inequality here uses the fact that $D \geq \|\mathbf{s}_{\mathtt{tca}}\|$, which in turn follows directly from $\mathit{Conflict}(\mathbf{s}, \mathbf{v})$. □

Theorem 57. *The algorithm MVP is coordinated for Repulsion.*

Proof. Lemma 56 implies that when the aircraft are in conflict, the algorithm MVP satisfies the criterion $\mathcal{R}^{\varepsilon}$, where $\varepsilon = -\mathtt{sign}(\mathbf{s} \cdot \mathbf{v}^{\perp})$. The result follows from Theorem 25 (Section 3.3) and Theorem 2 (Section 3.1). □

7.2 $\mathtt{GOtrack}_1^{\varepsilon}$ and $\mathtt{ACCoRDtrack}^{\varepsilon}$ are *ConflictFree*-Coordinated

This section proves that the algorithms $\mathtt{GOtrack}_1^{\varepsilon}$, and $\mathtt{ACCoRDtrack}^{\varepsilon}$ are *ConflictFree*-coordinated with themselves and with each. The proof follows from the fact that each of these algorithms satisfies the horizontal criterion $\mathcal{H}^{\varepsilon}$, defined in Section 5.1.

Lemma 58. *The algorithms $\mathtt{GOtrack}_1^{\varepsilon}$ and $\mathtt{ACCoRDtrack}^{\varepsilon}$ each satisfy the horizontal criterion $\mathcal{H}^{\varepsilon}$.*

Proof. If $\mathbf{v}'_o \in \mathtt{GOtrack}_1^{\varepsilon} \cup \mathtt{ACCoRDtrack}^{\varepsilon}$, then by Lemma 54 (Section 6.3) and Lemma 46 (Section 6.2), it follows that *LineSolution*$(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$ holds. The result follows directly from Lemma 33 (Section 5.3.1). □

Theorem 59. *The resolution algorithms $\mathtt{GOtrack}_1^{\varepsilon}$ and $\mathtt{ACCoRDtrack}^{\varepsilon}$ are coordinated with themselves and with each other for ConflictFree.*

Proof. This follows directly from Lemma 58 and Theorem 34 (Section 5.3.1). □

7.3 MVP is *ConflictFree*-Coordinated with $\mathtt{GOtrack}_1^{\varepsilon}$ and $\mathtt{ACCoRDtrack}^{\varepsilon}$

This section proves that the resolution algorithm MVP is *ConflictFree*-coordinated with the algorithms $\mathtt{GOtrack}_1^{\varepsilon}$, and $\mathtt{ACCoRDtrack}^{\varepsilon}$. This proof requires that the value ε is chosen such that $\varepsilon\, \mathbf{s} \cdot (\mathbf{v}_o - \mathbf{v}_i)^{\perp} \leq 0$. The result relies on Theorem 27 (Section 5.2).

Lemma 60. *If $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$, $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, and $\mathbf{v}'_o$ are vectors such that Conflict$(\mathbf{s}, \mathbf{v})$ holds, $\mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v} \neq \mathbf{0}$, $\mathbf{v}'_o \in$ MVP$(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$, and $\varepsilon\, \mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$, then $\mathbf{v}'_o - \mathbf{v}_o \in \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$.*

Proof. Since *Conflict*$(\mathbf{s}, \mathbf{v})$ holds, it is clear from the definition of the function tca that $\mathtt{tca}(\mathbf{s}, \mathbf{v})$ is positive. As noted in the proof of Lemma 56 (Section 7.1), it can be proved from the definition of MVP that

$$\mathbf{v}'_o - \mathbf{v}_i = \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right),$$

where $\mathbf{s}_{\mathtt{tca}} = \mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v})\, \mathbf{v}$. Thus,

$$\begin{aligned} \mathbf{v}'_o - \mathbf{v}_o &= (\mathbf{v}'_o - \mathbf{v}_i) - \mathbf{v} \\ &= \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right) - \mathbf{v} \\ &= \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}\left(\frac{D}{\|\mathbf{s}_{\mathtt{tca}}\|}\, \mathbf{s}_{\mathtt{tca}} - \mathbf{s}\right) - \frac{1}{\mathtt{tca}(\mathbf{s}, \mathbf{v})}(\mathbf{s}_{\mathtt{tca}} - \mathbf{s}) \\ &= \frac{D - \|\mathbf{s}_{\mathtt{tca}}\|}{\mathtt{tca}(\mathbf{s}, \mathbf{v})\|\mathbf{s}_{\mathtt{tca}}\|}\mathbf{s}_{\mathtt{tca}}. \end{aligned} \tag{32}$$

Since $\mathbf{s}_{\mathtt{tca}} \cdot \mathbf{v}^{\perp} = \mathbf{s} \cdot \mathbf{v}^{\perp}$, it can be proved using techniques from linear algebra that

$$\begin{aligned}\mathbf{s}_{\mathtt{tca}} &= \frac{-\varepsilon\, \mathbf{s}_{\mathtt{tca}} \cdot \mathbf{v}^{\perp}}{\|\mathbf{v}\|^2}\,(-\varepsilon\, \mathbf{v}^{\perp}) \\ &= \frac{-\varepsilon\, \mathbf{s} \cdot \mathbf{v}^{\perp}}{\|\mathbf{v}\|^2}\,(-\varepsilon\, \mathbf{v}^{\perp}).\end{aligned}$$

Further, by hypothesis, the coefficient of the vector $-\varepsilon\, \mathbf{v}^{\perp}$ in this equation is nonnegative. Hence, it follows from Equation (32) that there is a nonnegative real number $r \geq 0$ such that

$$\mathbf{v}'_o - \mathbf{v}_o = r\,(-\varepsilon\, \mathbf{v}^{\perp}).$$

By Lemma 32 (Section 5), it is easy to see that $\mathcal{H}^{\varepsilon}$ is closed under multiplication by a nonnegative scalar, so it suffices to prove that $-\varepsilon\, \mathbf{v}^{\perp} \in \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$. This always holds and can be proved from definitions using linear algebraic manipulations. □

Theorem 61. *Let* `cr` *be a resolution algorithm such that for all* $\mathbf{v}'_o \in \mathtt{cr}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$, $\mathbf{v}'_o \in \mathit{LineSolution}(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}'_o - \mathbf{v}_i, \varepsilon)$, *where* $\varepsilon\ \mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$. *The resolution algorithm* `MVP` *is ConflictFree-coordinated with* `cr`.

Proof. By Lemma 33 (Section 5.3.1) and Theorem 27 (Section 5.2), it suffices to prove that for all $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$,$\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, $\mathbf{v}'_i \in \mathtt{MVP}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$, if $\mathit{Conflict}(\mathbf{s}, \mathbf{v})$ holds, then $(\mathbf{v}'_i - \mathbf{v}_i) \in \mathcal{H}^{\varepsilon}_{\mathbf{s},\mathbf{v}}$. This follows directly from Lemma 60. □

Theorem 62. *The resolution algorithm* `MVP` *is ConflictFree-coordinated with both* $\mathtt{GOtrack}^{\varepsilon}_1$ *and* $\mathtt{ACCoRDtrack}^{\varepsilon}$, *if* ε *is chosen such that* $\varepsilon\ \mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$.

Proof. By Lemma 54 (Section 6.3) and Lemma 46 (Section 6.2), both $\mathtt{GOtrack}^{\varepsilon}_1$ and $\mathtt{ACCoRDtrack}^{\varepsilon}$ compute line solutions. The result follows from Theorem 61. □

7.4 `MVP` is Not *ConflictFree*-Coordinated

Theorems 57 and 61 state, respectively, that the Modified Voltage Potential algorithm `MVP` is *Repulsion*-coordinated and, furthermore, *ConflictFree*-coordinated with any algorithm that computes tangent trajectories to the protected zone. The numerical example in Section 6.4 suggest that `MVP` is also *ConflictFree*-Coordinated. However, as the following theorem shows, `MVP` is not *ConflictFree*-coordinated with itself. In other words, there exist scenarios where `MVP` does not achieve separation when both aircraft simultaneously maneuver according to the resolutions computed by the algorithm.

Theorem 63. *The algorithm* `MVP` *is not ConflictFree-coordinated.*

Proof. To prove this theorem, it suffices to show that there exists D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, $\mathbf{v}'_o$, and $\mathbf{v}'_i$ such that $\mathit{Conflict}(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$, $\mathbf{v}'_o \in \mathtt{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$, and $\mathbf{v}'_i \in \mathtt{MVP}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$, where $\mathit{Conflict}(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}'_o - \mathbf{v}'_i)$ holds.

Let D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, and $\mathbf{v}_i$ be defined as follows, where distances are in nautical miles and speeds are in knots.

$$\begin{aligned} D &= 5, \\ \mathbf{s}_o &= (0, 0.1), \\ \mathbf{s}_i &= (\sqrt{30.24}, 0), \\ \mathbf{v}_o &= (500, 0), \\ \mathbf{v}_i &= (250, 0). \end{aligned} \tag{33}$$

Simple algebraic manipulations can be used to show that vectors returned by the evaluations of $\texttt{MVP}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ and $\texttt{MVP}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$ are given by

$$\mathbf{v}'_o = (500, \frac{1225}{\sqrt{30.24}}), \tag{34}$$

$$\mathbf{v}'_i = (250, -\frac{1225}{\sqrt{30.24}}), \tag{35}$$

respectively.

The time at which the two aircraft achieve minimum separation when both aircraft simultaneously maneuver is computed by the function `tca` (Section 4.3), and is given by the following quotient.

$$\begin{aligned} \texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i) &= -\frac{(-\sqrt{30.24}, 0.1) \cdot (250, 2450/\sqrt{30.24})}{\|(250, 2450/\sqrt{30.24})\|^2} \\ &= -\frac{245/\sqrt{30.24} - 250\sqrt{30.24}}{250^2 + 2450^2/30.24}. \end{aligned}$$

It can be proved that the distance between the aircraft at this time is strictly less than 4.85 nautical miles, i.e., that the following inequality holds.

$$\|\mathbf{s} + \texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i)\,(\mathbf{v}'_o - \mathbf{v}'_i)\| < 4.85.$$

In contrast to the numerical example presented in Section 6.4, the arithmetic used in this proof is exact. Hence, this inequality formally proves that `MVP` is not *ConflictFree*-coordinated. □

7.5 $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ is Not *ConflictFree*-Coordinated

In [2], it is claimed that if $\texttt{GOtrack}^{\varepsilon}_{f_o}$ is *ConflictFree*-coordinated with $\texttt{GOtrack}^{\varepsilon}_{f_i}$, then $f_o + f_i \geq 1$. This section shows that the converse claim does not hold, i.e, the fact that $f_o + f_i \geq 1$ does not imply that the algorithms are *ConflictFree*-coordinated. Although it holds for particular values of f_o and f_i, e.g., when $f_o = 1$ and $f_i = 0$, it does not hold when $f_o = f_i = \frac{1}{2}$.

Theorem 64. *The algorithm* $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ *is not ConflictFree-coordinated.*

Proof. To prove this theorem, it suffices to show that there exists D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, ε, $\mathbf{v}'_o$, and $\mathbf{v}'_i$ such that *Conflict*$(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$, $\mathbf{v}'_o \in \mathtt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$, and $\mathbf{v}'_i \in \mathtt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$, where *Conflict*$(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}'_o - \mathbf{v}'_i)$ holds.

Let D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, and ε be defined as follows, where distances are in nautical miles and speeds are in knots.

$$\begin{aligned} D &= 5, \\ \mathbf{s}_o &= (0, 0), \\ \mathbf{s}_i &= (\frac{10}{\sqrt{3}}, 0), \\ \mathbf{v}_o &= (500, 0), \\ \mathbf{v}_i &= (250, 0), \\ \varepsilon &= -1. \end{aligned}$$

The following equalities follow directly from definitions.

$$\begin{aligned} \mathtt{track}(\mathbf{v}_o) &= \frac{\pi}{2}, \\ \mathtt{track}(\mathbf{v}_i) &= \frac{\pi}{2}, \\ \mathtt{track}(\mathbf{v}_o - \mathbf{v}_i) &= \frac{\pi}{2}, \\ \chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, 0, -1) &= \frac{\pi}{2}, \\ \chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, \frac{1}{2}, -1) &= \frac{2\pi}{3}, \\ \chi^*_{rel}(\mathbf{s}, \mathbf{v}_o - \mathbf{v}_i, 1, -1) &= \frac{5\pi}{6}. \end{aligned}$$

Vectors $\mathbf{v}'_o \in \mathtt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ and $\mathbf{v}'_i \in \mathtt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$ can be computed as follows.

$$\begin{aligned} \mathbf{v}'_o &= (\frac{125}{2}(1 + 3\sqrt{5}), \frac{125}{2}\sqrt{3}(1 - \sqrt{5})), \\ \mathbf{v}'_i &= (125, 125\sqrt{3}). \end{aligned}$$

Thus,

$$\mathbf{v}'_o - \mathbf{v}'_i = (\frac{125}{2}(3\sqrt{5} - 1), -\frac{125}{2}\sqrt{3}(1 + \sqrt{5})).$$

The relative velocity vector $\mathbf{v}'_o - \mathbf{v}'_i$ therefore has a norm equal to 500 knots. Hence, the time when the aircraft achieve minimum separation is given by

$$\begin{aligned} \mathtt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i) &= \frac{-1}{500^2}(\mathbf{s} \cdot (\mathbf{v}'_o - \mathbf{v}'_i)) \\ &= \frac{1}{500^2} 625 (\sqrt{15} - \frac{1}{\sqrt{3}}). \end{aligned}$$

The distance between the aircraft at this time satisfies the following inequality.

$$\|\mathbf{s} + \mathtt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i)(\mathbf{v}'_o - \mathbf{v}'_i)\| < 4.1.$$

In contrast to the numerical example presented in Section 6.4, the arithmetic used in this proof is exact. Hence, this inequality formally proves that $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ is not *ConflictFree*-coordinated. □

7.6 $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ and MVP are Not *ConflictFree*-Coordinated

This section presents numerical evidence that the resolution algorithm $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ is not *ConflictFree*-coordinated with MVP, not even when ε represents the opposite direction to the resolution returned by MVP. The result that follows uses values returned by an implementation of the algorithm $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ and, therefore, may be inaccurate. Since those values have not been formally checked in PVS, the following result does not qualify as a formal proof. To distinguish this result from all the other results presented in this paper, it is stated as a *conjecture* instead of a lemma or theorem.

The algorithm $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ has as a parameter $\varepsilon = \pm 1$, which refers to direction (left or right). The algorithm MVP does not have such a parameter, because it returns a vector for the ownship that yields a relative velocity vector that passes on the same side of the origin as the original relative velocity vector. Thus, when coordination is considered between the algorithms $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ and MVP, there is only one choice of the unit ε that could possibly provide coordination. This is the unit that satisfies $\varepsilon\,\mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$, where $\mathbf{s}$ and $\mathbf{v}$ are the current relative position and velocity vectors, respectively.

Conjecture 1. *The algorithms* $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$ *and* MVP *are not ConflictFree-coordinated, not even when the parameter* ε *is chosen so that* $\varepsilon\,\mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$.

To prove this conjecture, it suffices to show that there exists D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, $\mathbf{v}_i$, ε, $\mathbf{v}'_o$, and $\mathbf{v}'_i$ such that $\mathit{Conflict}(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}_o - \mathbf{v}_i)$, $\mathbf{v}'_o \in \texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$, and $\mathbf{v}'_i \in \texttt{MVP}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$, where $\mathit{Conflict}(\mathbf{s}_o - \mathbf{s}_i, \mathbf{v}'_o - \mathbf{v}'_i)$ holds.

An example is now given that numerically justifies that such a scenario does indeed exist. Let D, $\mathbf{s}_o$, $\mathbf{s}_i$, $\mathbf{v}_o$, and $\mathbf{v}_i$ be given as in Formula (33) and let ε be -1. It is easy to see that $\varepsilon\,\mathbf{s} \cdot \mathbf{v}^{\perp} \leq 0$. The vector $\mathbf{v}'_i \in \texttt{MVP}(\mathbf{s}_i, \mathbf{s}_o, \mathbf{v}_i, \mathbf{v}_o)$ is exactly defined by Formula (35). A vector $\mathbf{v}'_o \in \texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}(\mathbf{s}_o, \mathbf{s}_i, \mathbf{v}_o, \mathbf{v}_i)$ can be computed numerically and is approximately given by

$$\mathbf{v}'_o \approx (478.835, 143.993).$$

In this case, the time of closest approach if both aircraft maneuver, i.e., $\texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i)$, is approximately 0.00654 hours or about 23.544 seconds. At this time, the separation between the aircraft is approximately given by

$$\|\mathbf{s} + \texttt{tca}(\mathbf{s}, \mathbf{v}'_o - \mathbf{v}'_i)\,(\mathbf{v}'_o - \mathbf{v}'_i)\| \approx 4.718.$$

Thus, the aircraft are in conflict, and this completes the argument for Conjecture 1. The formal development of this argument is tedious although not necessarily difficult. The family of Geometric Optimization algorithms use analytical equations involving trigonometric functions. Trigonometric reasoning is not currently well-handled by state-of-the-art theorems provers.

	$\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$	MVP	$\texttt{GOtrack}^{\varepsilon}_{1}$	$\texttt{ACCoRDtrack}^{\varepsilon}$
$\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$	✗	✗*		
MVP	✗*	✗	✓	✓
$\texttt{GOtrack}^{\varepsilon}_{1}$		✓	✓	✓
$\texttt{ACCoRDtrack}^{\varepsilon}$		✓	✓	✓

Table 4. Coordination of $\texttt{GOtrack}^{\varepsilon}_{\frac{1}{2}}$, MVP, $\texttt{GOtrack}^{\varepsilon}_{1}$, and $\texttt{ACCoRDtrack}^{\varepsilon}$.

8 Conclusion

This paper proposed a general mathematical framework for studying implicit coordination in the context of state-based separation assurance systems. Implicit coordination has been formally defined before [3,4,6,11]. In those papers, the concept of coordination applies to a particular strategy for computing coordinated resolution maneuvers or to a specific conflict resolution algorithm. The work presented in this paper applies to any state-based separation assurance algorithm and to any type of safety property for which coordination needs to proved.

The framework is illustrated by formally studying coordination properties of well-known conflict resolution algorithms such as the Modified Voltage Potential algorithm (MVP), the Geometric Optimization algorithm for track angle maneuvers ($\texttt{GOtrack}^{\varepsilon}_{f}$), and ACCoRD's conflict resolution algorithm for track angle maneuvers ($\texttt{ACCoRDtrack}^{\varepsilon}$). Table 4 summarizes the main results where the intersection between a column and a row refers to *ConflictFree*-coordination for the given resolution algorithms. The symbol ✓ stands for cases where coordination has been formally proved in PVS. The symbol ✗ stands for cases where coordination has been formally disproved in PVS, i.e., a counterexample for coordination has been found and the claim that the counterexample does not satisfy coordination has been formally proved in PVS. The symbol ✗* stands for cases where a counterexample for coordination has been found and the claim that coordination does not hold has been numerically checked, but the formal proof of this claim is not provided. The remaining cases have not been studied, but they could be analyzed in the same way as the other cases presented in this paper.

The framework presented here relies on some physical and operational assumptions. For instance, the airspace is represented by a Euclidean geometry, where aircraft fly linear trajectories and maneuver instantaneously. These assumptions are common to state-based approaches. They allow for analytical solutions that yield efficient implementations. Despite its limitations, state-based CD&R is used in the self-separation concept [20] as a backup for more sophisticated separation assurance systems. Therefore, state-based CD&R is a critical component of this concept. The framework also assumes that resolution maneuvers are computed in a pairwise fashion. Although multiple simultaneous conflicts may be rare, they will exist and the safety case for a distributed air traffic concept of operations has to guarantee that they are correctly handled. Future work in this area will look at this

problem. In particular, it will be studied how the criteria concept can be integrated into prevention bands [14], a concept that naturally fuses conflict information for multiple aircraft.

In summary, the framework presented here is believed to be a fundamental step towards the understanding of how different state-based separation assurance algorithms can be deployed in the future airspace in a way that they safely interact with each other. This framework provides the mathematical basis for an approach to self-separation in NextGen that does not rely on a specifically mandated CD&R algorithm but on a criteria-based standard for conflict resolution [12].

The results presented in this paper have been mechanically checked using an interactive theorem prover, which provides strong guarantees that the mathematical development is correct. The use of a mechanical theorem prover requires a detailed description of the problem and a meticulous proof process. This level of rigor is justified by the critical role that aircraft separation plays in the overall safety of the next generation of air traffic management systems.

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REPORT DOCUMENTATION PAGE	Form Approved OMB No. 0704–0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
01-03-2011	Technical Publication	

4. TITLE AND SUBTITLE
State-Based Implicit Coordination and Applications

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Anthony J. Narkawicz and César A. Muñoz

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER
411931-02-51-07-01, 457280.02.07.07.07

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
NASA Langley Research Center
Hampton, Virginia 23681-2199

8. PERFORMING ORGANIZATION REPORT NUMBER
L–19959

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
Washington, DC 20546-0001

10. SPONSOR/MONITOR'S ACRONYM(S)
NASA

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
NASA/TP–2011–217067

12. DISTRIBUTION/AVAILABILITY STATEMENT
Unclassified-Unlimited
Subject Category 03
Availability: NASA CASI (443) 757-5802

13. SUPPLEMENTARY NOTES
An electronic version can be found at http://ntrs.nasa.gov.

14. ABSTRACT

In air traffic management, pairwise coordination is the ability to achieve separation requirements when conflicting aircraft simultaneously maneuver to solve a conflict. Resolution algorithms are implicitly coordinated if they provide coordinated resolution maneuvers to conflicting aircraft when only surveillance data, e.g., position and velocity vectors, is periodically broadcast by the aircraft. This paper proposes an abstract framework for reasoning about state-based implicit coordination. The framework consists of a formalized mathematical development that enables and simplifies the design and verification of implicitly coordinated state-based resolution algorithms. The use of the framework is illustrated with several examples of algorithms and formal proofs of their coordination properties. The work presented here supports the safety case for a distributed self-separation air traffic management concept where different aircraft may use different conflict resolution algorithms and be assured that separation will be maintained.

15. SUBJECT TERMS
Air traffic, avoidance, conflict, detection, resolution, prevention, formal verification

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			STI Help Desk (email: help@sti.nasa.gov)
U	U	U	UU	60	19b. TELEPHONE NUMBER *(Include area code)* (443) 757-5802