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# **Coupled Particle Transport and Pattern Formation in a Nonlinear Leaky-Box Model**

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***August 2009***

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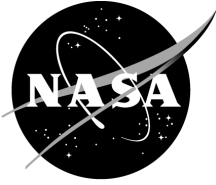
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## LIST OF ACRONYMS

ACR	anomalous cosmic ray
EGCR	extra galactic cosmic ray
GCR	galactic cosmic ray
SEP	solar energetic particle
TP	Technical Publication

## NOMENCLATURE

$A$	particle mass number
$a'$	constant (Thomas model)
$b'$	constant (Thomas model)
$c$	speed of light
$C(p)$	drift term in momentum space (Fokker-Planck model)
$D$	diffusion coefficient (Fokker-Planck model)
$D_0$	constant
$E'$	normalized energy
$F$	coupling function (Thomas model)
$f$	phase-space or number density function
$f(p,t)$	isotropic part of the particle's phase-space density
$G$	coupling function (Thomas model)
$K$	diffusion tensor
$m_0$	unit mass
$p$	particle's momentum
$Q$	source and sink term
$q$	particle charge
$V$	average propagation speed
$\vec{V}$	convection velocity
$v$	particle speed
$t$	time
$\ell$	characteristic length scale
$\alpha$	spectral index of magnetic turbulence



## NOMENCLATURE (Continued)

$\alpha'$	constant (Thomas model)
$\gamma$	coupling parameter (Thomas model)
$\kappa_0$	constant (diffusion coefficient)
$\kappa'$	constant (Thomas model)
$\kappa_{\parallel}$	spatial diffusion coefficient parallel to the magnetic field line
$\rho$	constant (Thomas model)
$\tau(p)$	momentum-dependent escape rate in the leaky-box model



## TECHNICAL PUBLICATION

# COUPLED PARTICLE TRANSPORT AND PATTERN FORMATION IN A NONLINEAR LEAKY-BOX MODEL

## 1. INTRODUCTION

In transport and acceleration of energetic (suprathermal) charged particles in space and astrophysical plasmas, the relevant length scales are typically much larger than the particle's Larmor radius. As a result, the motion of these particles is effectively determined by the structure of the electromagnetic field via both its regular and stochastic components. In the absence of particle-particle collisions (the low density limit), the electromagnetic field is also responsible for accelerating the particles up to relativistic energies.

The fluctuating component of the field can scatter particles especially when the particle's Larmor radius is comparable to the wavelength of the scattering hydromagnetic wave, 'resonant scattering.' If scattering is frequent and strong, it can isotropize the particle's density function and, in the diffusive limit, the motion of those particles can then be described statistically. The resulting kinetic equation<sup>1,2</sup> assumes a Fokker-Planck equation of the form

$$\frac{\partial f}{\partial t} = \nabla \cdot (K \nabla f) - \nabla \cdot (\vec{V} f) , \quad (1)$$

where  $f$  is the phase-space or number density function of the particle,  $K$  is a diffusion tensor, and  $\vec{V}$  is a convection velocity. The diffusion tensor can be formally<sup>3</sup> written in terms of the stochastic component of the magnetic field. With the addition of other terms like energy loss, e.g., due to ionization and synchrotron radiation, gain terms, e.g., due to the acceleration mechanism or mechanisms, and source and sink terms, equation (1) forms the basis of many phenomenological and computational kinetic studies of the transport and acceleration of energetic charged particles in space and astrophysical plasmas.

In case of solar energetic particles (SEPs),<sup>4</sup> recent and more realistic transport models<sup>5,6</sup> incorporate particle-wave coupling that requires self-consistent<sup>7</sup> solutions of the particle kinetic and wave propagation equations. Particles are scattered and accelerated by waves they themselves generate and/or amplify.<sup>8</sup> SEP observations not only support but appear to require this coupling between particles and waves in space.<sup>9</sup> For SEPs, as well as for various other energetic particles in disparate astrophysical situations—anomalous cosmic rays (ACRs) at the heliospheric termination shock,<sup>10</sup> galactic cosmic rays (GCRs) at supernovae shocks,<sup>11,12</sup> and even extra galactic cosmic rays (EGCRs) at galactic shocks<sup>13</sup>—diffusive shock acceleration<sup>14</sup> has been demonstrated to be a more efficient

(compared for example to Fermi’s original mechanism) and ‘natural’ process in accelerating suprathermal charged particles in space and astrophysical plasmas.

In addition to acceleration, large-scale transport in space plasmas is for most particles also diffusive since the diffusion length due to the fluctuating magnetic field tends to be smaller than the characteristic length of the field.<sup>3,15</sup> The magnetic field lines themselves can also wander diffusively and enhance the diffusion of the particles, affecting further their overall transport and acceleration characteristics.<sup>16–20</sup>

From a modeling perspective, coupling via any of the various terms that describe the particles’ acceleration, interactions, propagation, and even source terms, is likely to result in a nonlinear reactive-diffusive system of equations. Such systems are prone to diffusion-driven instabilities, which can give rise to steady-state (Turing) structures. These structures are brought about not necessarily due to one process (acceleration, transport, etc.) or another individually, but rather from the mutual interplay of these processes. Turing theory, or morphogenesis<sup>21</sup> based on diffusion-driven instability, has already found a number of suggestive applications in astrophysics,<sup>22–28</sup> statistical physics,<sup>29</sup> and condensed matter physics.<sup>30</sup> This is in addition to the more ‘natural’ applications to chemical and biological systems.<sup>31</sup>

Motivated in part by recent data<sup>32</sup> in which SEP characteristics appear to show patterns that are difficult to explain phenomenologically, this Technical Publication (TP) explores the effects of a specific form of coupling—via a nonlinear source term—on the coupled particles’ characteristics.

For purposes of this study, a reduced form of equation (1) ignoring explicit spatial dependence but including diffusion in momentum space can be simplified<sup>33,34</sup> to the following Fokker-Planck form (the leaky-box limit):

$$\frac{\partial f}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left( p^2 D \frac{\partial f}{\partial p} \right) - \frac{1}{p^2} \frac{\partial}{\partial p} (p^2 C f) - \frac{f}{\tau} + \text{sources} - \text{sinks} \quad , \quad (2)$$

where  $f(p,t)$  is understood to be the isotropic part of the particle’s phase-space density,  $p$  is the particle’s momentum,  $D$  the diffusion coefficient in momentum space,  $C$  describes the systematic momentum loss and gain rates, and  $\tau$  is an escape term in lieu of spatial diffusion assuming homogeneous spatial transport, i.e., in the limit  $\overline{\nabla \cdot (K \nabla f)} \rightarrow -f / \tau$  where  $\overline{(\dots)}$  denotes spatial averaging of some kind.

The stability of a system of two coupled equations based on equation (2) when treated as a nonlinear, reactive-diffusive system, and its implications on the characteristics of the coupled particles the system describes is being the focus here. Noncolliding particles can (mathematically) be coupled through any one of the terms in equation (2). Particle-wave coupling through the transport coefficients while the more physically compelling effect is difficult to reduce to amenable forms for purposes of this study. Instead, a mathematically well-studied model is relied on in which coupling is affected via a nonlinear source term.

The idealized physical picture here is one in which two particles are in a spatially homogeneous region with their momenta subject to both stochastic and systematic changes. The two particles are assumed coupled via a common, nonlinear source term. It is essentially a leaky-box limit description with an added caveat with regard to the nonlinear source term. While assumptions are not made about the physical processes that result in such a coupling in the source term, a specific form that is known to support steady-state structures under some conditions will be used.

The specific goals here are (1) to demonstrate that even a simple leaky-box type model, but with nonlinear coupling affected in some manner, can support steady-state structures under certain conditions, (2) to delineate the general physical conditions associated with these structures, and (3) to quantify their effects on modeled or observed particle characteristics.

While the applicability and potential implications of the formal concepts of Turing structures and diffusion-driven instabilities to energetic particles in space plasmas is only heuristically demonstrated here with the help of an idealized physical model, the insight gained may be of wider use and significance, especially to similar systems implementing more realistic coupling via, e.g., particle-wave interactions. Such applications of Turing morphogenesis theory can potentially have significant organizing benefits in space physics.

This TP is organized as follows: The two-particle Fokker-Planck model is described in section 2 and the prototypical nonlinear, reactive-diffusive system known as the Thomas system is introduced in section 3. In section 4, explicit correspondence between the idealized two-particle Fokker-Planck model and the Thomas system is derived. Physical significance of this correspondence is discussed in section 5 and sample application presented. Section 6 offers a brief summary and some concluding remarks.

## 2. A COUPLED, TWO-PARTICLE FOKKER-PLANCK MODEL

The acceleration and transport of two distinct charged-particle species are idealized as they interact with a collisionless, magnetized plasma. The particles are described by their respective distribution functions,  $f_{1,2}(\vec{p}, \vec{x}, t)$ , that evolve in momentum, space, and time due to their interactions with the plasma magnetic field. The magnetic field is assumed to have a regular and a stochastic component. The details of the regular component will be ignored here. The stochastic component is characterized by a spectral index and an energy level. The plasma is assumed to be spatially homogeneous, i.e., its spatial parameters change over timescales much larger than those that characterize  $f_{1,2}(\vec{p}, \vec{x}, t)$ . This allows us to ignore explicit spatial variations in the distribution function, i.e.,  $f(\vec{p}, \vec{x}, t) \approx f(\vec{p}, t)$ , and hence in all the particle properties described by it.

Also assumed is that magnetic turbulence scatters the particles sufficiently fast to give rise to diffusion both in coordinate and in momentum space. In a spherically symmetric momentum space assuming scattering off ‘hard spheres,’ the diffusion coefficient is typically written as<sup>7,34–36</sup>

$$D(p) = \frac{V^2 p^2}{9\kappa_{\parallel}(p)}, \quad (3)$$

where  $V$  is an average propagation speed of the scattering centers (Alfvén speed).  $\kappa_{\parallel}$ , the spatial diffusion coefficient parallel to the magnetic field line, is taken to be proportional to the rigidity (momentum  $p$  per charge  $q$ ) of the particle as

$$\kappa_{\parallel}(p) = \kappa_0 \left( \frac{p}{q} \right)^{2-\alpha}, \quad (4)$$

where  $\kappa_0$  is a constant inversely proportional to the level or energy of the magnetic turbulence (but establishes the energy scale) while  $\alpha$  is the spectral index of the turbulence. Diffusion in momentum (or energy) is an expression<sup>37</sup> of what is known as Fermi second-order acceleration process wherein the incremental change in momentum,  $\Delta p/p \propto (v/c)^2$ , where  $v$  is the particle speed and  $c$  is the speed of light.

In addition to diffusion in momentum space, also assumed is that the particles are subject to systematic momentum loss and gain processes like acceleration due to a shock (the so-called Fermi first-order acceleration) or deceleration due to passage through the plasma. However, the exact form of these terms is not crucial to our purposes here and will collectively be referred to as the ‘drift’ term in momentum space,  $C(p)$ .

Upon acceleration or deceleration, the particles are assumed to be able to escape the finite momentum region with a momentum-dependent escape rate,  $\tau(p) = \ell^2/\kappa_{\parallel}(p)$ , with  $\ell$  being a

characteristic length scale. This escape-rate term approximates diffusion in coordinate space in the homogeneous or leaky-box limit. Finally, the particles are assumed to share a common, but may be nonlinear, source and sink term,  $Q$ .

The above idealized particle transport picture can be described by the following set of Fokker-Planck equations in momentum space,

$$\frac{\partial f_1}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 D_1(p) \frac{\partial f_1}{\partial p} \right] - \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 C_1(p) f_1 \right] - \frac{f_1}{\tau_1(p)} + Q(f_1, f_2) \quad (5)$$

and

$$\frac{\partial f_2}{\partial t} = \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 D_2(p) \frac{\partial f_2}{\partial p} \right] - \frac{1}{p^2} \frac{\partial}{\partial p} \left[ p^2 C_2(p) f_2 \right] - \frac{f_2}{\tau_2(p)} + Q(f_1, f_2) . \quad (6)$$

Note that while the above system is an idealization of charged particle transport in space plasmas,<sup>33,34,38</sup> the simplifying assumptions made, i.e., leaky-box limit, should not take much away from the relevance and/or direct applicability of what follows to more realistic modeling involving other coupling venues. Moreover, outside of the assumption of a common nonlinear source term, there are no more (basic) assumptions in this two-particle model than what is typically made in a good number of similar physical models based on the leaky-box approximation.

To make contact with the so-called Thomas model to be discussed below, the above system is written in terms of the classical kinetic energy of the particle,  $E = p/(2m)$ , as

$$\frac{\partial f_1}{\partial t} = \frac{\partial^2}{\partial E^2} [D'_1(E) f_1] - \frac{\partial}{\partial E} [C'_1(E) f_1] - \frac{f_1}{\tau_1(E)} + Q(f_1, f_2) \quad (7)$$

and

$$\frac{\partial f_2}{\partial t} = \frac{\partial^2}{\partial E^2} [D'_2(E) f_2] - \frac{\partial}{\partial E} [C'_2(E) f_2] - \frac{f_2}{\tau_2(E)} + Q(f_1, f_2) . \quad (8)$$

The diffusion and drift coefficients in energy (in their canonical Fokker-Planck forms) are now written in terms of those appearing in equations (5) and (6) in momentum space as ( $i=1,2$ ):

$$D'_i(E) = \left( \frac{dE}{dp} \right)^2 D_i(p) \quad (9)$$

and

$$C'_i(E) = \left( \frac{dE}{dp} \right) C_i(p) . \quad (10)$$

Assumptions about the nature of particle-particle coupling in the above two-particle Fokker-Planck model will only be made by analogy with that appearing in the Thomas system, i.e., through the source term. No other coupling is assumed. Also, defining the appropriate initial and boundary conditions will be delayed until the model is transformed to a form suitable for direct comparison with the Thomas system, which is introduced and discussed in section 3.



### 3. A COUPLED, NONLINEAR REACTIVE-DIFFUSIVE MODEL

The Thomas model<sup>31</sup> is a coupled, nonlinear reactive diffusive model known to support Turing structures in its steady-state solutions. For two species described by densities  $f'_1(E', t)$  and  $f'_2(E', t)$ , assumed to be in nondimensionalized form, the model is described by the following coupled equations:

$$\frac{\partial f'_1}{\partial t'} = \frac{\partial^2 f'_1}{\partial E'^2} + \gamma F(f'_1, f'_2) \quad (11)$$

and

$$\frac{\partial f'_2}{\partial t'} = D \frac{\partial^2 f'_2}{\partial E'^2} + \gamma G(f'_1, f'_2) , \quad (12)$$

where  $D$  is the relative diffusion coefficient and  $\gamma$  is a coupling parameter. Functions  $F$  and  $G$  carry the ‘kinetics’ in this system. For the Thomas system, these are given by

$$F(f'_1, f'_2) = (a' - f'_1) - H(f'_1, f'_2) , \quad (13)$$

$$G(f'_1, f'_2) = \alpha'(b' - f'_2) - H(f'_1, f'_2) , \quad (14)$$

where

$$H(f'_1, f'_2) = \frac{\rho f'_1 f'_2}{1 + f'_1 + \kappa' f'_2} . \quad (15)$$

Parameters  $a'$ ,  $b'$ ,  $\alpha'$ ,  $\rho$ , and  $\kappa'$  are constants. The variable  $E' \in [0, 4]$  is taken to be a normalized ‘energy’ variable.

There are a number of conditions that the system described above must first satisfy before it can support Turing structures in steady state. These are related to the stability of the homogeneous solution to the introduction of small perturbations in the absence of diffusion, i.e., for  $t \rightarrow \infty$  with  $D = 0$ .

The reader is referred to reference 31 for a thorough treatment. For the particular set  $a' = 92$ ,  $b' = 64$ ,  $\alpha' = 1.5$ ,  $\rho = 18.5$ , and  $\kappa' = 0.1$ , it can be shown<sup>39</sup> that the above system must satisfy  $D \gtrsim 10$  and  $\gamma \gtrsim 8$  before it can support any structure. However, satisfying these two critical values for the relative diffusion and strength of the coupling does not in itself guarantee that the steady-state solutions of equations (11) and (12) exhibit Turing structures.

Figure 1 shows four steady-state numerical solutions<sup>40</sup> of equations (11) and (12) for four different sets of  $(\gamma, D)$  values but for the same values for the rest of the parameters mentioned above. Here, the initial conditions were taken to be random variations about the homogeneous solutions, i.e., when  $F=G=0$ , and the boundary conditions are such that  $\partial f'_1 / \partial E' = \partial f'_2 / \partial E' = 0$  at  $E' = (0, 4)$ . Note that all four pairs of  $(\gamma, D)$  satisfy the critical conditions that  $D \geq 10$  and  $\gamma \geq 8$ , yet only two of them, (10,10) and (12,9.8) actually show the sinusoidal structure. Note also that  $f'_1(E')$  and  $f'_2(E')$  are in phase over the entire domain of  $E'$ .

For the case of (12,9.8), which will be used for illustration later on, the sinusoidal structure is well approximated by the relation

$$f'(E') = a \cos(\omega E') + b, \quad (16)$$

with  $a = 2.28$ ,  $b = 9.8$  for  $f'_1(E')$ , and  $a = 0.43$ ,  $b = 9.3$  for  $f'_2(E')$ , with  $\omega = \pi$  for both.

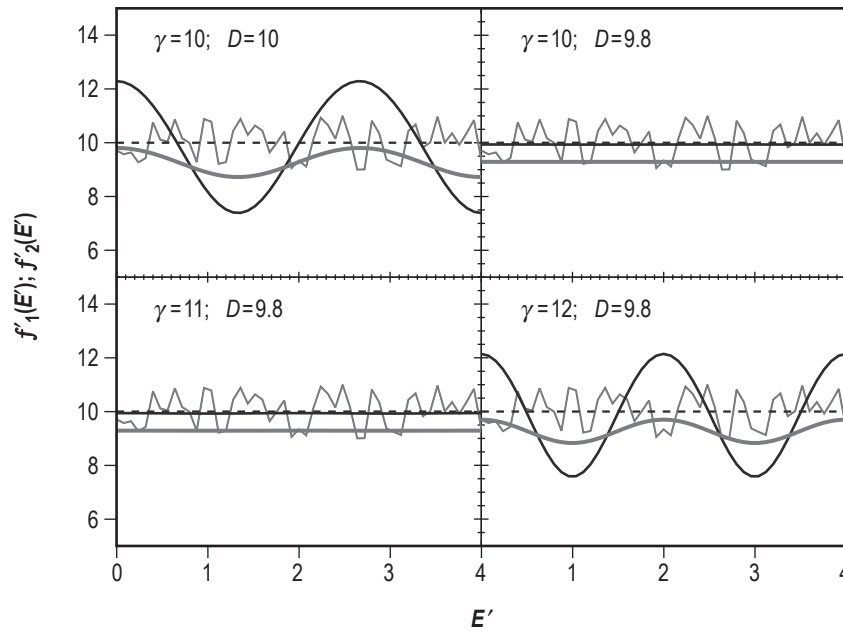


Figure 1. Four steady-state solutions of the Thomas model for various relative diffusion and coupling strengths—the rest of the parameters appearing in equations (11) and (12) are unchanged. Solid curve depicts  $f'_1(E')$  while the light curve depicts  $f'_2(E')$ . Light, jagged curve is the initial condition and dashed curve is the homogeneous solution. Sinusoidal steady-state patterns (Turing structures) can be seen for some values of the coupling strength parameter  $\gamma$  and the relative diffusion coefficient  $D$  but not for others.

#### 4. THE TWO-PARTICLE FOKKER-PLANCK MODEL AS A NONLINEAR, REACTIVE DIFFUSIVE MODEL

The system described by equations (7) and (8) must first be transformed to a constant diffusion, zero drift system and normalized before it can be made to correspond to the system in equations (11) and (12). Upon this second transformation, particle 1 will have a diffusion coefficient of unity and particle 2 will have a diffusion coefficient of  $D > 1$ . The transformed Fokker-Planck drift coefficients will both be zero. The transformed escape terms are structurally unaffected. The source term, which is taken to depend only on the independent variables, will assume its form by analogy with the Thomas system. It is in this regard that in this model, the particle-particle coupling is considered to be affected only via the source term.

The transformation requires that<sup>41</sup>

$$E'_1(E) = \int \frac{1}{\sqrt{D'_1(E)}} dE \quad (17)$$

and

$$E'_2(E) = \int \frac{\sqrt{D}}{\sqrt{D'_2(E)}} dE . \quad (18)$$

The transformed drift coefficients satisfy ( $i = 1, 2$ ):

$$C''_i(E'_i) = \frac{(1 \text{ or } \sqrt{D})}{\sqrt{D'_i}} \left[ C'_i - \frac{1}{2} \frac{dD'_i}{dE} \right] , \quad (19)$$

which can be made to vanish if

$$C'_i(E) = \frac{1}{2} \frac{dD'_i}{dE}(E) . \quad (20)$$

Requiring that  $f'_i dE'_i = f_i dE_i$ , the transformed distribution functions will then take the form

$$f'_i(E'_i, t) = \frac{\sqrt{D'_i}}{(1 \text{ or } \sqrt{D})} f_i(E, t) . \quad (21)$$

Note, however, that the above transformations apply to each particle separately since, in general,  $E'_1 \neq E'_2$ . Since a coupled, two-particle system is used, the requirement is that

$$E''_1(E) = E''_2(E) = E'(E) . \quad (22)$$

To that end, a typical form for the diffusion coefficient is assumed (cf., eqs. (3) and (9)):

$$D'_i(E) = D_{0,i} E^{2-\alpha/2} , \quad (23)$$

where, for  $i = 1, 2$ ,  $D_{0,i}$  can be written as,

$$D_{0,i} = D_0 \left( \frac{2^{\varepsilon_1} m_0^{\varepsilon_2} A_i}{q_i} \right)^{2-\alpha} , \quad (24)$$

with  $m_0$  being a unit mass (e.g., amu) that is normalized to unity here,  $A$  is mass number,  $\varepsilon_1 = (1 - \alpha/4)/(1 - \alpha/2)$ ,  $\varepsilon_2 = \varepsilon_1 - 1$ , and  $D_0$  is now a constant independent of energy as well as particle species. Since  $D_0 \propto 1/k_0$ , equation (3), and  $k_0$  is, in turn, inversely proportional to the strength of the turbulence,  $D_0$  is taken to characterize the level of magnetic turbulence. Normalization here then simply means  $D_0 = 1$ . With this normalization, the particles are characterized simply by their mass-to-charge ratios and the fluctuating magnetic field is characterized by its spectral index and energy level. Again,  $\alpha$  is the spectral index of the magnetic turbulence as in equation (4) and  $q$  is the particle charge.

The transformed energy  $E'$  that applies self-consistently to both particles must then take the form

$$E'(E) = \frac{4\alpha^{-1}}{\sqrt{D_{0,1} D_{0,2}}} E^{\alpha/4} . \quad (25)$$

Self-consistency also requires a transformation of the time variable as

$$t'(t) = \frac{t}{\left( D_{0,1} D_{0,2} \right)} . \quad (26)$$

Direct contact is then made with the Thomas system, equations (11) and (12), if the two-particle Fokker-Planck system, equations (7) and (8), is transformed according to table 1. For simplicity, a single common source, i.e.,  $Q_1 \approx Q_2 = Q$ , is assumed since in the Thomas system,  $\alpha' b' \approx a'$ .

Table 1. Correspondence between the two-particle Fokker-Planck model, equations (7) and (8), with that of the Thomas system, equations (11) and (12). Note that the Fokker-Planck model's equations have been transformed twice and then normalized.

Quantity in the Fokker-Planck Model	Quantity in the Thomas System
$f_i$	$f'_i (i=1,2)$
$E$	$E'$
$t$	$t'$
$D_1''$	1
$D_2''$	$D$
$C_i''$	0 ( $i=1,2$ )
$\tau_1$	$\gamma$
$\tau_1/\tau_2$	$\alpha'$
$Q \approx Q_1$	$\gamma(a'-H)$
$Q \approx Q_2$	$\gamma(\alpha'b'-H)$

Finally, since the ratio of the distribution functions in steady state is of interest, the following relation will be used:

$$\frac{f_1(E)}{f_2(E)} = \frac{1}{\sqrt{D}} \sqrt{\frac{D_{0,2}}{D_{0,1}}} \frac{f'_1[E'(E)]}{f'_2[E'(E)]}. \quad (27)$$

It is clear that any structure in the ratio  $f'_1/f'_2(E')$  will appear as a transformed structure in the ratio  $f_1/f_2(E)$ . The transformed structure in turn depends on two other factors: The first term in equation (27) simply multiplies it by a constant. The second term, though it does not depend on energy, it does depend on the coupled particles' mass-to-charge ratios, which appear in the transformation of energy, equation (25), as well as on the magnetic turbulence (characterized by  $D_0$  and  $\alpha$ ). As such, this factor also affects the transformed structure.

Hence, whenever a structure exists in  $E'$ -space due to particle-particle coupling, the transformed structure in  $E$ -space is expected to be shaped both by the property of the magnetic turbulence as well as by the two particles themselves through their mass-to-charge ratios. Next, the dependence of the structure on both of these factors is explored in more detail.

## 5. AN ILLUSTRATION

Using the two-particle Fokker-Planck model described above and assuming that the two particles are coupled via their common source term (cf., table 1), the steady-state solutions of the Thomas system can simply be treated by proxy as the steady-state solutions of the doubly transformed and normalized two-particle Fokker-Planck system. Hence, equation (16) can be used and transformed according to equation (25) to study the dependence of the structure on the attributes of the coupled particles and those of the magnetic turbulence.

Figure 2 shows three transformed solutions as functions of  $E$  for three different values of the spectral index: a Kolmogorov spectrum ( $\alpha = 5/3$ ), a Kraichnan spectrum ( $\alpha = 3/2$ ), and a third characterized by ( $\alpha = 4/3$ ). All three spectra, however, are characterized by the same turbulence level  $D_0 = 1$ . Particle 1 has an  $A/q = 1.5$  while particle 2 has  $A/q = 1$ .

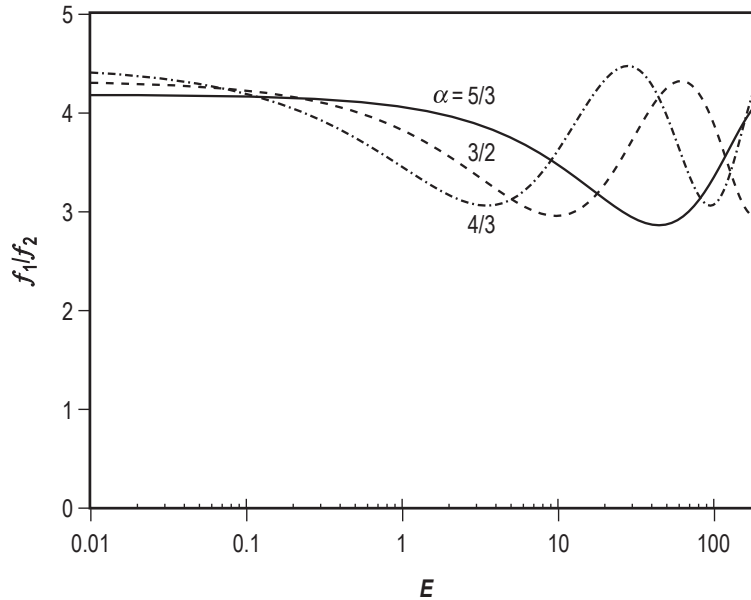


Figure 2. Three steady-state solutions, equation (16) transformed according to equation (25), as functions of  $E$  for three different values of magnetic turbulence spectral index,  $\alpha$ . All three spectra, however, are characterized by the same turbulence level of  $D_0 = 1$ . The turbulence index is seen to determine the relative ‘phase’ of the steady-state solution (structure).

The dependence of the steady-state solution (or structure) on the spectral index of the magnetic turbulence is seen; two different spectral indices can give rise to two different structures. The transformed sinusoidal structures appear to be simply out of phase with respect to each other as  $\alpha$

varies. This can be readily surmised from equation (25) and the fact that ratios of the solutions are plotted rather than the solutions themselves. The ‘phase’ of the structure appears to be predicated by the spectral index  $\alpha$ .

The physical significance of this dependence can be appreciated as follows. If the same particles are now decoupled but their transport is still assumed to take place in the same two magnetic fields, the ratio of their steady-state density functions will either be a monotonically decreasing or a monotonically increasing function of  $E$ . This is readily seen in the steady-state solutions of the now decoupled equations (11) and (12). The decoupled solutions for a given  $\alpha$  will share a common inverse power-law behavior in  $E$ , i.e.,  $f(E) \propto E^{-\eta}$ , where  $\eta \propto \alpha$ . As such, no ‘structure’ in the ratio of the two functions should be present.

Figure 3 shows three transformed sinusoidal solutions as functions of  $E$  for three different levels of the magnetic turbulence. Here, the spectral index of  $\alpha=5/3$  is the same for all three solutions. All other model parameters as well as the particles’ charge-to-mass ratios are the same as those used for figure 2. It can be seen that as the turbulence strength increases (i.e., higher  $D_0$  since  $D_0$  is directly proportional to the turbulence strength, equation (3), while  $\kappa_{\parallel}$  is inversely proportional to it, equation (4)), the structure becomes less pronounced. Equivalently, the structure appears to move to higher energy regions with stronger turbulence, and vice versa. Given that the structure in this model is simply a transformed sinusoidal function, this behavior is straightforward to appreciate both mathematically and physically, as follows.

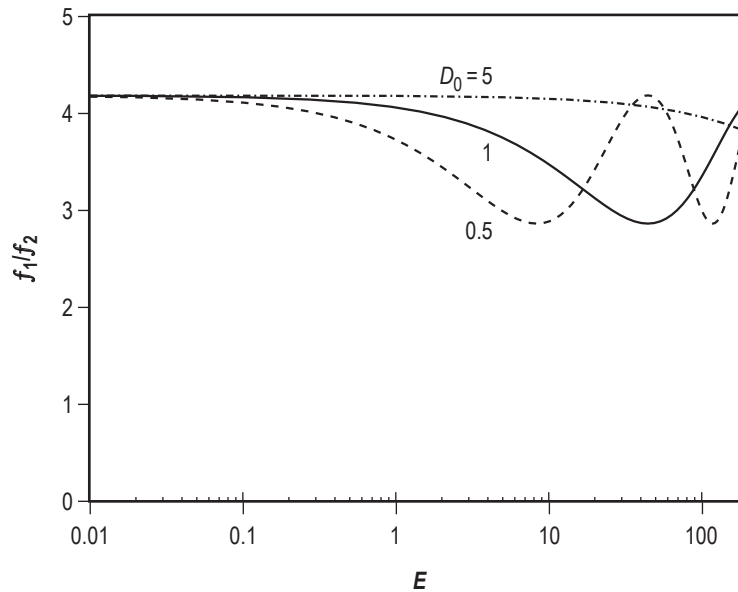


Figure 3. Three steady-state solutions, equation (16) transformed according to equation (25), as functions of  $E$  for three different levels of the magnetic turbulence. The spectral index of the turbulence,  $\alpha=5/3$ , is the same for all three solutions. All other parameters remain unchanged. The ‘structure’ is seen to become less pronounced (or, equivalently, move higher up in energy) with increasing level of turbulence.

As the turbulence level increases, the strength of the diffusion coefficient in momentum or energy space increases. In the coupled case, even though their relative diffusion remains the same, they both will have correspondingly shorter ‘mean free paths’ in energy space with increasing turbulence strength. If the two coupled particles are assigned a ‘correlation length’ in  $E$ -space, then when their ‘mean free paths’ become much shorter than this ‘correlation length,’ they can be considered effectively decoupled. In this case, any structure in  $E$ -space is not expected to be seen. With higher energy, however, the particles can have longer ‘mean free paths’ and hence if these happen to be comparable to or larger than their ‘correlation length’ the particles will remain coupled to each other and a structure can be expected in this higher energy region.

Finally, the dependence of the transformed sinusoidal structure on the mass-to-charge ratio of the particles has been examined. This dependence was found to be much weaker but similar, as in equation (25), to the effects of the strength of the magnetic turbulence. As a result, particles with somewhat different mass-to-charge ratios (e.g., SEP He<sup>+2</sup>-to- $p$  versus SEP Fe<sup>+14</sup>-to-O<sup>+2</sup>) are expected to show rather similar structures for the same magnetic turbulence whenever these structures can be attributed to particle-particle coupling.

Some recent SEP data<sup>32</sup> appear to have qualitatively similar ‘structures.’ The observed iron-to-oxygen ratio as a function of energy from two separate but similar solar particle events in 2002, for example, seem to follow each other closely up to a certain ‘critical’ energy after which they appear to diverge and separate widely. SEP iron and oxygen ions in these two large solar events are believed to have been accelerated by similar shocks driven by coronal mass ejections. One possible explanation<sup>42</sup> for this high-energy ‘variability’ invokes the changing angle between the magnetic field line and a direction normal to the shock as the shock propagates away from the lower corona and into the interplanetary space.

While this particular explanation for this observed SEP variability may be of limited applicability to the larger SEP database,<sup>5,43</sup> other ‘variabilities’ in SEP characteristics as well (e.g., elemental composition and charge state) still await more quantitative descriptions.



## 6. SUMMARY AND CONCLUSIONS

Motivated in part by modeling requirements in the description of energetic charged particles transport in space plasmas, an idealized two-particle Fokker-Planck model has been developed in order to explore the mathematical implications of particle-particle coupling on the coupled particles' characteristics. Such implications are expected to be manifest in the steady-state solutions of the systems as patterns or Turing structures.

In this two-particle leaky-box model, the particles' motion in momentum or energy space is entirely determined by the magnetic field's regular and stochastic components. Plasma properties are assumed fixed relative to particle properties.

Particle-particle coupling is assumed to be entirely due to a common nonlinear source term, but without deriving or promoting a specific form. The two-particle Fokker-Planck model is then made to correspond to the so-called Thomas model—a coupled, nonlinear, reactive-diffusive system that is known to support Turing structures in its steady-state solutions.

Relying on proxy steady-state solutions of the Thomas system, the dependence of the solutions on both the particles' and the magnetic field attributes is analyzed in some detail.

Particle-particle coupling (via a nonlinear common source term per the Thomas model) is seen to give rise to patterns (Turing structures) in the ratio of the particles' steady-state density functions. The structures are found to depend sensitively on the strength and spectral index of the magnetic turbulence, but only weakly on the coupled particles' mass-to-charge ratios. The spectral index of the turbulence is seen to determine the 'phase' of the structure, while the structure itself appears to become less pronounced and to move to higher energy regions with higher levels of turbulence.

While some distance away from direct correspondence to data, this study using a nonlinearly coupled leaky-box model demonstrates the possibility of describing observed structures in energetic particle characteristics as being due to reaction-diffusion-driven instabilities arising from, in this particular demonstration, particle-particle coupling. Putting this potentiality on firmer grounds—both mathematically and in connection with accepted phenomenology—could have a powerful organizing effect on an already rich but increasingly complex phenomenon.

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