NASA/TM–2010–216857

Time of Closest Approach in Three-Dimensional Airspace

César A. Muñoz and Anthony J. Narkawicz
Langley Research Center, Hampton, Virginia

October 2010

NASA STI Program . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA scientific and technical information (STI) program plays a key part in helping NASA maintain this important role.

The NASA STI Program operates under the auspices of the Agency Chief Information Officer. It collects, organizes, provides for archiving, and disseminates NASA's STI. The NASA STI Program provides access to the NASA Aeronautics and Space Database and its public interface, the NASA Technical Report Server, thus providing one of the largest collection of aeronautical and space science STI in the world. Results are published in both non-NASA channels and by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counterpart of peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services also include creating custom thesauri, building customized databases, and organizing and publishing research results.

For more information about the NASA STI Program, see the following:

- Access the NASA STI program home page at ***http://www.sti.nasa.gov***

- E-mail your question via the Internet to help@sti.nasa.gov

- Fax your question to the NASA STI Help Desk at 443-757-5803

- Phone the NASA STI Help Desk at 443-757-5802

- Write to:
  NASA STI Help Desk
  NASA Center for AeroSpace Information
  7115 Standard Drive
  Hanover, MD 21076–1320

NASA/TM–2010–216857

Time of Closest Approach in Three-Dimensional Airspace

César A. Muñoz and Anthony J. Narkawicz
Langley Research Center, Hampton, Virginia

National Aeronautics and
Space Administration

Langley Research Center
Hampton, Virginia 23681-2199

October 2010

Available from:

NASA Center for AeroSpace Information
7115 Standard Drive
Hanover, MD 21076-1320
443-757-5802

Abstract

In air traffic management, the aircraft separation requirement is defined by a minimum horizontal distance and a minimum vertical distance that the aircraft have to maintain. Since this requirement defines a cylinder around each aircraft rather than a sphere, the three-dimensional Euclidean distance does not provide an appropriate basis for the definition of *time of closest approach*. For instance, conflicting aircraft are not necessarily in loss of separation at the time of closest three-dimensional Euclidean distance. This paper proposes a definition of time of closest approach that characterizes conflicts in a three-dimensional airspace. The proposed time is defined as the time that minimizes a distance metric called cylindrical norm. An algorithm that computes the time of closest approach between two aircraft is provided and the formal verification of its main properties is reported.

1 Introduction

In a three-dimensional airspace, the separation requirement for two aircraft is specified as a minimum horizontal separation, which is typically 5 nautical miles, and a minimum vertical separation, which is typically 1000 feet. This requirement is graphically illustrated by a cylinder, called the *protected zone*, surrounding each aircraft. A *loss of separation* between two aircraft is an overlapping of the aircraft's protected zones. A *conflict* is a predicted loss of separation within a *lookahead time interval*, which is typically a few minutes long.

A natural question in the context of state-based separation assurance systems is, "What is the time of closest approach between two aircraft?" In a 2-dimensional airspace, the answer to this question is clear: the time of closest horizontal approach between two aircraft is the time when the Euclidean horizontal distance between the aircraft is minimal. The time of closest horizontal approach characterizes horizontal conflicts in the sense that two aircraft are in horizontal conflict if and only if they are in horizontal loss of separation at this time.

In a 3-dimensional airspace, the protected zone is a cylinder rather than a sphere. Typically, this cylinder is 30 times wider than high. Thus, the three-dimensional Euclidean distance does not properly reflect whether the aircraft are separated or not. For instance, vertical separation of one thousand feet may be safe, but horizontal separation of one thousand feet is not. Thus, in contrast to the 2-dimensional case, the time that minimizes the 3-dimensional Euclidean distance does not provide an appropriate definition for the time of closest approach between the aircraft.

The problem addressed by this paper is the mathematical and algorithmic definition of a time of closest approach between two aircraft in a 3-dimensional airspace. This time is defined as the time that minimizes a distance metric called *cylindrical norm*. As indicated by its name, the cylindrical norm accommodates the notion of aircraft distance to the fact that the protected zone is a cylinder. The rest of this paper is organized as follows. Section 2 explains the mathematical notation used in this paper. Section 3 reviews the definition of the time of closest horizontal approach. Section 4 proposes a definition of time of closest approach for a 3-dimensional airspace and states its main property, i.e., that the proposed time of closest approach correctly characterizes 3-dimensional conflicts. Section 5 summarizes this work and presents concluding remarks.

2 Mathematical Notation

The mathematical development presented in this paper has been fully formalized in the Prototype Verification System (PVS) [6]. PVS is a mechanical theorem prover that consists of an expressive specification language based on higher-order logic and a proof checker for this logic.[1] For readability, this paper uses standard mathematical notation instead of PVS syntax.

[1]PVS is electronically available from `http://pvs.csl.sri.com`. The development presented in this paper is electronically avaialable from `http://shemesh.larc.nasa.gov/people/cam/ACCoRD`.

The constructions in this paper consider an airspace with two distinguished aircraft, the *ownship* and the *intruder*, that are potentially in conflict. The *state* of an aircraft consists of its current position and velocity information, and it is represented by vectors in a Euclidean geometry. In a 2-dimensional airspace, the separation requirement is defined by minimum horizontal distance D. In a 3-dimensional airspace, the separation requirement is defined by a minimum horizontal distance D and a minimum vertical distance H. The values D and H are considered to be parametric constants.

The symbols $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$ represent the sets of real numbers, the set of 2-dimensional vectors, and the set of 3-dimensional vectors, respectively. Vector variables are written in **boldface** and can be denoted by their components. For example, if $\mathbf{w} \in \mathbb{R}^3$ and $\mathbf{u} \in \mathbb{R}^2$, then $\mathbf{w} = (w_x, w_y, w_z)$ and $\mathbf{u} = (u_x, u_y)$. The expression $\mathbf{w}_{(x,y)}$ denotes the projection of $\mathbf{w}$ in the horizontal plane, i.e.,[2]

$$\mathbf{w}_{(x,y)} \equiv (w_x, w_y).$$

The notation $\|\mathbf{w}\|$ refers to the Euclidean norm of the vector $\mathbf{w}$ and the notation $\mathbf{w} \cdot \mathbf{w}'$ refers to the dot product of the vectors $\mathbf{w}$ and $\mathbf{w}'$. The expression $\mathbf{0}$ represents the zero vector, i.e., the vector whose components are 0.

Aircraft trajectories are represented by a point moving at constant linear speed, i.e., if the current state of an aircraft is given by the position $\mathbf{s}$ and velocity vector $\mathbf{v}$, its predicted position at time t is $\mathbf{s} + t\,\mathbf{v}$. In this paper, the position vector $\mathbf{s}_o$ and the velocity vector $\mathbf{v}_o$ represent the current state of the ownship. Similarly, $\mathbf{s}_i$ and $\mathbf{v}_i$ represent the current position and velocity vectors of the intruder aircraft. As it simplifies the mathematical development, the formalization presented here usually considers a relative view where the intruder is fixed at the origin of the coordinate system. In this paper, the vectors $\mathbf{s}$ and $\mathbf{v}$ will commonly be used to denote the relative position $\mathbf{s}_o - \mathbf{s}_i$ and the relative velocity $\mathbf{v}_o - \mathbf{v}_i$, respectively.

Aircraft predicted trajectories are considered valid for a *lookahead time interval*, which is specified by a lower bound B and an upper bound T. The notation $[B, T]$, where $0 \leq B < T$, represents the interval of real numbers greater than or equal to B and less than or equal to T, i.e.,

$$[B, T] \equiv \{x \in \mathbb{R} \mid B \leq x \leq T\}.$$

The values B and T are parametric constants. The upper bound T is usually a real number. However, the formalism presented here also considers the special case where T has an infinite value, i.e., $T = \infty$. In that case, it is assumed that for all $x \in \mathbb{R}$, $x < \infty$ and $\min(x, \infty) = x$. It is remarked that this abuse of notation is a syntactical convenience and should not be understood as an extension of real arithmetic with infinite values. For instance, the value ∞ is not considered to be in $\mathbb{R}$. Hence, if $T = \infty$, the expression $[B, T]$ simply represents the interval of real numbers greater than or equal to B. Furthermore, particular care is taken in this paper to avoid the use of ∞ in arithmetic expressions other than $\min(x, \infty)$ and relational operations such as $x < \infty$.

[2]The symbol $\equiv$ is used in this paper to introduce mathematical definitions.

Finally, by convention, names of predicates and functions used in the specification of the problem are written in *italics*. Functions that represent algorithms to be implemented in a programming language are written in `typewriter` font.

3 Time of Closest Horizontal Approach

The horizontal separation requirement can be understood as an imaginary circle of diameter D around each aircraft and a horizontal conflict between two aircraft as a future overlapping of these circles. In this paper, an alternative but equivalent view is considered where the intruder is surrounded by a circle, called *protected zone*, of radius D. From this perspective, a horizontal conflict between these two aircraft is equivalent to the existence of a time within a lookahead time interval at which the ownship is in the interior of the intruder's protected zone.

3.1 Horizontal Conflict

Given a lookahead time interval $[B, T]$, where T is possibly infinite, a horizontal conflict between the ownship and the intruder aircraft occurs when there is a time $t \in [B, T]$ such that the horizontal distance between the aircraft is less than D, i.e.,

$$\|(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i)\| < D,$$

where $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i$, and $\mathbf{v}_i$ are all in $\mathbb{R}^2$. Since $(\mathbf{s}_o + t\,\mathbf{v}_o) - (\mathbf{s}_i + t\,\mathbf{v}_i) = (\mathbf{s}_o - \mathbf{s}_i) + t\,(\mathbf{v}_o - \mathbf{v}_i)$, the predicate that characterizes horizontal conflict can be defined in terms of the relative vectors $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$, i.e., the relative position and velocity vectors, respectively, of the ownship with respect to the intruder. The predicate $\mathit{HorizontalConflict?}_{[B,T]}$, parametric on the lookahead time interval $[B, T]$, is formally defined as follows.

$$\mathit{HorizontalConflict?}_{[B,T]}(\mathbf{s}, \mathbf{v}) \equiv \exists\, t \in [B, T] : \|\mathbf{s} + t\,\mathbf{v}\| < D. \tag{1}$$

3.2 Time of Closest Horizontal Approach

This section presents a mathematical derivation of the time of closest horizontal approach between two aircraft for a given lookahead time interval.

Definition 1. *The time of closest horizontal approach between the ownship and the intruder aircraft, for a lookahead time interval $[B, T]$, where T is possibly infinite, is the minimum time τ in the interval $[B, T]$ that satisfies*

$$\forall\, t \in [B, T] : \|\mathbf{s} + t\,\mathbf{v}\| \;\geq\; \|\mathbf{s} + \tau\,\mathbf{v}\|, \tag{2}$$

where $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$.

From the definition above, it is not clear that such time τ exists. The rest of this section provides an algorithm definition of time of closest horizontal approach and the proof that it satisfies Definition 1.

The norm $\|\mathbf{s}+t\,\mathbf{v}\|$ is minimized precisely when its square $\|\mathbf{s}+t\,\mathbf{v}\|^2$ is minimized. This squared norm is equal to $\|\mathbf{v}\|^2\,t^2 + (2\,\mathbf{s}\cdot\mathbf{v})\,t + \|\mathbf{s}\|^2$, which is a quadratic polynomial in the variable t. For any quadratic polynomial of the form $a\,t^2 + b\,t + c$ where $a > 0$, the minimum occurs at the point $t = -\frac{b}{2\,a}$. Thus, if $\mathbf{v} \neq \mathbf{0}$, then the minimum of the norm $\|\mathbf{s}+t\,\mathbf{v}\|$ is attained at time $-\frac{\mathbf{s}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$. If that time is less than B, then every $t \geq B$ satisfies $\|\mathbf{s}+t\,\mathbf{v}\| \geq \|\mathbf{s}+B\,\mathbf{v}\|$. Furthermore, if that time is greater than T, then every time $t \leq T$ satisfies $\|\mathbf{s}+t\,\mathbf{v}\| \geq \|\mathbf{s}+T\,\mathbf{v}\|$. If $\mathbf{v} = \mathbf{0}$, the distance between the aircraft remains constant. In that case, the minimum time that satisfies Formula (6) is B.

This motivates the definition of the function $\mathtt{tcha}_{[B,T]}$ that computes the time of closest horizontal approach for the lookahead time interval $[B,T]$.

$$\mathtt{tcha}_{[B,T]}(\mathbf{s},\mathbf{v}) \equiv \begin{cases} \min\left(T,\, \max\left(B,\, -\frac{\mathbf{s}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\right) & \text{if } \mathbf{v} \neq \mathbf{0}, \\ B & \text{otherwise.} \end{cases} \tag{3}$$

It has been proved in PVS, using basic algebra, that the time $\mathtt{tcha}_{[B,T]}(\mathbf{s},\mathbf{v})$ satisfies Definition 1.

Lemma 1. *For any $t \in [B,T]$, $\|\mathbf{s}+t\,\mathbf{v}\| \geq \|\mathbf{s}+\tau\,\mathbf{v}\|$, where $\tau = \mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v})$.*

The next theorem follows immediately.

Theorem 2. *HorizontalConflict?$_{[B,T]}(\mathbf{s},\mathbf{v})$ holds if and only if $\|\mathbf{s}+\tau\,\mathbf{v}\| < D$, where $\tau = \mathtt{tcha}_{[B,T]}(\mathbf{s},\mathbf{v})$.*

This theorem states that the ownship and intruder aircraft are in horizontal conflict within the lookahead time interval $[B,T]$ if and only if they are expected to be in horizontal loss of separation at the time of closest horizontal approach, which is given by the function $\mathtt{tcha}_{[B,T]}$.

4 Time of Closest Approach in a 3-Dimensional Airspace

In the relative frame of reference, the protected zone surrounding the intruder aircraft is a cylinder of half-height H and radius D. A conflict between the ownship and intruder aircraft occurs when there is a time within the lookahead time interval at which the ownship is in the interior of the intruder's protected zone.

4.1 Conflict

A mathematical definition of conflict is given as follows. The ownship and the intruder aircraft are in *conflict* during the time interval $[B,T]$, where T is possibly infinite, if there exists a time $t \in [B,T]$ when vertical separation is lost, i.e,

$$|(s_{oz} + t\,v_{oz}) - (s_{iz} + t\,v_{iz})| < H,$$

and horizontal separation is lost, i.e.,

$$\|(\mathbf{s}_o + t\,\mathbf{v}_o)_{(x,y)} - (\mathbf{s}_i + t\,\mathbf{v}_i)_{(x,y)}\| < D,$$

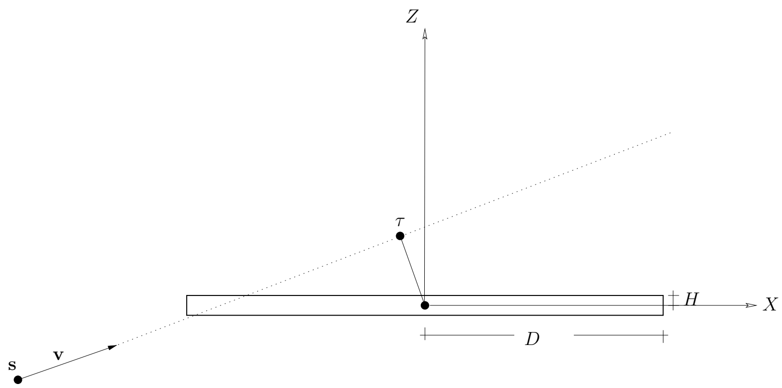

Figure 1. Time (τ) of closest Euclidean approach in a 3-dimensional airspace

where $\mathbf{s}_o, \mathbf{v}_o, \mathbf{s}_i$, and $\mathbf{v}_i$ are all in $\mathbb{R}^3$. As in the case of horizontal conflict in Formula (1), the predicate that characterizes conflict is defined on the relative vectors $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$.

$$\begin{aligned} \textit{Conflict?}(\mathbf{s}, \mathbf{v}) \equiv \exists\, t \in [B, T] : |s_z + t\, v_z| < H \ \ \texttt{and} \\ \|\mathbf{s}_{(x,y)} + t\, \mathbf{v}_{(x,y)}\| < D. \end{aligned} \tag{4}$$

4.2 Euclidean Distance

Identically to the construction in Section 3.2, the time $\min(T, \max(B, -\frac{\mathbf{s}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}))$, when $\mathbf{v} \neq \mathbf{0}$, minimizes the 3-dimensional Euclidean distance between the aircraft during the time interval $[B, T]$, where $\mathbf{s}$ and $\mathbf{v}$ are in $\mathbb{R}^3$. If the protected zone around an aircraft were a sphere instead of a cylinder, this formula would suffice as the definition of the time of closest approach. However, since the protected zone is a cylinder, this definition does not properly characterizes 3-dimensional conflicts. Indeed, it does not satisfy the property that the aircraft are in conflict during the time interval $[B, T]$ if and only if they are in loss of separation at this time.

If the dimensions of D and H were comparable, the minimal 3-dimensional Euclidean distance could be an indicator of the time of closest approach. However, the typical protected zone, i.e. where $D = 5$ nautical miles and $H = 1000$ feet, is 30 times wider than it is high. Figure 1 provides a side view of the protected zone on the X, Z-plane where the axes have the same scale. In Figure 1, assume that $v_y = 0$ and that τ is the time of minimum Euclidean distance for the relative position $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ and relative velocity vector $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$. This figure illustrates that although the aircraft are in conflict, they are not in loss of separation at time τ.

Hence, an appropriate definition of the 3-dimensional time of closest approach between two aircraft, which are surrounded by cylindrical protected zones, cannot be defined on terms of the minimum Euclidean distance between the aircraft.

4.3 Cylindrical Norm

An alternative notion of aircraft distance, which is based on *cylindrical norms*, is proposed in [3]. Given a cylinder of radius D and half-height H, the *cylindrical norm* of a vector $\mathbf{w} \in \mathbb{R}^3$, with respect to D and H, is the quantity

$$\|\mathbf{w}\|_{\text{cyl}} \equiv \max\left(\frac{|w_z|}{H}, \frac{\|\mathbf{w}_{(x,y)}\|}{D}\right). \tag{5}$$

With this norm, $\mathbb{R}^3$ is a metric space in the sense of real analysis [7]. That is, the cylindrical norm satisfies the following properties for any vectors $\mathbf{w}$ and $\mathbf{u}$ in $\mathbb{R}^3$.

- Positivity: $\|\mathbf{w}\|_{\text{cyl}} \geq 0$.
- Nullity: $\|\mathbf{w}\|_{\text{cyl}} = 0$ if and only if $\mathbf{w} = \mathbf{0}$
- Scalability: $\|k\,\mathbf{w}\|_{\text{cyl}} = |k|\,\|\mathbf{w}\|_{\text{cyl}}$ for any $k \in \mathbb{R}$
- Triangle Inequality: $\|\mathbf{w} + \mathbf{u}\|_{\text{cyl}} \leq \|\mathbf{w}\|_{\text{cyl}} + \|\mathbf{u}\|_{\text{cyl}}$

The *cylindrical distance* between the vectors $\mathbf{w}$ and $\mathbf{u}$ is defined as the cylindrical norm $\|\mathbf{w} - \mathbf{u}\|_{\text{cyl}}$.

The following lemma and theorem, which are proven in PVS, follow directly from Equation (4) and Equation (5).

Lemma 3. *The ownship and the intruder aircraft are in loss of separation if and only if* $\|\mathbf{s}_o - \mathbf{s}_i\|_{cyl} < 1$.

Theorem 4. *Let* $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *and* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$ *be in* $\mathbb{R}^3$*, Conflict?*$_{B,T}(\mathbf{s}, \mathbf{v})$ *holds if and only if there exists* $t \in [B, T]$ *such that* $\|\mathbf{s} + t\,\mathbf{v}\|_{cyl} < 1$.

4.4 Time of Closest Approach

Using the cylindrical norm, this section generalizes the construction in Section 3.2 to define the time of closest approach for a three dimensional airspace, where the protected zone is a cylinder.

Definition 2. *The time of closest approach between the ownship and the intruder aircraft, for a lookahead time interval* $[B, T]$*, where* T *is possibly infinite, is the minimum time* τ *in the interval* $[B, T]$ *that satisfies*

$$\forall\, t \in [B, T] : \|\mathbf{s} + t\,\mathbf{v}\|_{cyl} \geq \|\mathbf{s} + \tau\,\mathbf{v}\|_{cyl}, \tag{6}$$

where $\mathbf{s} = \mathbf{s}_o - \mathbf{s}_i$ *and* $\mathbf{v} = \mathbf{v}_o - \mathbf{v}_i$.

As in the case of Definition 1, the fact that there exists a time τ that satisfies Formula (6) is not obvious. The rest of this section provides an algorithmic definition of the time of closest approach and the proof that it satisfies Definition 2.

The time of closest horizontal approach is constructed by finding the minimum time in a time interval that minimizes the 2-dimensional Euclidean norm. This idea can be generalized to define the time of closest approach in a 3-dimensional airspace by finding the minimum time in $[B,T]$ that minimizes the cylindrical norm $\|\mathbf{s}+t\,\mathbf{v}\|_{\mathrm{cyl}}$. It suffices to minimize the square $\|\mathbf{s}+t\,\mathbf{v}\|_{\mathrm{cyl}}^2$ of this norm. This square can be written as the maximum of two quadratic polynomials as follows.

$$\begin{aligned}
\|\mathbf{s}+t\,\mathbf{v}\|_{\mathrm{cyl}}^2 &= \max\left(\frac{(s_z+t\,v_z)^2}{H^2}, \frac{\|\mathbf{s}_{(x,y)}+t\,\mathbf{v}_{(x,y)}\|^2}{D^2}\right) \\
&= \max\left(\frac{1}{H^2}\,(v_z^2\,t^2+(2\,s_z\,v_z)\,t+s_z^2),\right. \\
&\qquad \left.\frac{1}{D^2}\,(\|\mathbf{v}_{(x,y)}\|^2\,t^2+(2\,\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)})\,t+\|\mathbf{s}_{(x,y)}\|^2)\right).
\end{aligned} \tag{7}$$

It can be proved that the minimum of the function $\max(p(t),q(t))$, where t ranges over the interval $[B,T]$ and p and q are two quadratic polynomials of the form $p(t)=a\,t^2+b\,t+c$ and $q(t)=e\,t^2+f\,t+g$, is reached at a point τ that satisfies one of the following conditions.

- $\tau=B$, or
- $\tau=T$ and $T\neq\infty$, or
- τ is equal to either $-\frac{b}{2a}$ or $-\frac{f}{2e}$, or
- τ is such that $p(\tau)=q(\tau)$. If p and q are not equal and $a\neq e$, then there are at most two solutions to the equation $p(t)=q(t)$. If this equation has a solution, then the discriminant of the polynomial $(a-e)\,t^2+(b-f)\,t+(c-g)$ must be nonnegative. This discriminant is given by $(b-f)^2-4\,(a-e)(c-g)$. Alternatively, if $a=e$ and $b\neq f$, then there is exactly one solution to the equation $p(t)=q(t)$, namely $-\frac{c-g}{b-f}$. In both cases, it is easy to see that there are only finitely many possibilities for τ.

Applying this reasoning to the maximum in Equation (7), there are at most six possibilities for the time τ. The first four possible times are B, T, $-\frac{s_z}{v_z}$, and $-\frac{\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)}}{\|\mathbf{v}_{(x,y)}\|^2}$. The other two possibilites are the solutions to the equation

$$\frac{1}{H^2}\,(v_z^2\,t^2+(2\,s_z\,v_z)\,t+s_z^2)=\frac{1}{D^2}\,(\|\mathbf{v}_{(x,y)}\|^2\,t^2+(2\,\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)})\,t+\|\mathbf{s}_{(x,y)}\|^2). \tag{8}$$

Equation (8) can be rewritten as

$$A\,t^2+B\,t+C=0, \tag{9}$$

where

$$A=\frac{v_z^2}{H^2}-\frac{\|\mathbf{v}_{(x,y)}\|^2}{D^2}, \tag{10}$$

$$B=\frac{2\,s_z\,v_z}{H^2}-\frac{2\,\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)}}{D^2}, \tag{11}$$

$$C=\frac{s_z^2}{H^2}-\frac{\|\mathbf{s}_{(x,y)}\|^2}{D^2}. \tag{12}$$

Equation (9) has a solutions for t precisely when the discriminant $B^2 - 4\,A\,C$ is nonnegative. In this case, there are two possible solutions, which are equal if this discriminant is zero, and these solutions are given by the quadratic formula. This motivates the following definition of the function $\mathtt{tca}_{[B,T]}$.

$$
\begin{array}{l}
\mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v}) \equiv \\
\quad \mathtt{let} \\
\qquad A = \dfrac{v_z^2}{H^2} - \dfrac{\|\mathbf{v}_{(x,y)}\|^2}{D^2}, \\
\qquad B = \dfrac{2\,s_z\,v_z}{H^2} - \dfrac{2\,\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)}}{D^2}, \\
\qquad C = \dfrac{s_z^2}{H^2} - \dfrac{\|\mathbf{s}_{(x,y)}\|^2}{D^2}\ \mathtt{in} \\
\quad \mathcal{T} := \emptyset; \\
\quad \mathtt{if}\ \mathbf{v}_{(x,y)} \neq \mathbf{0}\ \mathtt{then} \\
\qquad \mathcal{T} := \mathcal{T} \cup \left\{ -\dfrac{\mathbf{s}_{(x,y)}\cdot\mathbf{v}_{(x,y)}}{\|\mathbf{v}_{(x,y)}\|^2} \right\}; \\
\quad \mathtt{endif}; \\
\quad \mathtt{if}\ v_z \neq 0\ \mathtt{then} \\
\qquad \mathcal{T} := \mathcal{T} \cup \left\{ -\dfrac{s_z}{v_z} \right\}; \\
\quad \mathtt{endif}; \\
\quad \mathtt{if}\ A \neq 0\ \ \mathtt{and}\ \ B^2 - 4\,A\,C \geq 0\ \mathtt{then} \\
\qquad \mathtt{for}\ \iota \in \{1,-1\}\ \mathtt{do} \\
\qquad\quad \mathcal{T} := \mathcal{T} \cup \left\{ \dfrac{-B + \iota\,\sqrt{B^2 - 4\,A\,C}}{2\,A} \right\}; \\
\qquad \mathtt{endfor}; \\
\quad \mathtt{elsif}\ A = 0\ \ \mathtt{and}\ \ B \neq 0\ \mathtt{then} \\
\qquad\quad \mathcal{T} := \mathcal{T} \cup \left\{ -\dfrac{C}{B} \right\}; \\
\quad \mathtt{endif}; \\
\quad \tau := B; \\
\quad \mathtt{for}\ t \in \mathcal{T}\ \mathtt{do} \\
\qquad \mathtt{if}\ t > B\ \ \mathtt{and}\ \ \|\mathbf{s} + t\,\mathbf{v}\|_{\mathrm{cyl}} < \|\mathbf{s} + \tau\,\mathbf{v}\|_{\mathrm{cyl}}\ \mathtt{then} \\
\qquad\quad \tau := t; \\
\qquad \mathtt{endif}; \\
\quad \mathtt{endfor}; \\
\quad \mathtt{return}\,\min(\tau, T);
\end{array}
\tag{13}
$$

It has been proved in PVS that the time $\mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v})$ satisfies Definition 2.

Lemma 5. *For any* $t \in [B,T]$, $\|\mathbf{s} + t\,\mathbf{v}\|_{cyl} \geq \|\mathbf{s} + \tau\,\mathbf{v}\|_{cyl}$, *where* $\tau = \mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v})$.

The proof of Lemma 5 uses the fact that the quantity $\|\mathbf{s} + t\,\mathbf{v}\|_{\mathrm{cyl}}$ is a convex function of t, which can be proved using basic algebra. From that lemma, the next theorem follows immediatly.

Theorem 6. *Conflict?*$(\mathbf{s},\mathbf{v})$ *holds if and only if* $|s_z + \tau\,v_z| < H$ *and* $\|\mathbf{s}_{(x,y)} + \tau\,\mathbf{v}_{(x,y)}\| < D$, *where* $\tau = \mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v})$.

Theorem 6 states that $\mathtt{tca}_{[B,T]}$ properly characterizes 3-dimensional conflicts, i.e., the ownship and the intruder are in conflict during the time interval $[B,T]$ if and only if they are expected to be in loss of separation at time $\mathtt{tca}_{[B,T]}(\mathbf{s},\mathbf{v})$.

5 Conclusion

Mathematical and algorithmic definitions of time of closest approach have been presented for two and three-dimensional airspace geometries. Although the definition of time of closest horizontal approach is well-known (see for example [4]), the fact that this definition characterizes horizontal conflicts appears to be a novel approach to this subject. The major contribution of this paper is the definition of time of closest approach for a three-dimensional airspace and the formal proof of its properties. The definition of this time is based on the notion of cylindrical distance, originally proposed in [3].

Since the typical protected zone is very flat, researchers in state-based separation assurance systems usually define the time of closest approach in the horizontal plane. This is the approach followed for example by [4], where a 3-dimensional conflict detection and resolution algorithm is proposed. In the case of conflicting aircraft, special care has to be taken when the time of closest horizontal approach does not lie in the interval of conflict.

The cylindrical distance at time of closest separation can be used as a measure of threat severity in a 3-dimensional airspace. Indeed, this non-dimensional value is less than 1 in the case of a conflict. The closer this value is to 1, the closer the relative trajectory is to the envelope of the protected zone. A similar non-dimensional value ε is proposed in [2] and used in [1] as a measure of conflict risk. This value ε represents the ellipse distance of an ellipsoid enclosed within the cylindrical separation zone.

Neither the time of closest horizontal approach nor the value ε characterize three-dimensional conflicts as $\mathtt{tca}_{[B,T]}$ does. It is believed that the proposed time of closest separation allows for simpler separation assurance algorithms and yields a notion of distance in a 3-dimensional airspace that is more appropriate for a cylindrical protected zone. Indeed, this notion is at the base of the formally verified conflict prevention algorithm presented in [5].

References

1. Richard Barhydt, Todd M. Eischeid, Michael T. Palmer, and David J. Wing. Regaining lost separation in a piloted simulation of autonomous aircraft operations.

In *Proceedings of the 5th USA/Europe Air Traffic Management R&D Seminar*, Budapest, Hungary, June 2003.

2. Wim den Braven. Analysis of aircraft/air traffic control system performance. In *Proceedings of the AIAA Guidance, Navigation and Control Conference*, number AIAA-95-3363, pages 1721–1733, Baltimore, Maryland, USA, August 1995.

3. Gilles Dowek and César Muñoz. Conflict detection and resolution for 1,2,...N aircraft. In *Proceedings of the 6th AIAA Aviation Technology, Integration and Operations Conference (ATIO)*, number ATIO-2007-7737, Belfast, Northern Ireland, September 2007.

4. Jacco Hoekstra. Designing for safety: The free flight air traffic management concept. Technical Report 90-806343-2-8, Technische Universiteir Delft, November 2001.

5. Anthony Narkawicz, César Muñoz, and Gilles Dowek. Formal verification of air traffic prevention bands algorithms. Technical Memorandum NASA/TM-2010-216706, NASA, Langley Research Center, Hampton VA 23681-2199, USA, June 2010.

6. Sam Owre, John Rushby, and Natarajan Shankar. PVS: A prototype verification system. In Deepak Kapur, editor, *Proceeding of the 11th International Conference on Automated Deductioncade*, volume 607 of *Lecture Notes in Artificial Intelligence*, pages 748–752. Springer, June 1992.

7. Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, third edition, 1976.

REPORT DOCUMENTATION PAGE	*Form Approved OMB No. 0704–0188*

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
01-10-2010	Technical Memorandum	

4. TITLE AND SUBTITLE	**5a. CONTRACT NUMBER**
Time of Closest Approach in Three-Dimensional Airspace	
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	**5d. PROJECT NUMBER**
César A. Muñoz and Anthony J. Narkawicz	
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
	411931-02-51-07-01

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)	**8. PERFORMING ORGANIZATION REPORT NUMBER**
NASA Langley Research Center Hampton, Virginia 23681-2199	L–19933
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	**10. SPONSOR/MONITOR'S ACRONYM(S)**
National Aeronautics and Space Administration Washington, DC 20546-0001	NASA
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)
	NASA/TM–2010–216857

12. DISTRIBUTION/AVAILABILITY STATEMENT

Unclassified-Unlimited
Subject Category 03
Availability: NASA CASI (443) 757-5802

13. SUPPLEMENTARY NOTES

An electronic version can be found at http://ntrs.nasa.gov.

14. ABSTRACT

In air traffic management, the aircraft separation requirement is defined by a minimum horizontal distance and a minimum vertical distance that the aircraft have to maintain. Since this requirement defines a cylinder around each aircraft rather than a sphere, the three-dimensional Euclidean distance does not provide an appropriate basis for the definition of *time of closest approach*. For instance, conflicting aircraft are not necessarily in loss of separation at the time of closest three-dimensional Euclidean distance. This paper proposes a definition of time of closest approach that characterizes conflicts in a three-dimensional airspace. The proposed time is defined as the time that minimizes a distance metric called cylindrical norm. An algorithm that computes the time of closest approach between two aircraft is provided and the formal verification of its main properties is reported.

15. SUBJECT TERMS

Air traffic, conflict detection, time of predicted closest approach, distance at time of predicted closest approach, 3-dimensional airspace

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**			STI Help Desk (email: help@sti.nasa.gov)
					19b. TELEPHONE NUMBER ***(Include area code)***
U	U	U	UU	19	(443) 757-5802