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A Model of the THUNDER Actuator

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1   Introduction

A THUNDER actuator is a composite of three thin layers, a metal base, a piezoelectric wafer and a metal top cover, bonded together under pressure and at high temperature with the LaRC SI polyimid adhesive. When a voltage is applied between the metal layers across the PZT the actuator will bend and can generate a force. This document develops and describes an analytical model the transduction properties of THUNDER actuators. The model development is divided into three sections. First, a static model is described that relates internal stresses and strains and external displacements to the thermal pre-stress and applied voltage. Second, a dynamic energy based model is described that allows calculation of the resonance frequencies, developed force and electrical input impedance. Finally, a fully coupled electro-mechanical transducer model is described.

The model development proceeds by assuming that both the thermal pre-stress and the piezoelectric actuation cause the actuator to deform in a pure bend in a single plane. It is useful to think of this as a two step process, the actuator is held flat, differential stresses induce a bending moment, the actuator is released and it bends.

The thermal pre-stress is caused by the different amounts that the constituent layers shrink due to their different coefficients of thermal expansion. The adhesive between layers sets at a high temperature and as the actuator cools, the metal layers shrink more than the PZT. The PZT layer is put into compression while the metal layers are in tension.

The piezoelectric actuation has a similar effect. An applied voltage causes the PZT layer to strain, which in turn strains the two metal layers. If the PZT layer expands it will put the metal layers into tension and PZT layer into compression.

In both cases, if shear force effects are neglected, the actuator assembly will experience a uniform in-plane strain. As the materials each have a different elastic modulus, different stresses will develop in each layer and these stresses will induce a bending moment. When the actuator is released from its flat configuration, the differential stresses are relieved as the actuator bends.

2   Static Model

The first step in making a model of the THUNDER actuator is to find expressions for the relationship between the the force and displacement developed when a DC voltage is applied.

The actuator has three structural layers, a metal base, a PZT middle layer and a metal top cover. Let these three layers be labeled
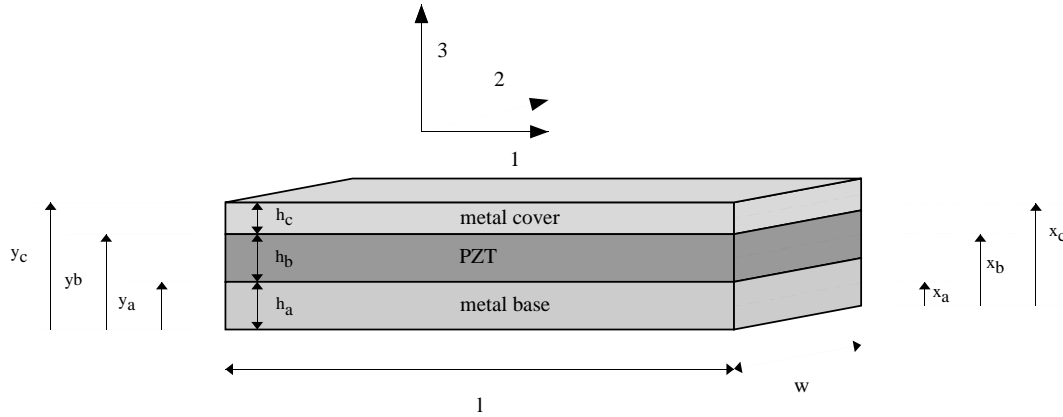
Figure 1: The three layers of the actuator with the coordinates used in the model

c   top cover
b   PZT
a   metal base

A schematic of the layers of the actuator is shown in Figure 1 where the co-ordinate system is chosen to be compatible with the IEEE Standard on Piezoelectricity[1]. The actuator is initially flat and oriented parallel to the 12 plane with the 3 axis perpendicular to the plane of the layers. The 3 axis is also assumed to be the direction of polarization of the PZT. The bottom surface of the metal base is aligned with $y = 0$. The length of the actuator is l, the width is w, the coordinate of the beam in the 1 direction is z and in the 3 direction is y.

Let the thickness of the three layers be h_a, h_b and h_c. Introduce the auxiliary variables y_a, y_b and y_c which are the coordinates of the layer boundaries from the lower surface of the metal base, and x_a, x_b and x_c which are the coordinates of the centerlines of each layer.

$$
\begin{aligned}
y_a &= h_a \\
y_b &= h_a + h_b \\
y_c &= h_a + h_b + h_c \\
x_a &= y_a/2 \\
x_b &= (y_a + y_b)/2 \\
x_c &= (y_a + y_b + y_c)/2
\end{aligned}
$$

2.1   Simple Bending

To fully describe the elastic deformation of each layer we would need to specify the three stresses T_1, T_2 and T_3 and (using the indicial notation from the IEEE standard [1]) the three shear stresses T_4, T_5 and T_6 at every location in body of the actuator.

However, if we make the assumption that the layers are thin and that they are unable to support stresses in the 3 direction, then $T_3 = 0$. In the simple bending model, we also assume that that actuator only bends in the 13 plane and that $T_2 = 0$. In reality, both the thermal strain and the piezoelectric actuation will cause both bending and stress in the 23 plane. We also neglect the effect of shear stresses, that is assume that plane cross-sections of the actuator remain plane under bending, a commonly used approximation that holds for moderate shear loads [3]. The one dimensional pure bending model gives simple formulas that aid an engineering understanding of the mechanisms and parameters of the actuator. An extension that includes the out-of-plane stresses is possible if a little involved.

2.2   Assumed Strain Model

The strain in all layers due to bending, thermal pre-stress and PZT actuation will be a combination of a uniform strain and a bending strain. We will assume that the actuator bends with a radius of curvature R or equivalently with a curvature $c = R^{-1}$ and that the deformation is pure bending, i.e. that plane cross-sections remain plane. (Positive curvatures are assumed, by convention, to be concave upward and so we expect to have negative curvature under the thermal pre-stress.) The beam will then have a neutral plane, a distance y_0 from the lowest surface, at which the strain is just the uniform strain S_0. The strain in each layer is then a linear function of its distance from the neutral plane.

$$S_1(y) = S_0 - (y - y_0)/R \tag{1}$$

The actuator will bend until the internal stresses are in equilibrium. For pure bending, the equations of elasticity are reduced to a single stress

$$T_1 = Y S_1$$

where Y is the Young's modulus of the material.

Thermal strains are caused by the different coefficients of thermal expansion of the three layers. Let these coefficients be α_a, α_b and α_c. If the temperature at which the adhesive layers in the actuator set is θ degrees above ambient, then the thermal strain induced in each layer is

$$S_1 = -\alpha\theta$$

If a potential v is applied between the top metal layer and the metal base, then an electrical field is applied across the PZT. In the simple model, this field is assumed to be uniform and is given by

$$E_3 = -v/h_b$$

If the PZT has been polarized in the 3 direction then an electrical field applied in the 3 direction induces a strain in the 1 direction given by

$$S_1 = d_{31} E_3 = -d_{31} v / h_b$$

where d_{31} is a piezoelectric coefficient of the material.

We now have three sources of strain, bending, thermal and piezoelectric. The resulting strains in each layer must combine to give the induced strain of Equation 1,

$$S_0 - \frac{y - y_0}{R} = \begin{cases} T_a/Y_a - \alpha_a \theta & 0 \leq y < y_a \\ T_b/Y_b - \alpha_b \theta + d_{31} E_3 & y_a \leq y < y_b \\ T_c/Y_c - \alpha_c \theta & y_b \leq y < y_c \end{cases} \tag{2}$$

These equations can be rearranged to give the stresses induced by the thermal and piezoelectric strains for a given radius of curvature R and neutral axis location y_0,

$$\begin{aligned} T_a &= Y_a \left(\alpha_a \theta + S_0 - \frac{y - y_0}{R} \right) \\ T_b &= Y_b \left(\alpha_b \theta - d_{31} E_3 + S_0 - \frac{y - y_0}{R} \right) \\ T_c &= Y_c \left(\alpha_c \theta + S_0 - \frac{y - y_0}{R} \right) \end{aligned} \tag{3}$$

or if we let $\hat{y}_0$ be the location of the plane where $S_1(y) = 0$,

$$\hat{y}_0 = y_0 + R S_0 \tag{4}$$

then

$$\begin{aligned} T_a &= Y_a \left(\alpha_a \theta - \frac{y - \hat{y}_0}{R} \right) \\ T_b &= Y_b \left(\alpha_b \theta - d_{31} E_3 - \frac{y - \hat{y}_0}{R} \right) \\ T_c &= Y_c \left(\alpha_c \theta - \frac{y - \hat{y}_0}{R} \right) \end{aligned} \tag{5}$$

Note that due the uniform strain induced by the thermal and PZT effects, the plane of zero strain does not occur at the bending neutral axis y_0.

To solve these equations we will need to find the two unknowns, R and $\hat{y}_0$ and to this end, we will use two equilibrium equations for the net force and net moment at a given section of the actuator.

We consider first the static case, with no inertial forces. At any cross-section of the actuator, the forces and moments must be in equilibrium. Thus

$$F = \int_0^{y_a} T_a \, dy + \int_{y_a}^{y_b} T_b \, dy + \int_{y_b}^{y_c} T_c \, dy = 0 \tag{6}$$

$$M = -\int_0^{y_a} T_a y\,dy - \int_{y_a}^{y_b} T_b y\,dy - \int_{y_b}^{y_c} T_c y\,dy = 0 \tag{7}$$

where F and M are the force and bending moment per unit width of the beam. We proceed by substituting the expressions for stress and performing these integrations to get two equations with the unknowns R and $\hat{y}_0$. As the algebra here can get rather messy it is wise to proceed in steps, starting with the case of no thermal or piezoelectric actuation and an applied moment M_b.

2.3   Applied Bending Moment

If a bending moment is M_b applied at both ends of the beam with $\theta = E_3 = 0$, (S_0 is also zero by definition for pure bending) then force equilibrium gives

$$
\begin{aligned}
-FR &= \int_0^{y_a} Y_a(y - y_0)dy + \int_{y_a}^{y_b} Y_b(y - y_0)dy + \int_{y_b}^{y_c} Y_c(y - y_0)dy \\
&= Y_a\left[\frac{y^2}{2} - yy_0\right]_0^{y_a} + Y_b\left[\frac{y^2}{2} - yy_0\right]_{y_a}^{y_b} \\
&\quad + Y_c\left[\frac{y^2}{2} - yy_0\right]_{y_b}^{y_c} = 0
\end{aligned}
\tag{8}
$$

and taking a representative term

$$
\begin{aligned}
\left[\frac{y^2}{2} - yy_0\right]_{y_a}^{y_b} &= \frac{y_b^2 - y_a^2}{2} - (y_b - y_a)y_0 \\
&= h_b(x_b - y_0)
\end{aligned}
\tag{9}
$$

therefore

$$
\begin{aligned}
FR &= Y_a h_a x_a + Y_b h_b x_b + Y_c h_c x_c \\
&\quad -(Y_a h_a + Y_b h_b + Y_c h_c)y_0 = 0
\end{aligned}
\tag{10}
$$

and

$$y_0 = \frac{Y_a h_a x_a + Y_b h_b x_b + Y_c h_c x_c}{Y_a h_a + Y_b h_b + Y_c h_c} \tag{11}$$

Expressions like this will be more compact if we introduce the indicial notation, where a repeated index i in an expression indicates that the expression is summed over all values of i. Thus

$$Y_i h_i = Y_a h_a + Y_b h_b + Y_c h_c \tag{12}$$

and

$$Y_i h_i x_i = Y_a h_a x_a + Y_b h_b x_b + Y_c h_c x_c \tag{13}$$

We can then write an expression for the neutral axis under bending as

$$y_0 = \frac{Y_i h_i x_i}{Y_i h_i} \tag{14}$$

Now, if the applied bending moment is M_b and moments are taken about $y = 0$,

$$\begin{aligned} M_b R &= \int_0^{y_a} Y_a(y - y_0) y \, dy + \cdots \\ &= Y_a \left[\frac{y^3}{3} - \frac{y^2 y_0}{2} \right]_0^{y_a} + \cdots \end{aligned} \tag{15}$$

Again, taking a representative term

$$\begin{aligned} \left[\frac{y^3}{3} - \frac{y^2 y_0}{2} \right]_{y_a}^{y_b} &= \frac{y_b^3 - y_a^3}{3} - \frac{y_b^2 - y_a^2}{2} y_0 \\ &= \frac{h_b^3}{12} + h_b x_b^2 - h_b x_b y_0 \end{aligned} \tag{16}$$

it can be shown that

$$M_b R = \frac{Y_i h_i^3}{12} + Y_i h_i x_i^2 - (Y_i h_i x_i) y_0 \tag{17}$$

The right hand side of this equation is the bending stiffness B of the composite beam (B is the bending stiffness per unit width as M_b is the bending moment per unit width.) Let

$$\begin{aligned} B &= \frac{Y_i h_i^3}{12} + Y_i h_i x_i^2 - (Y_i h_i x_i) y_0 \\ &= \frac{Y_i h_i^3}{12} + Y_i h_i x_i^2 - \frac{(Y_i h_i x_i)^2}{Y_i h_i} \end{aligned} \tag{18}$$

so that

$$R = \frac{B}{M_b}$$

More manipulation will clarify the moment equilibrium equation if we introduce the auxiliary variables $w_i = x_i - y_0$ when,

$$B = \frac{Y_i h_i^3}{12} + Y_i h_i w_i^2 \tag{19}$$

2.4   Thermal Pre-Stress

Now consider the case of a THUNDER actuator including the thermal pre-stress but with no applied voltage. The equations for the three layer stresses reduce to

$$
\begin{aligned}
T_a &= Y_a\left(\alpha_a\theta - \frac{y-\hat{y}_0}{R}\right) \\
T_b &= Y_b\left(\alpha_b\theta - \frac{y-\hat{y}_0}{R}\right) \\
T_c &= Y_c\left(\alpha_c\theta - \frac{y-\hat{y}_0}{R}\right)
\end{aligned}
\tag{20}
$$

We now need to substitute these equations into the force and moment equilibrium equations to find R and $\hat{y}_0$ The force equilibrium equation is

$$
\begin{aligned}
0 &= Y_a\left[y(\alpha_a R\theta + \hat{y}_0) - \frac{y^2}{2}\right]_0^{y_a} + \cdots \\
&= Y_i h_i \alpha_i R\theta + Y_i h_i \hat{y}_0 - Y_i h_i x_i
\end{aligned}
\tag{21}
$$

If this expression is solved for $\hat{y}_0$ we have

$$
\hat{y}_0 = \frac{Y_i h_i x_i - Y_i h_i \alpha_i R\theta}{Y_i h_i} = y_0 - \frac{Y_i h_i \alpha_i}{Y_i h_i}R\theta
\tag{22}
$$

To find the static strain induced by the thermal pre-stress, we can apply a bending moment to the beam to counteract the thermal bending moment and flatten the actuator. The component of the force equilibrium equation due to bending forces and the neutral axis then is zero and comparing Equation 22 with Equation 4 we have

$$
S_{0,\theta} = -\frac{Y_i h_i \alpha_i}{Y_i h_i}\theta
\tag{23}
$$

The negative stress implies that the thermal stress produces a uniform contraction of the actuator as expected.

The moment equilibrium equation is now

$$
\begin{aligned}
0 &= Y_a\left[\frac{y^2}{2}(\alpha_a R\theta + \hat{y}_0) - \frac{y^3}{3}\right]_0^{y_a} + \cdots \\
&= Y_i h_i x_i \alpha_i R\theta + Y_i h_i x_i \hat{y}_0 - \frac{Y_i h_i^3}{12} - Y_i h_i x_i^2
\end{aligned}
\tag{24}
$$

Substitution of $\hat{y}_0$ from Equation 22 gives

$$
\left(Y_i h_i x_i \alpha_i - \frac{Y_i h_i x_i . Y_i h_i \alpha_i}{Y_i h_i}\right)R\theta = \frac{Y_i h_i^3}{12} + Y_i h_i x_i^2 - \frac{(Y_i h_i x_i)^2}{Y_i h_i}
\tag{25}
$$

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or

$$M_\theta = \frac{B}{R} = (Y_i h_i x_i \alpha_i - \frac{Y_i h_i x_i . Y_i h_i \alpha_i}{Y_i h_i})\theta = Y_i h_i w_i \alpha_i \theta \tag{26}$$

Note that the bending stiffness B is independent of the effect of the thermal pre-stress. The same expression for M_b can be found by solving for the bending moment required to keep the actuator flat. The moment equation for zero curvature is

$$
\begin{aligned}
M_\theta &= Y_a \left[\frac{y^2}{2}(\alpha_a \theta + S_0) \right]_0^{y_a} + \cdots \\
&= Y_i h_i x_i \alpha_i \theta + Y_i h_i x_i S_0
\end{aligned}
\tag{27}
$$

and when the static thermal strain is substituted we have the same result as before.

2.5   Piezoelectric Actuation

The effect of a voltage applied to the PZT layer is to strain that layer which results in a compressive stress in the PZT and tensile stresses in the metal layers. By a similar reasoning to that used in the previous section, the bending stiffness of the actuator will not change under this load. We can calculate the static strain and the effective bending moment of the piezoelectric effect directly by assuming that an applied moment keeps the actuator flat and by solving for the required moment. The stresses in a flat actuator with a piezoelectric actuation are then

$$
\begin{aligned}
T_a &= Y_a S_0 \\
T_b &= Y_b(S_0 - d_{31} E_3) \\
T_c &= Y_c S_0
\end{aligned}
\tag{28}
$$

the force equilibrium equation is

$$
\begin{aligned}
0 &= Y_a \left[S_0 y \right]_0^{y_a} + Y_b \left[(S_0 - d_{31} E_3) y \right]_{y_a}^{y_b} + Y_c \left[S_0 y \right]_{y_b}^{y_c} \\
&= Y_i h_i S_0 - Y_b h_b d_{31} E_3
\end{aligned}
\tag{29}
$$

and therefore the uniform piezoelectric strain is

$$S_{0,E} = \frac{Y_b h_b d_{31} E_3}{Y_i h_i} \tag{30}$$

and the piezoelectric bending moment is

$$M_p = Y_a \left[\frac{y^2}{2} S_0 \right]_0^{y_a} + Y_b \left[\frac{y^2}{2}(S_0 - d_{31} E_3) \right]_{y_a}^{y_b} + Y_c \left[\frac{y^2}{2}(S_0) \right]_{y_b}^{y_c}$$
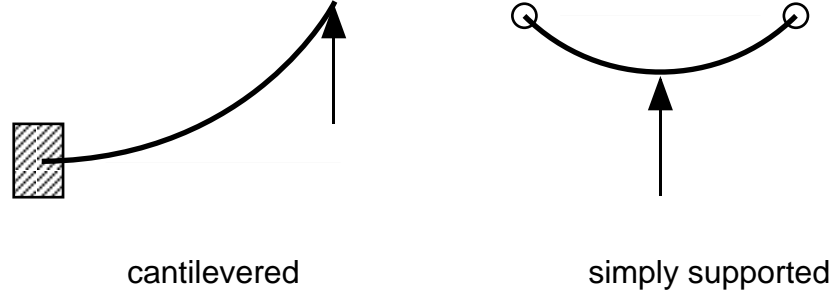
cantilevered   simply supported

Figure 2: Cantilevered and Simply Support Actuator Configurations

$$
\begin{aligned}
&= Y_i h_i x_i S_0 - Y_b h_b x_b d_{31} E_3 \\
&= (\frac{Y_i h_i x_i}{Y_i h_i} - x_b) Y_b h_b d_{31} E_3 \\
&= (y_0 - x_b) Y_b h_b d_{31} E_3 \\
&= -Y_b h_b w_b d_{31} E_3
\end{aligned}
\tag{31}
$$

2.6   Displacement

There are many possible ways of mounting and loading a THUNDER actuator, each of which will result in a slightly different force/displacement characteristic. For the purposes of modeling, we assume that the actuator is mounted in one of two ways, either simply supported when we will be concerned with displacements at the center of the actuator or cantilevered when we will be concerned about displacements and forces at the free end. Figure 2 illustrates these cases. An advantage of considering these cases is that the curvature is uniform along the length of the actuator under the induced moments of thermal and piezoelectric actuation. Other boundary conditions, such as clamped ends, introduce reaction moments that make the curvature a function of location. In addition, these reaction moments will result in reduced displacements of the actuator and so are undesirable in practice.

The displacement of the actuator is a function of the induced curvature. For small displacements we shall assume that this function is linear. THUNDER actuators can exhibit large displacements compared to their length and it may be necessary to use the nonlinear function for accurate predictions. However, in most cases the linear function will give sufficient accuracy.

The curvature c is given in terms of the actuator displacement u and location z

by

$$c = \frac{1}{R} = \frac{d^2 u / dz^2}{\left[1 + (du/dz)^2\right]^{3/2}} \tag{32}$$

and for small values the linear approximation is

$$c \approx \frac{d^2 u}{dz^2} \tag{33}$$

The displacement of the cantilevered actuator can then be found as a function of z

$$u(z) = \frac{z^2}{2R} \tag{34}$$

and the displacement at the free end of an actuator of length l is

$$u_{CF} = \frac{l^2}{2R}$$

For the simply supported actuator

$$u(z) = \frac{z^2 - zl}{2R}$$

and so the displacement at the center is

$$u_{SS} = -\frac{l^2}{8R}$$

In order to find the blocked force developed by the actuator, we could apply the force necessary to return the actuator to a neutral displacement. Alternatively, we can use the principle of virtual work (in the static case) or Hamilton's principle (in the dynamic case) to solve for both the force and the electrical parameters, charge and current. These models are developed in the next section.

2.7 Accounting for the Adhesive Layers

In the development so far we have neglected the effect of the adhesive layers between the metal base and cover layers and the PZT layer. In thin THUNDER actuators these layers can make up a substantial component of the total thickness. Although the adhesive layer is thin and relatively compliant and so does not support significant in-plane stress, it does separate the active layers, moving them further from the bending neutral axis and so can affect the bending stiffness and induced moments. Fortunately, these layers are simple to include in the model. Expressions such as Equation 19 for the bending stiffness are evaluated for the five layers, metal base, adhesive, PZT adhesive and metal cover rather than just the three original layers. This and other expressions can be used assuming summation over the appropriate indices for the five layers.

3 Dynamic and Energy Based Modeling

The next step in the model development requires the introduction of dynamics. The aim is to find expressions for the resonance frequency, the developed force and the electrical input impedance. The most useful approach is to use energy based modeling using the principle of virtual work or Hamilton's principle. This approach is also more convenient when including effect of more complicated actuator geometries as it leads to a quicker formulation of the appropriate relationships.

3.1 Energy and the Hamiltonian

The THUNDER actuator is a coupled electro-mechanical system. As magnetic effects can be ignored, Hamilton's principle applied to the actuator can be written [2]

$$\int_{t1}^{t2} \left[\delta(T - U + W) + f\delta u - q\delta\phi \right] dt = 0 \tag{35}$$

where U is the mechanical potential energy stored in the actuator, T is the stored kinetic energy and W is the stored electrical energy. f is the force applied at a location where the displacement is u and q is the charge applied at a location where the electrical potential is ϕ. The stored energies are functions of S the full vector of strains in a given layer, T the full vector of stresses, E the full vector of electrical field strengths, D the full vector of electrical displacements (charge/area) and u̇ the vector of local mechanical velocities.

$$T = \frac{1}{2} \int_V \rho \dot{\mathbf{u}}' \dot{\mathbf{u}} \, dV \tag{36}$$

$$U = \frac{1}{2} \int_V \mathbf{S}'\mathbf{T} \, dV \tag{37}$$

$$W = \frac{1}{2} \int_V \mathbf{E}'\mathbf{D} \, dV \tag{38}$$

The prime denotes a transpose and the integrals are taken over the volume of the actuator.

3.2 Stress and Strain

For a general piezoelectric material the electrical displacement and the mechanical stress can be written as linear functions of the applied electrical field and mechanical strain [1]

$$\mathbf{T} = \mathbf{c}^E\mathbf{S} - \mathbf{e}'\mathbf{E} \tag{39}$$

$$\mathbf{D} = \mathbf{e}\mathbf{S} + \epsilon^S\mathbf{E} \tag{40}$$

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The parameters in this equation are the matrix of material compliances at a fixed field c^E, the matrix of clamped dielectric constants ϵ^S and a matrix of piezoelectric constants e.

For the simple bending model considered so far these equations reduce to functions of the stress and strain in the 1 direction and the electrical field and displacement in the 3 direction.

$$T_1 = Y S_1 - Y d_{31} E_3 \tag{41}$$

$$D_3 = \epsilon_{33}^S E_3 + Y d_{31} S_1 \tag{42}$$

The first of these equations is familiar from Equation 29.

3.3 Generalized Variables

The analysis proceeds by finding expressions for the external variables displacement, force, charge and voltage and the internal variables S_1,T_1, E_3 and D_3 in terms of a set for generalized variables that meet the requirements of the boundary conditions of a given actuator. If, as is common in practice, we are concerned about the low frequency response of the actuator below its first bending resonance frequency, then a single displacement variable is sufficient. For this case, and with cantilevered or simply supported boundaries, a suitable choice of displacement variable is the curvature $c = R^{-1}$. The voltage variable is simply the voltage applied to the terminals v. We can then write the various internal and external variables in terms of c and v. The two independent internal variables are S_1 and E_3

$$S_1 = c(y - \hat{y}_0) = c y' \tag{43}$$

$$E_3 = \begin{cases} -v/h_b & \text{in the PZT layer} \\ 0 & \text{elsewhere} \end{cases} \tag{44}$$

The independent external variables will depend on the choice of boundary conditions. Let $u(z)$ be the displacement of the actuator as a function of location. For the cantilevered actuator

$$u(z) = c\, \Phi_{CF}(z) = c z^2/2$$

and for the simply supported actuator

$$u(z) = c\, \Phi_{SS}(z) = c(z^2 - zl)/2$$

3.4 Transducer Equations

The transducer equations are a pair of coupled dynamic equations that relate the the two electrical variables voltage v and the charge q to the mechanical variables

displacement u and force f. They can be obtained by variational methods by substitution of the equations given above into the expressions for energy and taking variations by Equation 35. The method, described in more detail in a paper by Hagood et al. [2], results in the equations

$$
\begin{aligned}
M\ddot{c} + Kc - \Theta v &= \Phi(z_0)f \\
\Theta c + Cv &= q
\end{aligned}
\tag{45}
$$

where M is a mass, K is a stiffness, C is a capacitance and Θ is a transduction constant each of which is found by integrating energy expressions over the volume of the actuator. In contrast to the previous section the following expressions now include the width, w, and the area, $A = lw$, of the actuator.

K is simply a reformulation of the bending stiffness of Equation 19.

$$
K = A \int_y Y y'^2 dy = AB
\tag{46}
$$

C is the clamped capacitance

$$
C = \frac{\epsilon_{33}^S A}{h_b}
\tag{47}
$$

Θ is a reformulation of the PZT actuation moment of Equation 31

$$
\Theta = \int_{\mathrm{PZT}} A \frac{Y_b d_{31}}{h_b} y' dy = A(y_0 - x_b)Y_b d_{31}
\tag{48}
$$

The mass M and the function $\Phi(z_0)$ depend on the boundary conditions and the location of the applied force.

$$
M = \int_V \rho \Phi^2(z) dV = \rho_i h_i w \int_0^l \Phi^2(z) dz
$$

where ρ_i is the density of the layer i and $\Phi(z)$ is chosen appropriately for the boundary conditions. $\Phi_0 = \Phi(z_0)$ is the value of the appropriate Φ at the location z_0 at which the force f is applied. For the cantilevered actuator

$$
M_{CF} = \rho_i h_i w l^5 / 20 = ml^4 / 20
$$

where m is the mass of the actuator, and if the force is applied at the free end

$$
\Phi_{CF}(z_0) = l^2 / 2
$$

For the simply supported actuator

$$
M_{SS} = \rho_i h_i w l^5 / 120 = ml^4 / 120
$$

and if the force is applied at the center

$$
\Phi_{SS}(z_0) = -l^2 / 8
$$

3.5   Resonance Frequencies

The first resonance frequency of the actuator is given by

$$\omega_1 = \sqrt{K/M}$$

and so for the cantilevered actuator

$$w_{CF} = \sqrt{\frac{20AB}{\rho_i h_i w l^5}} = \frac{4.472}{l^2}\sqrt{\frac{B}{\rho_i h_i}}$$

and for the simply supported actuator

$$w_{SS} = \sqrt{\frac{120AB}{\rho_i h_i w l^5}} = \frac{10.95}{l^2}\sqrt{\frac{B}{\rho_i h_i}}$$

If the concern is model accuracy around the first resonance frequency of the actuator, An alternative choice of generalized displacement variable is the amplitude of the first mode of vibration. The first mode shape is then be used to calculate the integrals for M and K and we obtain the slightly lower first resonance frequencies [4]

$$
\begin{aligned}
w_{CF} &= \frac{3.52}{l^2}\sqrt{\frac{B}{\rho_i h_i}} \\
w_{SS} &= \frac{9.87}{l^2}\sqrt{\frac{B}{\rho_i h_i}}
\end{aligned}
\tag{49}
$$

If the concern is the behavior of the actuator above the first resonance then the model should use a set of generalized displacement variables corresponding to the amplitudes of the modes of vibration that cover the frequency range desired.

3.6   Alternate Formulations

If the excitation is sinusoidal with frequency ω and using the relationship $u = c\,\Phi_0$ then the transducer equations, Equations 45 can be written

$$(K - \omega^2 M)\, u - \Phi_0 \Theta\, v \;=\; \Phi_0^2\, f \tag{50}$$

$$\Theta\, u + \Phi_0 C\, v \;=\; \Phi_0\, q \tag{51}$$

In this form of these equations the input variables are f and q and that the output variables are u and v. There are four possible alternative forms of the transducer equations depending on the choice of inputs and outputs.

Inputs	Outputs
force, charge	displacement, voltage
force, voltage	displacement, charge
displacement, charge	force, voltage
displacement, voltage	force, charge

In total there are four possible equations relating each of the three triplets of variables (u, v, f), (u, v, q), (f, q, u) and f, q, v. The first two of these are given by Equations 50 and 51. Some algebra gives the other two which are

$$\Phi_0^2 C\ f + \Phi_0 \Theta\ q = ((K - \omega^2 M)C + \Theta^2)\ u \tag{52}$$

$$-\Phi_0\ f + (K - \omega^2 M)\ q = ((K - \omega^2 M)C + \Theta^2)\ v \tag{53}$$

4 Model Extensions

4.1 Plate Bending

The analysis used so far has used a beam model of the actuator. This assumes that the effect of the PZT is to bend the actuator in the 13 plane only (Figure 1). In reality, both the thermal pre-stress and the PZT actuation cause three dimensional strain, with bending in both 13 and 23 planes. A full analysis of this behavior is possible but for engineering purposes it is sufficient to modify the beam model by suitable approximations to account for these three dimensional effects.

The first effect of note is perhaps slight. The bending stiffness of equation 19 was derived assuming the the stress $T_2 = 0$. Thin plate theory allows for stresses in the other plane, results in a bending stiffness modified by the Poisson's ratio σ. Of course, each material will have a different value of σ and the full analysis of the multi-layered thin plate will take this into account. However, a good engineering approximation is to assume that they are all equal to their average weighed by the thickness of the layers. Let σ be this average Poisson's ratio, then the bending stiffness of the multi-layered plate is

$$B\big|_{\text{plate}} = \frac{B}{(1 - \sigma^2)} \tag{54}$$

As σ is usually about 0.3, the effect is to raise the stiffness by about 10% and the first natural frequency by about 5%.

4.2 Static Shell Curvature

Of perhaps greater significance is the effect of static curvature on the bending stiffness. It is well known that curvature in the 23 plane will increase the stiffness of

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a plate in the 13 plane and *vice versa*. The curvature of the thermal pre-stress has the effect of stiffening the actuator for the PZT induced bending, raising the first resonance frequency and reducing the displacement. Again, a full analysis of this effect is possible, but a useful engineering approximation is fortunately available. The theory of shallow spherical shells [6] gives a formula for the increase in natural frequency of a curved homogeneous plate as a function of the radius of curvature R.

$$\omega_n^2|_{\text{shell}} = \omega_n^2|_{\text{plate}} + \frac{Y}{\rho R^2} \tag{55}$$

where Y is the young's modulus, ρ is the density. The equivalent bending stiffness is then

$$B|_{\text{shell}} = B|_{\text{plate}} + \frac{Y h l^4}{\beta^2 R^2} \tag{56}$$

where the constant β^2 depends on the choice of boundary conditions. A major complication occurs when we wish to calculate the shell bending stiffness due to thermal pre-stress as the radius of curvature R itself depends on the bending stiffness. By equation 26

$$R = B/M_\theta \tag{57}$$

where M_θ is the thermally induced bending moment. We thus have a cubic equation which must be solved for the shell bending stiffness

$$B^3|_{\text{shell}} = B^2|_{\text{shell}} B|_{\text{plate}} + \frac{Y h l^4 M_\theta^2}{\beta^2} \tag{58}$$

4.3   Circular Actuators

A full analysis of a circular actuator from first principles is possible using a similar analysis to that used above for beam and plate actuators. However, as in the previous section, it is sufficient for engineering purposes to modify the plate model by suitable approximations.

It was shown in sections 2.4 and 2.5 that the effect of both thermal pre-stress and piezoelectric actuation is to induce a bending moment. An equivalent effect to the moments induced by distributed action of thermal and piezoelectric strain, can be achieved by applying an equal line moment to the edge of the plate. Thus, if the rectangular plate is transformed into a circular plate, the line moments at the edges are transformed into a line moment around the circumference and the formulae of equations 27 and 31 are still valid. As a circular disk certainly does not support plane bending, the appropriate bending stiffness is that of equation 54. The displacement at the center of a simply supported disk of radius a with a

line moment M_0 applied on its circumference is [5]

$$u_0 = \frac{M_0 a^2 (1 - \sigma^2)}{2B(1 + \sigma)} \tag{59}$$

and the first resonance frequency is given by [6]

$$\omega_1 = \frac{4.98}{a^2} \sqrt{\frac{B}{\rho h}} \tag{60}$$

The mass term in the transducer model can be found by integrating the square of the generalized displacement function $\phi^2(r)$ over the area of the disk where

$$\phi(r) = (r - a)(r + a)/2 \tag{61}$$

and so

$$M = \int_A \rho_i h_i \phi(r) dA = \rho_i h_i a^6 2\pi/24 = ma^4/12 \tag{62}$$

and the other parameters can be readily derived.

5   Equivalent Circuit Transducer Model

It was demonstrated in previous sections that the parameters of the transducer model depend on the particular boundary conditions chosen for a given application. In practice, the pure boundary conditions of the analytical model are difficult and impractical to implement and the actuator mounting is chosen to meet the mechanical requirements of the application while attempting to optimize one or more performance variables. For example, if the THUNDER actuator is to be used as a sound source, a mechanical requirement is that a good air seal is obtained from the front to the back of the actuator and that this seal does not fail or fatigue during operation. Good acoustic performance requires that the volume displacement of the actuator at the design frequency be optimized. If the actuator is to be used as a source of force, the mounting must allow free movement while transmitting the force to the supporting structure.

Another major practical effect is the static curvature caused by the thermal pre-stress which occurs both across the length and width. This shell curvature will stiffen the actuator leading to higher resonance frequencies and lower displacements. The actual curvature of an installed actuator is hard to accurately predict as it depends on both the details of the mounting and the manufacturing process of the actuator. Small variations in each can lead to large variations in resonance frequency.
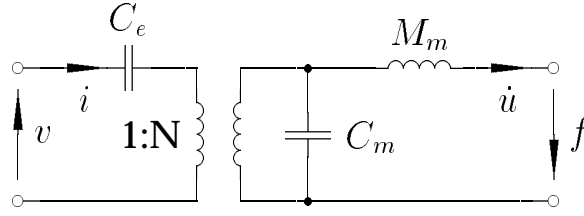
Figure 3: An equivalent circuit for a THUNDER actuator

Thus while the analytical model is useful for understanding the relationship between design variables (material choices and dimensions) and performance variables (force and displacement per volt or ampere), it can only provide approximate predictions of actual performance variables. An experimentally based model is then useful for designing practical actuators and comparing their performance.

When designing a THUNDER actuator for a particular application, it is useful to have a transducer model that relates the electrical input power variables v and i to the mechanical input power variables f and $\dot{u}$ using parameters that can be experimentally measured and hence allow candidate designs to be compared. A useful tool for this purpose is the equivalent electrical circuit shown in figure 3.

In the center of the diagram is an ideal transformer with a 'turns ratio' of N. N is the key transduction constant that relates input current to velocity (or charge to displacement). Turning the circuit around would give the sensor relationships of volts per force and charge per displacement. C_e represents the electrical capacitance for the stress free actuator. C_m represents the mechanical compliance for a open circuit actuator

Solving for the circuit variables gives the transducer equations in terms of N, C_e, C_m and M_m.

$$\left(1 - \omega^2 (C_m + N^2 C_e) M_m\right) u - N C_e\ v \quad = \quad \left(C_m + N^2 C_e\right) f \tag{63}$$

$$N C_e\ u + C_m C_e\ v \quad = \quad \left(C_m + N^2 C_e\right) q \tag{64}$$

$$C_m\ f + N\ q \quad = \quad \left(1 - \omega^2 C_m M_m\right) u \tag{65}$$

$$-N C_e\ f + \left(1 - \omega^2 (C_m + N^2 C_e) M_m\right) q \quad = \quad C_e (1 - \omega^2 C_m M_m)\ v \tag{66}$$

If the transducer equations derived from the equivalent circuit are compared to those derived from the analytical model (Equations 50 to 53) then the following equivalences can be found

$$N = \frac{\Theta \Phi_0}{KC} \left(\frac{1}{1 + \Theta^2/KC}\right) \tag{67}$$

$$C_m = \frac{\Phi_0^2}{K} \cdot \left(\frac{1}{1 + \Theta^2/KC}\right) \tag{68}$$

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$$C_e = C(1 + \Theta^2/KC) \tag{69}$$

and

$$M_m = \frac{M}{\Phi_0^2} \tag{70}$$

The first resonance frequency is given by

$$\omega_1^2 = \frac{1}{(C_m + N^2 C_e)M_m} = \frac{K}{M} \tag{71}$$

which is the resonance frequency for an open circuit actuator $q = 0$. There is also a second slightly higher frequency given by

$$\omega_2^2 = \frac{1}{C_m M_m} = \frac{KC + \Theta^2}{MC} \tag{72}$$

which is the resonance frequency for a shorted actuator $v = 0$. A final but important quantity is the coupling factor k which is a measure of the efficiency of the conversion of electrical energy to mechanical energy. Assuming that a force and voltage are applied such that u and q are both zero. Then by Equation 63

$$-NC_e v = (C_m + N^2 C_e)f \tag{73}$$

If a small DC charge Δq is applied then by Equation 65

$$\Delta u = N \Delta q \tag{74}$$

and the ratio of electrical energy supplied to the actuator to the mechanical energy supplied by the actuator is

$$k^2 = \frac{-f \Delta u}{v \Delta q} = \frac{N^2 C_e}{C_m + N^2 C_e} = \frac{\Theta^2}{KC + \Theta^2} = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2} \tag{75}$$

6 Design Approximations

When designing an actuator for a given application, we would like to know the effect of particular choices of model parameters on the performance of the actuator. The equations describing the THUNDER model are moderately complicated and the effect of varying a parameter such as the thickness of the base layer on a performance variable such as the displacement per volt, is not readily apparent.

This section discusses a series of approximations or design guidelines that are useful when selecting actuator parameters for a particular design. In general, we would like to design a transducer that is matched to mechanical impedance of the

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medium into which it works. A THUNDER actuator uses the bending process as a mechanical amplifier to achieve more displacement than a direct strain PZT device, and so displacement performance parameters tend to be of greater importance than force performance parameters .

Assuming then that the objective of the designer is to make a THUNDER device with the best displacement output, the most important performance parameters are the displacement per coulomb N, the displacement per volt NC_e, and the first resonant frequency w_1.

6.1   PZT **thickness**

In general, a thinner PZT results in greater actuator displacement at the expense of greater actuator force. In a typical application, displacement is of greater importance that force and so the designer should select the thinnest PZT material available. The thickness of available PZT wafers is determined both by the manufacturing process of the wafer itself and by the need to handle the wafer for subsequent processing (as a ceramic PZT is fragile and fractures easily). The thickness of the PZT chosen for incorporation into a THUNDER device is thus likely to be fixed by its availability.

6.2   **Base Thickness**

The optimum thickness of the metal base layer is a complex function of the selection of materials and thickness of the other layers of the actuator. The effect of base layer thickness on displacement can be seen by considering its effect on the bending stiffness B and the induced piezoelectric moment M_p.

$$B = \frac{Y_i h_i^3}{12} + Y_i h_i w_i^2 \tag{76}$$

$$M_p = -w_b Y_b h_b d_{31} E_3 \tag{77}$$

and the curvature is the ratio

$$c = M_p / B \tag{78}$$

The bending stiffness obviously increases as the base thickness increases. M_p also increases through the effect of moving the centerline of the PZT x_b further from the bending neutral axis y_0. As a rule of thumb, the maximum curvature occurs when

$$Y_a h_a = Y_b h_b \tag{79}$$

and

$$w_a = -w_b \approx h_b / 3 \tag{80}$$

that is the bending neutral axis is just in the PZT (most of the compression strain of the PZT is acting on the same side of the neutral axis but no effort is wasted squeezing base material). With this choice of base thickness, useful approximations of B and M_p are possible. As PZT has a lower modulus than most metals, the base layer thickness will be smaller than the PZT layer thickness and

$$B \approx \frac{Y_b h_b^3}{3} \tag{81}$$

Similarly,

$$M_p \approx \frac{h_b^2}{3} Y_b d_{31} E_3 \tag{82}$$

and so

$$c = -\frac{M_p}{B} \approx -\frac{d_{31} E_3}{h_b} \tag{83}$$

6.3   Actuator Length

Having selected the PZT and base layer thicknesses, the next step is to choose the length of the actuator. The displacement output increases with the square of the length of the actuator and so longer actuators are preferred. However, the first resonance frequency of the actuator is inversely proportional to the square of the length of the actuator and, in general we will want to operate the actuator at frequencies below the resonance. At and near the resonance, although the increased output is attractive, the performance of the actuator is unpredictable and the phase shift accompanying the resonance makes the actuator difficult to control. The design choice is then to make the actuator as long as possible such that the operating bandwidth of the actuator falls below the first resonance.

As the thickest layer, the PZT will also dominate the expression for the surface density, $\rho_i h_i \approx \rho_b h_b$ and so the first resonance frequency can be approximated by

$$\omega_1 = \frac{\beta}{l^2} \sqrt{\frac{B}{\rho_i h_i}} \approx= \frac{\beta h_b}{\sqrt{3} l^2} \sqrt{\frac{Y_b}{\rho_b}} \tag{84}$$

where the constant β depends on the choice of boundary conditions, bearing in mind that the stiffness of the actuator will, in practice, depend on the specific details of both the mounting and the static curvature induced by the thermal prestress.

If ω_0 is the greatest frequency at which the actuator will be operated, then we require that

$$\omega_0 \leq \omega_1 \tag{85}$$

Thus the length of the actuator is given by

$$l^2 \leq \frac{\beta h_b}{\sqrt{3}\omega_0}\sqrt{\frac{Y_b}{\rho_b}} \tag{86}$$

and so the maximum displacement

$$u_{\mathbf{max}} \propto cl^2 \propto d_{31}E_3\frac{\beta}{\omega_0}\sqrt{\frac{Y_b}{\rho_b}} \tag{87}$$

6.4   Transducer Parameters

Using the approximations for B and w_b given above, approximations for the transducer parameters K and Θ can be found

$$K \approx \frac{Y_b h_b^3 A}{3} \tag{88}$$

$$C = \frac{\epsilon_{33}A}{h_b} \tag{89}$$

$$\Theta \approx \frac{Y_b h_b A d_{31}}{3} \tag{90}$$

$$M_{CF} = \frac{ml^4}{20} \quad M_{SS} = \frac{ml^4}{120} \tag{91}$$

and

$$\omega_{1,CF} = 2.6\frac{h_b}{l^2}\frac{Y_b}{h_b} \quad \omega_{1,SS} = 6.3\frac{h_b}{l^2}\frac{Y_b}{h_b} \tag{92}$$

6.5   Circuit Parameters

Using the approximations for K and Θ given above, approximations for the circuit parameters can be found

$$N \approx \frac{\Phi_0 d_{31}}{h_b A e_{33}} \tag{93}$$

The ratio d_{31}/e_{33} is the g_{31} constant for the PZT material.

$$C_e \approx \frac{\epsilon_{33}A}{h_b} \tag{94}$$

$$C_m \approx \frac{3}{Y_b h_b^3 A} \tag{95}$$

and

$$M_m = \frac{M}{\phi_0^2} \tag{96}$$

6.6   Static Thermal Curvature

Accurate estimates of the static thermal curvature are difficult as as the eventual curvature depends on both the shell bending stiffness and the precise details of the mounting of the actuator,

An approximation for the thermal bending moment is

$$M_\theta = Y_b h_b^2 (\alpha_a - \alpha_b)\theta/3$$

where α_a and α_b are the coefficients of thermal expansion for the base and PZT layers and θ is the temperature (above ambient) at which the actuator adhesive set. An estimate of the curvature is then

$$c = M_\theta/B = (\alpha_a - \alpha_b)\theta/h_b$$

If the curvature is large then the actuator will be stiffer than predicted by a factor x where

$$x \approx \left(\frac{l^2 (\alpha_a - \alpha_b)\theta}{h_b^3 \beta} \right)^{2/3} \tag{97}$$

The bending stiffness B and the stiffness K must be increased and the compliance C_m must be decreased by this factor.

Notation

B .... bending stiffness

C .... capacitance

C_e .... transduction free capacitance

C_m .... transduction open circuit compliance

D .... electric displacement

E_3 .... electric field strength in 3 direction

F .... force

K .... stiffness

M .... mass

M .... moment

M_b .... applied bending line moment

M_m ...... transduction mass

M_p .... piezoelectric actuation line moment

M_{theta} . . . . thermal actuation line moment

N . . . . transduction constant

R . . . . radius of curvature

S . . . . mechanical strain

S_0 . . . . uniform or static strain

T_1 . . . . mechanical stress in 1 direction

T_i . . . . mechanical stress in 1 direction in layer i

Y . . . . mechanical modulus

Y_i . . . . mechanical modulus of layer i

a . . . . radius

c . . . . curvature

d_{31} . . . . . . piezoelectric constant

f . . . . force

h_i . . . . thickness of layer i

l . . . . length

q . . . . charge

u . . . . actuator displacement

v . . . . voltage

w . . . . width

x_i . . . . centerline of layer i

y_0 . . . . bending neutral axis

y_i . . . . upper boundary of layer i

α_i . . . . coefficient of thermal expansion of layer i

β . . . . boundary condition frequency constant

ϵ_{33} . . . . . . dielectric constant

$\hat{y}_0$ . . . . effective neutral axis

ω_1 . . . . . . first natural frequency

ϕ . . . . mode shape

ρ_i . . . . material density of layer i

σ . . . . Poisson's ratio

θ . . . . temperature above ambient at which adhesive set

θ . . . . transduction constant

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