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High Speed Solution of Spacecraft Trajectory Problems Using Taylor Series Integration

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Taylor series integration is implemented in a spacecraft trajectory analysis code – the Spacecraft N-body Analysis Program (SNAP) – and compared with the code's existing eighth-order Runge-Kutta Fehlberg time integration scheme. Nine trajectory problems, including near Earth, lunar, Mars and Europa missions, are analyzed. Head-to-head comparison at five different error tolerances shows that, on average, Taylor series is faster than Runge-Kutta Fehlberg by a factor of 15.8. Results further show that Taylor series has superior convergence properties. Taylor series integration proves that it can provide rapid, highly accurate solutions to spacecraft trajectory problems.

Nomenclature

x_1	=	x component of spacecraft position relative to inertial frame centered at the central body
x_2	=	y component of spacecraft position relative to inertial frame centered at the central body
x_3	=	z component of spacecraft position relative to inertial frame centered at the central body
x_4	=	x component of spacecraft velocity relative to inertial frame centered at the central body
x_5	=	y component of spacecraft velocity relative to inertial frame centered at the central body
x_6	=	z component of spacecraft velocity relative to inertial frame centered at the central body
x_7	=	spacecraft mass
$\overline{X}$	=	$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ = spacecraft state vector
$\vec{x}$	=	(x_1, x_2, x_3) = spacecraft position vector
$\vec{v}$	=	$(x_4, x_5, x_6) = (v_1, v_2, v_3)$ = spacecraft velocity vector
t	=	time
$'$	=	*d/dt* = derivative with respect to time
$\dot{}$	=	*d/dt* = derivative with respect to time
$\dot{m}$	=	mass flow rate
T	=	thrust magnitude
a_1	=	x component of spacecraft acceleration relative to inertial frame centered at the central body
a_2	=	y component of spacecraft acceleration relative to inertial frame centered at the central body
a_3	=	z component of spacecraft acceleration relative to inertial frame centered at the central body

$\vec{a}$	=	(a_1, a_2, a_3) = acceleration vector
$\vec{x}_j$	=	position vector of the j^{th} other body relative to the central body
G	=	gravitational constant
M	=	body mass
h	=	step size
τ	=	local error tolerance
η	=	step multiplication factor

I. Introduction

The advantages of Taylor series integration in solving ordinary differential equations have been known for some time [1-25]. Foremost among these is the ability to maintain very high computational efficiency while achieving high accuracy. In fact, comparisons with other methods have shown that Taylor series integration can be faster by a factor of twenty or more [20].

The success of the method depends on recasting the governing differential system into a canonical form whereby system derivatives can be obtained to arbitrary order through recursion. This obviates the need to directly calculate derivatives, and makes it possible to obtain derivative information cheaply. The system variables can thus be expanded in a highly accurate series at each time level at minimal cost.

It turns out that most differential systems can be recast into the required canonical form in a straightforward manner. This has led a number of authors to develop general purpose software which can recast an arbitrary system into canonical form and solve the resulting equations automatically [10, 11, 15-18, 22-24]. Taylor series integration can thus be used as both a general purpose solver and also for specific applications.

The focus here is on calculating spacecraft trajectories. Previous work using Taylor series to calculate trajectories includes that in Refs. 21 and 22. Unlike Refs. 21 and 22, which used an automated Taylor series package [16,22] to propagate trajectories in Earth orbit, the present work focuses on using Taylor series integration in an existing trajectory analysis code, SNAP (Spacecraft N-body Analysis Program) [26]. Developed at NASA's Glenn Research Center, SNAP is a high fidelity trajectory propagation program that can propagate the trajectory of a spacecraft about virtually any body in the solar system. The equations of motion include the effects of central body gravitation with N x N harmonics, other body gravitation with N x N harmonics, solar radiation pressure, atmospheric drag (for Earth orbits) and spacecraft thrusting (including shadowing). The equations are solved using an eighth-order Runge-Kutta Fehlberg (RKF) [27] single step method with variable step size control.

The purpose of this paper is to demonstrate the use of Taylor series (TS) integration in a high fidelity trajectory analysis code, SNAP, and to provide a detailed comparison of TS performance to eighth-order RKF [27]. Section II presents the equations of motion, Section III describes the TS formulation and Section IV discusses the numerical implementation. Section V compares TS and RKF on a representative set of spacecraft trajectory problems, including near Earth, lunar, Mars and Europa missions. It is shown that TS is faster than RKF by an average factor of 15.8, while simultaneously improving accuracy.

II. Equations of Motion

Let $\overline{X} = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ denote the spacecraft state vector, where $(x_1, x_2, x_3) = \vec{x}$ is the spacecraft position in Cartesian coordinates relative to an inertial frame centered at the central body, $(x_4, x_5, x_6) = \vec{v}$ is the spacecraft velocity relative to an inertial frame centered at the central body, and x_7 is the spacecraft mass. The equations of motion are

$$\begin{aligned} x_1' &= x_4 \\ x_2' &= x_5 \\ x_3' &= x_6 \end{aligned} \tag{1}$$

$$
\begin{aligned}
x_4' &= a_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \\
x_5' &= a_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \\
x_6' &= a_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, t) \\
x_7' &= -\dot{m}(t)
\end{aligned}
$$

where a_i is the acceleration in the i^{th} coordinate direction and $\dot{m}$ is the mass flow rate. The acceleration is a function of the forces acting on the spacecraft. Forces included in SNAP are central body, other body, thrust, atmospheric drag (for low Earth orbits), solar radiation pressure, oblateness effects of Earth, and oblateness effects of other bodies, so that

$$
\vec{a} = \vec{a}_{cb} + \vec{a}_{ob} + \vec{a}_{th} + \vec{a}_{d} + \vec{a}_{srp} + \vec{a}_{obe} + \vec{a}_{obo} \tag{2}
$$

This paper considers only the first four acceleration terms, which are given below.

$$
a_{cb,i} = -G\,M_{cb} \frac{x_i}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}} \quad := \; -G\,M_{cb} \frac{x_i}{|\vec{x}|^3} \tag{3}
$$

where G is the gravitational constant, M_{cb} is the central body mass, i = 1,2,3 denotes the coordinate direction, and the spacecraft mass is much less than the central body mass.

$$
a_{ob,i} = \sum_j -G\,M_j \left(\frac{x_i - x_{i,j}}{|\vec{x} - \vec{x}_j|^3} + \frac{x_{i,j}}{|\vec{x}_j|^3} \right) \tag{4}
$$

where j denotes the j^{th} body and $|\vec{x}_j| := \left(x_{1,j}^2 + x_{2,j}^2 + x_{3,j}^2\right)^{\frac{1}{2}}$.

$$
a_{th,i} = T \frac{v_i}{|\vec{v}|} \frac{1}{x_7} \tag{5}
$$

where T is the constant thrust magnitude and the thrust direction is parallel to the velocity vector.

$$
a_{d,1} = c_1 \left[(x_4 + c_2 x_2)^2 + (x_5 - c_2 x_1)^2 + x_6^2 \right]^{\frac{1}{2}} (x_4 + c_2 x_2) / x_7 \tag{6}
$$

$$
a_{d,2} = c_1 \left[(x_4 + c_2 x_2)^2 + (x_5 - c_2 x_1)^2 + x_6^2 \right]^{\frac{1}{2}} (x_5 - c_2 x_1) / x_7 \tag{7}
$$

$$
a_{d,3} = c_1 \left[(x_4 + c_2 x_2)^2 + (x_5 - c_2 x_1)^2 + x_6^2 \right]^{\frac{1}{2}} x_6 / x_7 \tag{8}
$$

where $c_1 = -\frac{1}{2} C_D \cdot (\textit{spacecraft area}) \cdot (\textit{atmospheric density})$, c_2 = rotation rate of Earth and C_D is the drag coefficient.

III. Taylor Series Formulation

Let the state vector $\overline{X}$ have initial condition $\overline{X}_0$. Within the radius of convergence, the system variables $x_n(t)$ can be expanded in a Taylor series,

$$x_n(t) = \sum_{k=0}^{\infty} \frac{x_n^{(k)}(t_0)}{k!}(t - t_0)^k , \qquad n = 1,...,7 \tag{9}$$

where the derivatives $x_n^{(k)}$ are obtained by successively differentiating the right hand side of Eqs. (1). This can be efficiently accomplished using recurrence relations, as follows. Consider only the central body acceleration term, so that

$$\begin{aligned}
x_1' &= x_4 \\
x_2' &= x_5 \\
x_3' &= x_6 \\
x_4' &= -G\,M_{cb} \frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}} \\
x_5' &= -G\,M_{cb} \frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}} \\
x_6' &= -G\,M_{cb} \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{\frac{3}{2}}} \\
x_7' &= 0
\end{aligned} \tag{10}$$

Introduce x_8 and x_9,

$$x_8 = x_1^2 + x_2^2 + x_3^2 \tag{11}$$

$$x_9 = x_8^{3/2} \tag{12}$$

Eqs. (10) become

$$\begin{aligned}
x_1' &= x_4 \\
x_2' &= x_5 \\
x_3' &= x_6 \\
x_4' &= -G\,M_{cb} \frac{x_1}{x_9} \\
x_5' &= -G\,M_{cb} \frac{x_2}{x_9}
\end{aligned} \tag{13}$$

$$x_6' = -G\,M_{cb}\,\frac{x_3}{x_9}$$

$$x_7' = 0$$

and two auxiliary equations are added to the system:

$$x_8' = 2x_1x_4 + 2x_2x_5 + 2x_3x_6 \tag{14}$$

$$x_9' = \frac{3}{2}\frac{x_9x_8'}{x_8} \tag{15}$$

The right hand side of the new system, Eqs. (13) – (15), can now be differentiated using recurrence relations for products and quotients.

For a function $w(t) = f(t)\,g(t)$, the Leibnitz rule for differentiating products gives [16]

$$W(k) = \sum_{j=0}^{k} F(j)\,G(k-j) \tag{16}$$

where $W(k) := \dfrac{w^{(k)}(t_0)}{k!}$, $F(j) := \dfrac{f^{(j)}(t_0)}{j!}$ and $G(k-j) := \dfrac{g^{(k-j)}(t_0)}{(k-j)!}$ are reduced derivatives.

For a function $w(t) = \dfrac{f(t)}{g(t)}$, the recurrence relation for quotients gives [11]

$$W(k) = \frac{1}{g}\left[F(k) - \sum_{j=1}^{k} G(j)W(k-j)\right] \tag{17}$$

where *W, F* and *G* are reduced derivatives as above.

The recurrence relations are derived as follows. Let

$$u_n = x_n', \qquad n = 1,\ldots,9 \tag{18}$$

Eqs. (13) – (15) become

$$u_1 = x_4 \tag{19}$$

$$u_2 = x_5 \tag{20}$$

$$u_3 = x_6 \tag{21}$$

$$u_4 = -G\,M_{cb}\,\frac{x_1}{x_9} \tag{22}$$

$$u_5 = -G\,M_{cb}\,\frac{x_2}{x_9} \tag{23}$$

$$u_6 = -G\,M_{cb}\,\frac{x_3}{x_9} \tag{24}$$

$$u_7 = 0 \tag{25}$$

$$u_8 = 2x_1x_4 + 2x_2x_5 + 2x_3x_6 \tag{26}$$

$$u_9 = \frac{3}{2}\frac{x_9u_8}{x_8} \tag{27}$$

Introduce auxiliary functions $w_4 = \dfrac{x_1}{x_9}$, $w_5 = \dfrac{x_2}{x_9}$, $w_6 = \dfrac{x_3}{x_9}$, $w_{8,1} = x_1x_4$, $w_{8,2} = x_2x_5$, $w_{8,3} = x_3x_6$, and $w_9 = \dfrac{x_9u_8}{x_8}$.

Using Eq. (18) one obtains

$$u_n^{(k-1)} = x_n^{(k)}, \quad k \geq 1 \tag{28}$$

from which

$$\frac{u_n^{(k-1)}}{(k-1)!} = \frac{x_n^{(k)}}{(k-1)!}, \qquad k \geq 1 \tag{29}$$

$$\Rightarrow\ U_n(k-1) = k\,X_n(k) \ \Rightarrow\ X_n(k) = \frac{U_n(k-1)}{k}, \qquad k \geq 1 \tag{30}$$

The recurrence relations are then, for all $k \geq 1$,

$$U_1(k) = X_4(k) = \frac{U_4(k-1)}{k} \tag{31}$$

$$U_2(k) = X_5(k) = \frac{U_5(k-1)}{k} \tag{32}$$

$$U_3(k) = X_6(k) = \frac{U_6(k-1)}{k} \tag{33}$$

$$U_4(k) = -G\,M_{cb}\,W_4(k) \tag{34}$$

$$U_5(k) = -G\,M_{cb}\,W_5(k) \tag{35}$$

$$U_6(k) = -G\,M_{cb}\,W_6(k) \tag{36}$$

$$U_7(k) = 0 \tag{37}$$

$$U_8(k) = 2\,W_{8,1}(k) + 2\,W_{8,2}(k) + 2\,W_{8,3}(k) \tag{38}$$

$$U_9(k) = \frac{3}{2} W_9(k) \tag{39}$$

where

$$W_4(k) = \frac{1}{x_9}\left[X_1(k) - \sum_{j=1}^{k} X_9(j) W_4(k-j)\right] \;=\; \frac{1}{x_9}\left[\frac{U_1(k-1)}{k} - \sum_{j=1}^{k} \frac{U_9(j-1)}{j} W_4(k-j)\right] \tag{40}$$

$$W_{8,1}(k) = \sum_{j=0}^{k} X_1(j)\, X_4(k-j) \;=\; x_1\,\frac{U_4(k-1)}{k} + \frac{U_1(k-1)}{k}\,x_4 + \sum_{j=1}^{k-1} \frac{U_1(j-1)}{j}\,\frac{U_4(k-j-1)}{k-j} \tag{41}$$

etc., and a similar expression can be derived for W_9.

The Taylor series coefficients are then

$$\frac{x_n^{(k)}}{k!} := X_n(k) = \frac{U_n(k-1)}{k}, \qquad 1 \le k \le K \tag{42}$$

and the local series solution is

$$x_n(t) = \sum_{k=0}^{K} \frac{x_n^{(k)}(t_0)}{k!}(t-t_0)^k \;+\; T_{n,K} \tag{43}$$

where $U_n(k)$ is defined by Eqs. (31) – (39), $U_n(0)$ is defined by Eqs. (19) – (27), K is the number of terms in the series and $T_{n,K}$ is the truncation error.

The other acceleration terms can be handled similarly. Only other body acceleration, Eq. (4), requires special consideration, due to the need for the motion of other bodies. This can generally be obtained from ephemeris files. However, integration by Taylor series requires derivatives not available from ephemeris files. It is thus necessary to integrate the other body motion as part of the governing differential system. This leads to a substantially larger system of equations, but fortunately can still be integrated efficiently.

Once the recurrence relations are derived for all acceleration terms and the state vector specified, Eqs. (42) – (43) are used to expand the system variables in a series from t_0 to t_1, where the step size $h_1 := t_1 - t_0$ is determined to meet the local error tolerance. From t_1, the variables are expanded in a new series to t_2, and so forth. Thus, by a process of "analytic continuation," one obtains a set of overlapping series solutions that cover the integration domain.

IV. Numerical Implementation

Taylor series integration was implemented in SNAP by making some minor modifications to existing source code and adding three additional subroutines - a driver routine which automatically introduces auxiliary variables, sets up initial conditions and integrates; a routine which calculates system reduced derivatives using recurrence relations for the following quotients and products: $\frac{x_m}{x_n}$, $x_m x_n$, $\frac{x_m x_n'}{x_n}$, $x_m x_m'$, $\frac{x_m'}{x_n}$, $\frac{x_m x_n}{x_l}$ and $x_m x_n'$; and a routine which determines the step size and sums the series. The number of series terms is variable up to a maximum of 30, but remains constant throughout the integration. Positive and negative terms are summed separately to avoid cancellation of significant digits.

The step size can be determined from the standard formula [28]

$$h_{\text{next}} = \eta\, h \left(\frac{\tau}{e_{\max}} \right)^{\frac{1}{M}} \tag{44}$$

where h denotes the current step, τ the local error tolerance, $e_{\max}$ the estimate of maximum truncation error, M the order of the maximum truncation error estimate and $\eta < 1$ the step multiplication factor. Eq. (44) is more or less restrictive depending on η and the truncation error estimate $e_{\max}$. Generally, $e_{\max}$ should not be calculated from the next series term, due to the extra computation required and the fact that it is not a reliable error estimate [29]. A conservative approach which takes advantage of the series terms already computed leads to

$$e_{\max} = \text{Max}_n \left[\left| X_n(K-1) \right| h^{K-1} + \left| X_n(K) \right| h^K \right] \tag{45}$$

where the expression in brackets is derived by subtracting the Taylor series solution of degree K - 2 from the solution of degree K and taking absolute values of individual terms. Eq. (45) can be viewed as a truncation error estimate for the series of degree $K - 2$ which is then applied to the more accurate series of degree K.

An alternative to Eqs. (44) – (45) is to simply require h to be small enough that the system variables directly satisfy the absolute error tolerance requirement

$$\left| X_n(K-1) \right| h^{K-1} + \left| X_n(K) \right| h^K \le \tau \tag{46}$$

for all n. Eq. (46) can be solved by fixed point iteration,

$$h_{l+1} = \exp\left(\frac{1}{K-1} \quad \ln \frac{\tau}{\left| X_n(K-1) \right| + h_l \left| X_n(K) \right|} \right) \tag{47}$$

The smallest h is chosen over all n, and multiplied by the step multiplication factor η. This approach offers the advantage of directly calculating the step size without the need for a previous step, and guarantees that the error tolerance is met. Eq. (44), on the other hand, requires a previous step and will require a repeat step whenever $e_{\max} > \tau$.

The step selection methods above performed very similarly in the current study. Both methods provided stable, accurate solutions and used approximately the same number of time steps in head-to-head calculations.

V. Results

We compare RKF and TS performance on the trajectory problems in Table 1. All calculations were run on a Dell PowerEdge 2600 with two 3.066 GHz processors and four GB of RAM. Source code was compiled using the Absoft Fortran 90 compiler without optimization. SNAP was run with all intermediate print and stop options turned off. All TS calculations used a series with 20 terms and a variable step size determined by Eq. (47) with a step multiplication factor that ranged from 0.75 to 0.9.

Table 1

Problem	Title	Description	Central Body	Other Bodies
1	Satellite in low Earth orbit (LEO)	A 10,000 kg satellite orbits Earth for 10 days at an inclination of 28.45 degrees.	Earth	Moon
2	Satellite in LEO with drag	A 10,000 kg satellite orbits Earth for 10 days with constant drag at an inclination of 28.45 degrees.	Earth	Moon
3	Spacecraft spiraling out of Earth's gravity well	A 10,000 kg spacecraft spirals out of Earth's gravity well in a low thrust trajectory. Calculation stops when the semi-major axis of trajectory equals 40,000 km.	Earth	Sun, Moon
4	Spacecraft from near Earth to lunar orbit	A 3580 kg spacecraft 400 km above Earth has been propelled with sufficient energy to reach the Moon. Spacecraft coasts to Moon, performs insertion burn, propagates to apolune, and performs final burn to achieve 500 km by 10,000 km polar lunar orbit with an argument of perilune equal to 90 degrees. See Fig. 1.	Moon	Earth, Sun
5	Spacecraft in lunar orbit	Spacecraft with 2848.56 kg mass coasts for 10 days in 500 km by 10,000 km polar lunar orbit with an argument of perilune equal to 90 degrees. See Fig. 1.	Moon	Earth, Sun
6	Spacecraft thrusting from near Earth to Mars coast	A 585 kg spacecraft near Earth thrusts for 38.45 days to achieve sufficient energy to coast to Mars. See Fig. 2.	Sun	Earth, Moon, Venus, Mars, Jupiter barycenter, Saturn barycenter
7	Spacecraft coast to Mars flyby	A 555.66 kg spacecraft coasts to Mars flyby for 161.55 days. See Fig. 2.	Sun	Earth, Moon, Venus, Mars, Jupiter barycenter, Saturn barycenter
8	Spacecraft thrusting tangentially out of Europa orbit	A 10,000 kg spacecraft in Europa orbit thrusts tangentially to spiral out until the semi-major axis equals 10,000 km.	Europa	Jupiter, Sun, Ganymede, Io Callisto
9	Spacecraft coast near Europa	A 9800.49 kg spacecraft coasts for one day after spiraling out of Europa orbit.	Europa	Jupiter, Sun, Ganymede, Io Callisto

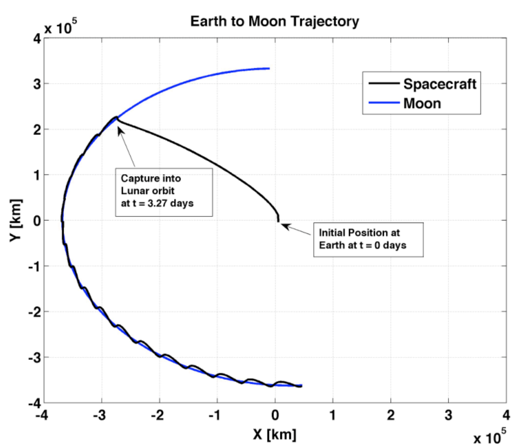

Figure 1. Earth to Moon Trajectory

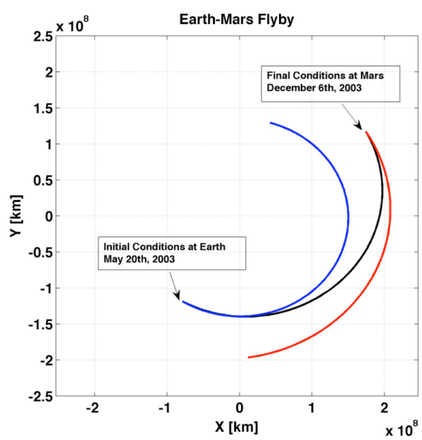

Figure 2. Earth to Mars Flyby

Each trajectory was integrated at five error tolerances from 10^{-10} to 10^{-14}. Tables 2 - 10 summarize the results. RKF results are shown on top and TS on bottom. Spacecraft positions are in kilometers and CPU ratio is RKF/TS. RKF and TS velocities agreed equally well as spacecraft position, but are omitted here for brevity.

Table 2 Results for Problem 1

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	-4505.4956174253	4460.8015923075	2416.2243468200	6.91	
1.E-11	-4505.4976241104	4460.8000212437	2416.2234952381	9.10	
1.E-12	-4505.4977855238	4460.7998948715	2416.2234267390	12.23	
1.E-13	-4505.4977981582	4460.7998849798	2416.2234213773	16.05	
1.E-14	-4505.4977991371	4460.7998842135	2416.2234209620	21.17	
1.E-10	-4505.4893402433	4460.8066582279	2416.2273550484	0.20	34.6
1.E-11	-4505.4893401709	4460.8066582847	2416.2273550792	0.22	41.4
1.E-12	-4505.4893402598	4460.8066582151	2416.2273550415	0.25	48.9
1.E-13	-4505.4893403875	4460.8066581153	2416.2273549873	0.28	57.3
1.E-14	-4505.4893402869	4460.8066581940	2416.2273550300	0.31	68.3

Table 3 Results for Problem 2

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	-6286.5348347365	2234.8075149479	1209.8215168263	7.04	
1.E-11	-6286.5358380683	2234.8053241518	1209.8203296255	9.16	
1.E-12	-6286.5359189241	2234.8051476038	1209.8202339534	12.18	
1.E-13	-6286.5359251975	2234.8051339059	1209.8202265305	16.04	
1.E-14	-6286.5359257481	2234.8051327038	1209.8202258791	21.28	
1.E-10	-6286.5317837499	2234.8146866042	1209.8256504332	0.26	27.1
1.E-11	-6286.5317836592	2234.8146868021	1209.8256505404	0.29	31.6
1.E-12	-6286.5317837023	2234.8146867081	1209.8256504895	0.33	36.9
1.E-13	-6286.5317837303	2234.8146866471	1209.8256504564	0.37	43.4
1.E-14	-6286.5317836427	2234.8146868382	1209.8256505600	0.42	50.7

Table 4 Results for Problem 3

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	21783.589218926	30426.566335107	14117.151742798	44.01	
1.E-11	21783.168251826	30426.814392403	14117.266771932	58.17	
1.E-12	21783.134612346	30426.834214343	14117.275963762	77.72	
1.E-13	21783.131993250	30426.835757635	14117.276679417	102.06	
1.E-14	21783.131799858	30426.835871591	14117.276732261	134.87	
1.E-10	21783.126013948	30426.839470527	14117.277961164	2.29	19.2
1.E-11	21783.126025067	30426.839463974	14117.277958125	2.57	22.6
1.E-12	21783.126012532	30426.839471362	14117.277961551	2.88	27.0
1.E-13	21783.126009565	30426.839473099	14117.277962357	3.27	31.2
1.E-14	21783.126006877	30426.839474690	14117.277963095	3.69	36.6

Table 5 Results for Problem 4

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	-178.63870529793	4876.3161604664	-10650.392195425	0.44	
1.E-11	-178.63855098380	4876.3170369698	-10650.391796702	0.51	
1.E-12	-178.63855046608	4876.3170506570	-10650.391790443	0.59	
1.E-13	-178.63855601675	4876.3170085371	-10650.391809627	0.71	
1.E-14	-178.63854695266	4876.3170607540	-10650.391785873	0.87	
1.E-10	-178.71939471463	4876.1033547743	-10650.488272923	0.19	2.31
1.E-11	-178.71939481530	4876.1033542962	-10650.488273141	0.20	2.55
1.E-12	-178.71939478451	4876.1033544409	-10650.488273078	0.20	2.95
1.E-13	-178.71939478450	4876.1033544292	-10650.488273079	0.20	3.55
1.E-14	-178.71939478714	4876.1033544962	-10650.488273051	0.20	4.35

Table 6 Results for Problem 5

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	-215.32731201862	-1650.3697144277	1635.1626824928	1.99	
1.E-11	-215.32794605537	-1650.3715299629	1635.1612149497	2.64	
1.E-12	-215.32799474610	-1650.37166939046	1635.1611022567	3.49	
1.E-13	-215.32799868206	-1650.3716806616	1635.1610931478	4.62	
1.E-14	-215.32799901817	-1650.3716816241	1635.1610923701	6.15	
1.E-10	-215.32849813488	-1650.3731189359	1635.1600219626	0.32	6.22
1.E-11	-215.32849812774	-1650.3731189154	1635.1600219790	0.35	7.54
1.E-12	-215.32849812980	-1650.3731189214	1635.1600219744	0.40	8.73
1.E-13	-215.32849813332	-1650.3731189310	1635.1600219653	0.44	10.5
1.E-14	-215.32849813453	-1650.3731189349	1635.1600219635	0.49	12.6

Table 7 Results for Problem 6

	Spacecraft Position at End of Calculation			CPU	CPU
τ	x	y	z	(sec)	ratio
1.E-10	21572817.105605	-139377155.25969	-62943596.519662	0.14	
1.E-11	21572817.105556	-139377155.25962	-62943596.519125	0.15	
1.E-12	21572817.105508	-139377155.25961	-62943596.519004	0.21	
1.E-13	21572817.105509	-139377155.25960	-62943596.518999	0.25	
1.E-14	21572817.105496	-139377155.25961	-62943596.518987	0.34	
1.E-10	21572816.904607	-139377156.15407	-62943596.9802248	0.11	1.27
1.E-11	21572816.904606	-139377156.15407	-62943596.9802247	0.12	1.25
1.E-12	21572816.904607	-139377156.15407	62943596.9802248	0.14	1.50
1.E-13	21572816.904607	-139377156.15407	-62943596.9802248	0.16	1.56
1.E-14	21572816.904607	-139377156.15407	-62943596.9802247	0.17	2.00

Table 8 Results for Problem 7

τ	Spacecraft Position at End of Calculation x	y	z	CPU (sec)	CPU ratio
1.E-10	174298859.53574	117412103.45296	49149031.016983	0.14	
1.E-11	174298859.53061	117412103.45425	49149031.017954	0.15	
1.E-12	174298859.53010	117412103.45436	49149031.018040	0.17	
1.E-13	174298859.53005	117412103.45437	49149031.018048	0.18	
1.E-14	174298859.53003	117412103.45437	49149031.018052	0.23	
1.E-10	174298891.93452	117412111.58059	49149027.497160	0.11	1.27
1.E-11	174298891.93452	117412111.58059	49149027.497159	0.13	1.15
1.E-12	174298891.93452	117412111.58059	49149027.497160	0.15	1.13
1.E-13	174298891.93453	117412111.58059	49149027.497159	0.17	1.06
1.E-14	174298891.93453	117412111.58059	49149027.497159	0.17	1.35

Table 9 Results for Problem 8

τ	Spacecraft Position at End of Calculation x	y	z	CPU (sec)	CPU ratio
1.E-10	7528.3536043710	1375.7425889427	-383.13557482604	0.79	
1.E-11	7528.3536031572	1375.7425956652	-383.13557244841	1.00	
1.E-12	7528.3536030460	1375.7425962755	-383.13557223246	1.33	
1.E-13	7528.3536030359	1375.7425963294	-383.13557221337	1.74	
1.E-14	7528.3536030345	1375.7425963361	-383.13557221099	2.25	
1.E-10	7528.7471022132	1375.7729665868	-383.16004903160	0.14	5.64
1.E-11	7528.7471022129	1375.7729665694	-383.16004903738	0.15	6.67
1.E-12	7528.7471022110	1375.7729665782	-383.16004903433	0.16	8.31
1.E-13	7528.7471022134	1375.7729665759	-383.16004903524	0.18	9.67
1.E-14	7528.7471022136	1375.7729665815	-383.16004903345	0.21	10.7

Table 10 Results for Problem 9

τ	Spacecraft Position at End of Calculation x	y	z	CPU (sec)	CPU ratio
1.E-10	-6179.7835988316	19484.717456953	11599.814016718	0.12	
1.E-11	-6179.7835986778	19484.717455196	11599.814015865	0.14	
1.E-12	-6179.7835986665	19484.717454928	11599.814015733	0.17	
1.E-13	-6179.7835986627	19484.717454902	11599.814015720	0.22	
1.E-14	-6179.7835986621	19484.717454906	11599.814015723	0.27	
1.E-10	-6179.2994738523	19483.004328592	11598.975228083	0.04	3.00
1.E-11	-6179.2994738523	19483.004328592	11598.975228083	0.05	2.80
1.E-12	-6179.2994738525	19483.004328591	11598.975228082	0.05	3.40
1.E-13	-6179.2994738524	19483.004328591	11598.975228083	0.04	5.50
1.E-14	-6179.2994738522	19483.004328591	11598.975228083	0.05	5.40

Table 11 summarizes the CPU ratios. TS is faster than RKF by more than an order of magnitude in 18 of 45 cases. The average speedup is 15.8. For the interplanetary trajectory problems (4-9) the average speedup is 4.53. The gain here is smaller due to the additional equations that TS must solve to account for other body motion. As noted previously, TS must integrate other body motion as part of the differential system, whereas RKF obtains other body motion from ephemeris files. This difference in the integration methods explains the small differences in spacecraft positions observed in Tables 2-10.

Table 11 RKF/TS CPU ratios

τ	Problem 1	2	3	4	5	6	7	8	9
1.E-10	34.6	27.1	19.2	2.31	6.22	1.27	1.27	5.64	3.00
1.E-11	41.4	31.6	22.6	2.55	7.54	1.25	1.15	6.67	2.80
1.E-12	48.9	36.9	27.0	2.95	8.73	1.50	1.13	8.31	3.40
1.E-13	57.3	43.4	31.2	3.55	10.5	1.56	1.06	9.67	5.50
1.E-14	68.3	50.7	36.6	4.35	12.6	2.00	1.35	10.7	5.40

Another important property to consider is convergence. Tables 12 - 20 present the number of converged digits obtained for each spacecraft coordinate at each error tolerance, where the $\tau = 10^{-14}$ case was used as the fully converged solution. RKF results are on top and TS on bottom. TS has more converged digits than RKF in 103 out of 108 cases, while RKF has more converged digits in one case. On average, TS has 2.63 more converged digits per case. The results also indicate that TS solutions are nearly fully converged at all error tolerances, suggesting that the step selection method may be too conservative. Finally, it should be noted that convergence itself does not necessarily imply accuracy. However, it does indicate that a necessary condition for accuracy is satisfied.

Table 12 Number of Converged Digits for Problem 1

τ	x	y	z
1.E-10	6	6	6
1.E-11	7	7	7
1.E-12	8	8	8
1.E-13	9	9	10
1.E-10	10	11	11
1.E-11	10	10	10
1.E-12	11	11	11
1.E-13	10	10	11

Table 13 Number of Converged Digits for Problem 2

τ	x	y	z
1.E-10	6	6	6
1.E-11	7	7	7
1.E-12	8	8	9
1.E-13	9	9	9
1.E-10	10	10	9
1.E-11	10	11	10
1.E-12	10	10	9
1.E-13	10	10	9

Table 14 Number of Converged Digits for Problem 3

τ	x	y	z
1.E-10	4	5	5
1.E-11	5	6	6
1.E-12	7	6	7
1.E-13	8	8	9
1.E-10	10	10	10
1.E-11	9	9	10
1.E-12	10	10	10
1.E-13	10	10	10

Table 15 Number of Converged Digits for Problem 4

τ	x	y	z
1.E-10	6	6	8
1.E-11	6	7	9
1.E-12	6	8	10
1.E-13	6	7	9
1.E-10	9	9	11
1.E-11	10	10	12
1.E-12	10	10	12
1.E-13	10	10	12

Table 16 Number of Converged Digits for Problem 5

τ	x	y	z
1.E-10	5	6	6
1.E-11	6	7	7
1.E-12	7	8	8
1.E-13	9	9	9
1.E-10	12	11	13
1.E-11	11	11	11
1.E-12	11	11	11
1.E-13	11	12	11

Table 17 Number of Converged Digits for Problem 6

τ	x	y	z
1.E-10	10	12	10
1.E-11	10	13	11
1.E-12	10	14	12
1.E-13	12	13	12
1.E-10	14	14	14
1.E-11	13	14	14
1.E-12	14	14	14
1.E-13	14	14	14

Table 18 Number of Converged Digits for Problem 7

τ	x	y	z
1.E-10	10	11	10
1.E-11	11	12	11
1.E-12	12	13	11
1.E-13	12	14	13
1.E-10	13	14	13
1.E-11	13	14	14
1.E-12	13	14	13
1.E-13	14	14	14

Table 19 Number of Converged Digits for Problem 8

τ	x	y	z
1.E-10	9	8	8
1.E-11	10	10	9
1.E-12	11	11	10
1.E-13	11	11	11
1.E-10	12	11	11
1.E-11	12	11	10
1.E-12	12	12	11
1.E-13	12	12	10

Table 20 Number of Converged Digits for Problem 9

τ	x	y	z
1.E-10	10	9	10
1.E-11	11	11	11
1.E-12	11	12	12
1.E-13	12	12	13
1.E-10	13	13	14
1.E-11	13	13	14
1.E-12	12	14	13
1.E-13	13	14	14

VI. Conclusion

Taylor series integration was implemented in a high fidelity trajectory analysis code (SNAP) and compared with 8th order Runge-Kutta Fehlberg on a representative set of trajectory problems. On average, TS was more than an order of magnitude faster than RKF. TS also showed superior convergence properties, having more converged digits than RKF in 103 out of 108 cases. Taylor series integration thus proved that it can provide rapid, highly accurate solutions to spacecraft trajectory problems. This is consistent with other reports which have found Taylor series integration to be superior to conventional methods in both speed and accuracy [11,16,20].

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13. SUPPLEMENTARY NOTES

14. ABSTRACT
Taylor series integration is implemented in a spacecraft trajectory analysis code – the Spacecraft N-body Analysis Program (SNAP) – and compared with the code's existing eighth-order Runge-Kutta Fehlberg time integration scheme. Nine trajectory problems, including near Earth, lunar, Mars and Europa missions, are analyzed. Head-to-head comparison at five different error tolerances shows that, on average, Taylor series is faster than Runge-Kutta Fehlberg by a factor of 15.8. Results further show that Taylor series has superior convergence properties. Taylor series integration proves that it can provide rapid, highly accurate solutions to spacecraft trajectory problems.

15. SUBJECT TERMS
Taylor series; Spacecraft trajectory; Runge-Kutta method; Trajectory analysis; SNAP; Numerical integration; High order

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