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On an Asymptotically Consistent Unsteady Interacting Boundary Layer

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On an Asymptotically Consistent Unsteady Interacting Boundary Layer

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Abstract

This paper develops the asymptotic matching of an unsteady compressible boundary layer to an inviscid flow. Of particular importance is the velocity injection or transpiration boundary condition derived by this theory. It is found that in general the transpiration will contain a slope of the displacement thickness and a time derivative of a density integral. The conditions under which the second term may be neglected, and its consistency with the established results of interacting boundary layer are discussed.

Introduction

A variety of viscous/inviscid interaction schemes have been developed for a steady or unsteady boundary layer coupled with steady or unsteady inviscid flow. For various reasons (e.g. expediency, robustness, efficiency), engineering codes have frequently combined unsteady inviscid and steady boundary layer flow solvers. In such cases the coupling typically is treated as quasi-steady. The attempt will be made here to show that under the correct circumstances this approach is approximately valid. The larger question is the proper way to derive the inviscid velocity boundary condition imposed by an unsteady boundary layer. Heuristic treatments appeal to the similarity between the displacement thickness slope derived from boundary layer theory and the surface slope derived from thin airfoil theory. Based on this similarity, a time derivative of the displacement thickness is sometimes added to the viscous/inviscid coupling. This formulation has shown up from time to time, as for example a viscous-inviscid interaction model for cascades, [1] or as a recent publication apparently illustrates.[2] In contrast to this approach, the attempt will be made here to show that the method of matched asymptotic expansions applied to an unsteady interacting boundary layer (IBL) yields only a time derivative due to the density variation through the boundary layer. This result was first presented by LeBalleur, [3] and later by Bartels [4] [5] [6], and Epureanu [7].

The paper will derive the transpiration boundary condition for an unsteady compressible laminar boundary layer. The same form will be shown to apply to an unsteady turbulent boundary layer. The transpiration boundary condition will be derived for a surface with time and spatially varying height using the Prandtl transposition. Reduced forms of the transpiration condition will be shown in the limits of incompressibility and steady state, which are then shown to merge with other well known interacting boundary layer results. Finally, its relation to the IBL theory of boundary layer separation and other IBL methods will be discussed.

Laminar boundary layer matching

A laminar two-dimensional boundary layer is composed of two regions, an inner viscous and an outer inviscid but rotational region. The leading order boundary layer equations, matching conditions and the resulting injection velocity will be derived using the method of matched asymptotic expansions, developed for fluid dynamics by Van Dyke.[8]

The Navier-Stokes equations are non-dimensionalized with free stream density, velocity and characteristic length l that is of $O(1)$. The length scale on which the present interaction is derived is $x \sim O(1)$. The Reynolds number is defined such that $\mathrm{Re} = \rho u_0 l / \mu$. The non-dimensional inviscid flow field density, x and y velocities, pressure and temperature are expanded in the following way.

$$\begin{aligned}
\rho &\sim \rho_1(x,y,t) + \epsilon \rho_2(x,y,t) + \cdots && (1)\\
u &\sim u_1(x,y,t) + \epsilon u_2(x,y,t) + \cdots && (2)\\
v &\sim v_1(x,y,t) + \epsilon v_2(x,y,t) + \cdots && (3)\\
p &\sim p_1(x,y,t) + \epsilon p_2(x,y,t) + \cdots && (4)\\
T &\sim T_1(x,y,t) + \epsilon T_2(x,y,t) + \cdots
\end{aligned}$$

The scale parameter ϵ for a laminar boundary layer is $\mathrm{Re}^{-1/2}$. The vertical scale of the inviscid flow is $y \sim O(1)$. The inviscid mass and momentum continuity equations at leading order are

$$\begin{aligned}
\rho_{1t} + (\rho_1 u_1)_x + (\rho_1 v_1)_y &= 0 && (5)\\
(\rho_1 u_1)_t + (\rho_1 u_1^2)_x + (\rho_1 v_1 u_1)_y + p_{1x} &= 0 && (6)\\
(\rho_1 v_1)_t + (\rho_1 u_1 v_1)_x + (\rho_1 v_1^2)_y + p_{1y} &= 0 && (7)
\end{aligned}$$

Athough the energy equation can be similarly expanded, it is not necessary for the present purpose and will not be included. The boundary layer expansions have the following form

$$\begin{aligned}
\rho &\sim R_1(x,Y,t) + \epsilon R_2(x,Y,t) + \cdots && (8)\\
u &\sim U_1(x,Y,t) + \epsilon U_2(x,Y,t) + \cdots && (9)\\
v &\sim \epsilon V_1(x,Y,t) + \epsilon^2 V_2(x,Y,t) + \cdots && (10)\\
p &\sim p_1(x,Y,t) + \epsilon p_2(x,Y,t) + \cdots && (11)\\
T &\sim \Theta_1(x,Y,t) + \epsilon \Theta_2(x,Y,t) + \cdots
\end{aligned}$$

where the boundary layer scaled coordinate $Y = y\epsilon^{-1}$ is used. The leading order boundary layer equations are

$$\begin{aligned}
R_{1t} + (R_1 U_1)_x + (R_1 V_1)_Y &= 0 && (12)\\
(R_1 U_1)_t + (R_1 U_1^2)_x + (R_1 V_1 U_1)_Y + p_{1x} &= (\mu U_{1Y})_Y && (13)\\
p_{1Y} &= 0 && (14)
\end{aligned}$$

The method of matched asymptotic expansions requires that the Taylor series of the inviscid flow quantities as $y \to 0$ is matched with the boundary layer expansion as $Y \to \infty$. [8] The

Taylor series of the inviscid expansion is

$$\begin{aligned}
\rho &\sim \rho_1(x,0,t) + y\rho_{1y}(x,0,t) + \epsilon\rho_2(x,0,t) + \cdots \\
&\sim \rho_1(x,0,t) + \epsilon\left(Y\rho_{1y}(x,0,t) + \rho_2(x,0,t)\right) + \cdots \\
u &\sim u_1(x,0,t) + yu_{1y}(x,0,t) + \epsilon u_2(x,0,t) + \cdots \\
&\sim u_1(x,0,t) + \epsilon\left(Yu_{1y}(x,0,t) + u_2(x,0,t)\right) + \cdots \\
v &\sim v_1(x,0,t) + yv_{1y}(x,0,t) + \epsilon v_2(x,0,t) + \cdots \\
&\sim v_1(x,0,t) + \epsilon\left(Yv_{1y}(x,0,t) + v_2(x,0,t)\right) + \cdots \\
& \quad \mathit{etc...}
\end{aligned}$$

Matching of each quantity at successive orders as $y \to 0$ with the corresponding boundary layer quantities as $Y \to \infty$ yields the matching at leading order

$$\begin{aligned}
\rho_1(x,0,t) &= R_1(x,Y,t) && (15) \\
u_1(x,0,t) &= U_1(x,Y,t) && (16) \\
v_1(x,0,t) &= 0 && (17) \\
& \quad \mathit{etc...}
\end{aligned}$$

and

$$\begin{aligned}
\rho_2(x,0,t) + Y\rho_{1y}(x,0,t) &= R_2(x,Y,t) && (18) \\
u_2(x,0,t) + Yu_{1y}(x,0,t) &= U_2(x,Y,t) && (19) \\
v_2(x,0,t) + Yv_{1y}(x,0,t) &= V_1(x,Y,t) && (20) \\
& \quad \mathit{etc...}
\end{aligned}$$

Note that

$$R_{1Y} \to 0 \qquad (21)$$

as $Y \to \infty$ for density to remain finite in the limit. In view of equations 15, 16 and 21, the correspondence can be made between the continuity equations in the dual limits as $y \to 0$ and $Y \to \infty$ that

$$\begin{aligned}
-R_1V_{1Y} &= R_{1t} + (R_1U_1)_x && \mathit{in\ the\ Inner\ Limit} && (22) \\
&= \rho_{1t} + (\rho_1 u_1)_x = -(\rho_1 v_1)_y && \mathit{in\ the\ Outer\ Limit} && (23)
\end{aligned}$$

By making use of equations 15 and 17, equations 22 and 23 can be written

$$R_1V_{1Y} = \rho_1 v_{1y} \qquad (24)$$

or finally that

$$V_{1Y}(x,Y,t) = v_{1y}(x,0,t) \qquad (25)$$

as $Y \to \infty$. The expressions for velocity matching at leading order, equations 16 and 20 can be written

$$\begin{aligned}
u_1(x,0,t) &= U_1(x,Y,t) && (26) \\
v_2(x,0,t) &= V_1(x,Y,t) - YV_{1Y}(x,Y,t) && (27)
\end{aligned}$$

This is identical to the leading order velocity matching for a boundary layer derived by Van Dyke.[9] These two matching conditions are finite in the limit $Y \to \infty$ and provide, along with expressions for density, pressure and temperature, the necessary inviscid/viscous matching for a compressible unsteady boundary layer. The first of these states the widely recognized matching of the x-component of the inviscid and viscous velocities at the boundary layer edge. The second is the boundary layer contribution to the injection or transpiration velocity into the inviscid flow. The injection velocity can easily be rewritten into a more usable form. Integrating the boundary layer continuity equation 12 yields the result

$$\rho_1(x,0,t)V_1(x,Y,t) = R_1(x,Y,t)V_1(x,Y,t) = -\int_0^Y (R_{1t} + (R_1U_1)_x)d\widehat{Y} \tag{28}$$

in the limit as $Y \to \infty$. This integral is infinite in the limit as $\mathrm{Re} \to \infty$. However, by using the velocity matching of equation 27 combined with equations 5, 28 and 25, the finite condition

$$\begin{aligned} v_2(x,0,t) &= V_1(x,Y,t) - YV_{1Y}(x,Y,t) \\ &= \frac{1}{\rho_1}\left(\int_0^\infty ((\rho_1u_1)_x - (R_1U_1)_x)dY + \int_0^\infty (\rho_{1t} - R_{1t})dY\right) \\ &= \frac{1}{\rho_1}\left(\frac{\partial}{\partial x}\int_0^\infty (\rho_1u_1 - R_1U_1)dY + \frac{\partial}{\partial t}\int_0^\infty (\rho_1 - R_1)dY\right) \end{aligned}$$

is obtained. Using the definition of *displacement thickness* δ^*

$$\delta^* = \int_0^\infty (1 - \frac{R_1U_1}{\rho_1u_1})dY$$

and defining the new term, *density thickness* δ_R

$$\delta_R = \int_0^\infty (1 - \frac{R_1}{\rho_1})dY$$

the resulting injection velocity of an unsteady compressible boundary layer into the inviscid flow is

$$v_2(x,0,t) = \frac{1}{\rho_1}\left(\frac{\partial(\rho_1u_1\delta^*)}{\partial x} + \frac{\partial(\rho_1\delta_R)}{\partial t}\right) \tag{29}$$

This important result presents an asymptotically consistent treatment of the influence of an unsteady boundary layer on the outer inviscid flow. It is identical in form to the matching derived by Le Balleur [3] and later by Bartels [4], [6], [5] and Epureanu. [7] It states that the inviscid flow sees an injection velocity due to the sum of the slope of the displacement thickness and the time variation of the integral of density through the boundary layer.

Now several simplifications of this theory can be found. If the flow is unsteady but incompressible

$$\frac{\partial(\rho_1\delta_R)}{\partial t} = 0$$

and the injection velocity can be written

$$v_2(x,0,t) = \frac{\partial(u_1\delta^*)}{\partial x} \tag{30}$$

where the incompressible displacement thickness is

$$\delta^* = \int_0^\infty (1 - \frac{U_1}{u_1}) dY \ . \tag{31}$$

This is the widely used *incompressible* boundary layer velocity matching condition. If the density variation through the boundary layer is small or the time scale of the variation is long (e.g. low frequency or quasi-steady) such that

$$\frac{\partial(\rho_1 \delta_R)}{\partial t} \sim O(\epsilon)$$

the injection velocity can be written

$$v_2(x, 0, t) \approx \frac{1}{\rho_1} \frac{\partial(\rho_1 u_1 \delta^*)}{\partial x} \tag{32}$$

This is the widely used *quasi-steady* interaction. It is a reasonable approximation for many subsonic and transonic adiabatic flows. It is identical to the velocity condition for a steady compressible subsonic or supersonic boundary layer used by Davis [10].

Turbulent boundary layer matching

In contrast to the two layer structure of a laminar boundary layer, a turbulent boundary layer has three layers in the limit of large Reynolds number. Nevertheless, if the necessary equations of turbulent theory are identical to the corresponding equations of laminar theory it is possible to utilize the same matching conditions between the appropriate regions for laminar and turbulent boundary layers. This will be shown here. The present formulation follows that of Mellor. [11] (See also ref. [12])The turbulent boundary layer is expanded in the small parameter $\epsilon = u_t/u_0$ where u_t is friction velocity or turbulent velocity at some designated point and u_0 is a characteristic flow velocity, e.g. free stream mean flow velocity. The outer inviscid region is similar in nature to the inviscid region of a laminar boundary layer. It is expanded in a form identical to that of equations 1-4.

$$\begin{aligned}
\rho &\sim \rho_1(x, y, t) + \epsilon \rho_2(x, y, t) + \cdots && (33) \\
u &\sim u_1(x, y, t) + \epsilon u_2(x, y, t) + \cdots && (34) \\
v &\sim v_1(x, y, t) + \epsilon v_2(x, y, t) + \cdots && (35) \\
p &\sim p_1(x, y, t) + \epsilon p_2(x, y, t) + \cdots && (36) \\
T &\sim T_1(x, y, t) + \epsilon T_2(x, y, t) + \cdots && (37)
\end{aligned}$$

This expansion follows that of Mellor [11], except that temperature and density are also expanded. The turbulent stress expanded in ref. [11] in the inviscid, defect and viscous regions is not necessary for the present purpose. The first order inviscid equations are identical to equations 5 - 7.

$$\begin{aligned}
\rho_{1t} + (\rho_1 u_1)_x + (\rho_1 v_1)_y &= 0 && (38) \\
(\rho_1 u_1)_t + (\rho_1 u_1^2)_x + (\rho_1 v_1 u_1)_y + p_{1x} &= 0 && (39) \\
(\rho_1 v_1)_t + (\rho_1 u_1 v_1)_x + (\rho_1 v_1^2)_y + p_{1y} &= 0 && (40)
\end{aligned}$$

The term by term expansions in the defect region follows the form shown in equations 8-11.

$$\begin{aligned}
\rho &\sim R_1(x,Y,t)+\epsilon R_2(x,Y,t)+\cdots && (41)\\
u &\sim U_1(x,Y,t)+\epsilon U_2(x,Y,t)+\cdots && (42)\\
v &\sim \epsilon V_1(x,Y,t)+\epsilon^2 V_2(x,Y,t)+\cdots && (43)\\
p &\sim p_1(x,Y,t)+\epsilon p_2(x,Y,t)+\cdots && (44)\\
T &\sim \Theta_1(x,Y,t)+\epsilon\Theta_2(x,Y,t)+\cdots && (45)
\end{aligned}$$

As with the inviscid expansion, this follows that of Mellor [11] with the addition of temperature and density expansions. The coordinate normal to the wall is stretched according to $Y = y\epsilon^{-1}$. The leading order equations in the defect layer are

$$\begin{aligned}
R_{1t}+(R_1U_1)_x+(R_1V_1)_Y &= 0 && (46)\\
(R_1U_1)_t+(R_1U_1^2)_x+(R_1V_1U_1)_Y+p_{1x} &= 0 && (47)\\
p_{1Y} &= 0 && (48)
\end{aligned}$$

The defect layer equations have a zero normal pressure gradient as do the laminar boundary layer equations. The defect layer equations are inviscid at leading order and thus, in structure, appear as a subset of the laminar equations. In contrast to the inviscid outer region, it contains Reynolds stress terms at second order. The viscous layer at leading order also preserves zero normal pressure gradient. This is the layer from which the log law for a turbulent boundary layer is derived.

The matching of the inviscid and defect layers is the result of interest. At leading order it can be written, following Mellor [11]

$$\begin{aligned}
u_1(x,0,t) &= U_1(x,Y,t)\\
v_1(x,0,t) &= 0\\
v_2(x,0,t)+Yv_{1y}(x,0,t) &= V_1(x,Y,t)
\end{aligned}$$

For a compressible flow the matching of density, among other quantities, is also required. At leading order

$$\rho_1(x,0,t) = R_1(x,Y,t)$$

The matching conditions at leading order between the inviscid and defect layer for a turbulent boundary layer are thus identical to those matching the inviscid with a laminar boundary layer. Furthermore, since the mass continuity equations of the inviscid and defect layers for a turbulent boundary layer (equations 38 and 46) are identical to those for a laminar boundary layer (equations 5 and 12), the end result is that the injection velocity due to a turbulent boundary layer will have a form identical to that for a laminar boundary layer.

Boundary layer over an airfoil and the transfer of the transpiration boundary condition to the X-axis

This formulation follows that in Davis and Werle in which the baseline to which the boundary layer equations are transfered is the x-axis. See [13] , [14] and [15] for details. The geometry and coordinate systems used in the Prandtl transposition are shown in Figure 1.

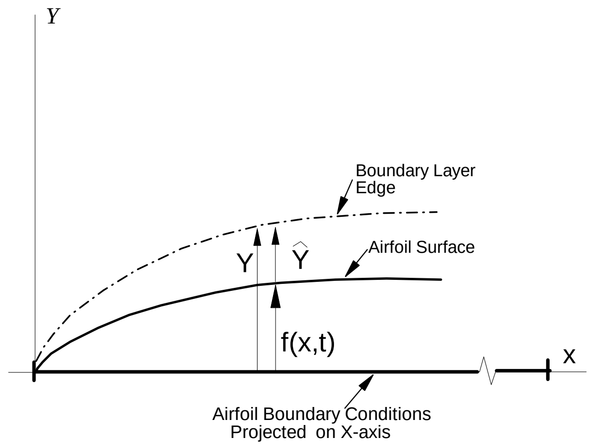

Figure 1: Geometry for Prandtl transposition

The Prandtl shift of the boundary layer to a boundary layer along the x-axis requires definition of the normal coordinate

$$Y = \widehat{Y} + f(x,t)$$

along with the normal boundary layer velocity

$$V(x,\widehat{Y},t) = V_1(x,Y,t) - f_x U_1(x,Y,t) - f_t$$

The continuity, momentum and energy equations can be shown to have the same form when transformed. The matching condition (equation 29) as $Y \to \infty$ is written

$$v_2(x,0,t) = V_1 - Y V_{1Y} = V - \widehat{Y} V_{\widehat{Y}} + u_1 f_x + f_t - f V_{\widehat{Y}}$$

But since it can be shown that

$$V - \widehat{Y} V_{\widehat{Y}} = \frac{1}{\rho_1}\left(\frac{\partial(\rho_1 u_1 \delta^*)}{\partial x} + \frac{\partial(\rho_1 \delta_R)}{\partial t}\right)$$

and

$$u_1 f_x + f_t - f V_{\widehat{Y}} = \frac{1}{\rho_1}\left[\frac{\partial(\rho_1 u_1 f)}{\partial x} + \frac{\partial(\rho_1 f)}{\partial t}\right]$$

as $Y \to \infty$, the resulting transpiration boundary condition is

$$v_2(x,0,t) = \frac{1}{\rho_1}\left(\frac{\partial(\rho_1 u_1(\delta^* + f))}{\partial x} + \frac{\partial(\rho_1(\delta_R + f))}{\partial t}\right)$$

This is the transpiration condition for a surface with height f transformed using the Prandtl transposition. For small time variation in the density and small spatial variation in $\rho_1 u_1$ this can be approximated as

$$v_2(x,0,t) = \frac{1}{\rho_1}\left(\frac{\partial(\rho_1 u_1 \delta^*)}{\partial x} + \frac{\partial(\rho_1 \delta_R)}{\partial t}\right) + u_1 f_x + f_t$$

Consistency with unsteady interactive boundary layer theory

It is useful to show that a proposed theory is consistent with other established results. Comparisons with the results of steady boundary layer asymptotics have been made earlier in this paper. Additional comparisons can be made with theories developed for unsteady boundary layer interaction. The present paper predicts for $\delta_R = 0$ that the unsteady laminar transpiration velocity for a problem with $x \sim O(1)$ is given by the slope of the displacement thickness only, i.e.

$$v(x,0,t) = \frac{1}{\mathrm{Re}^{-1/2}} \frac{\partial(u\delta^*)}{\partial x} \ .$$

This is consistent with the unsteady laminar IBL on the short length scale of a separation. On this scale an incompressible viscous lower deck interacts with the outer inviscid flow through the pressure-displacement relation

$$p(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A_s(s,t)}{x-s} ds$$

for a subsonic incompressible or compressible free stream involving the slope of the unknown displacement function $A(x,t)$. [16] For a supersonic free stream the pressure-displacement relation is $p = -A_x(x,t)$. [16], [17], [18]. This is also the interaction of an unsteady boundary layer separation at the incipient development of Tollmien-Schlichting waves. [14] Peridier, Smith and Walker apply this IBL theory to an $x \sim O(1)$ problem of a laminar incompressible boundary layer interacting with a vortex. [19], [20] To simplify somewhat the form, their equations 72-74 (ref. [20]) are rewritten here for a flat plate boundary layer in the absence of external excitation. The interaction at leading order is

$$u(x,0,t) = u_1(x,0,t) + \cdots = 1 + \frac{1}{\pi\,\mathrm{Re}^{-1/2}} \int_{-\infty}^{\infty} \frac{F(s,t)}{x-s} ds \qquad (49)$$

where the outer flow injection is induced by the displacement slope, i.e.

$$F(x,t) = \frac{\partial}{\partial x}\left(u_1(x,0,t)\delta^*(x,t)\right) \ . \qquad (50)$$

The displacement slope $F(x,t)$ is reproduced from ref. [20]. At the same time it and the velocity-displacement relationship (equation 49) are directly obtainable from the results derived in the present paper for a flat plate boundary layer in which $\delta_R = 0$.

The present theory is in agreement with the viscous-inviscid overlay or "defect formulation" of Le Balleur. [3] The velocity boundary condition to the inviscid flow is derived for an unsteady compressible flow. In the notation of ref. [3] the transpiration velocity is $w(x,0,t)$, boundary layer edge velocity and density are ρ and u. Using the inviscid and viscous continuity equations, the result presented on page 24 of ref. [3] is

$$\rho w(x,0,t) = \frac{\partial}{\partial t}\int_0^{\infty} (\rho - \overline{\rho}) dz + \frac{\partial}{\partial x}\int_0^{\infty} (\rho u - \overline{\rho u}) dz$$

where the quantities with the overbar are due to the boundary layer. Epureanu [7] presents the transpiration flux Q_{bl}

$$Q_{bl} = \frac{\partial(\rho_e \delta_\rho)}{\partial t} + \frac{\partial(\rho_e u_e \delta^*)}{\partial x} \ .$$

Other than notational differences ($\delta_\rho = \delta_R$ in the present notation), these expressions are equivalent and identical to equation 29 of the present paper.

Concluding remarks

This paper attempts to clear up misconceptions regarding the proper approach to coupling an unsteady boundary layer and inviscid flow solver. The general form of the transpiration velocity has been derived for laminar and turbulent boundary layers. Various simplifications of the transpiration velocity are also derived. The most general form of transpiration as well as the simplified forms are all demonstrated to be consistent with the results of other well established methods and are fully consistent with the asymptotic theory of unsteady incompressible and compressible boundary layer separation.

References

[1] Cizmas, P. G. A., Hall, K. C., "A Viscous-Inviscid Model of Unsteady Small-Disturbance Flows in Cascades," 31st AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, July 10-12, 1995, San Diego, CA.

[2] Batina, J., Advanced Small Perturbation Potential Flow Theory for Unsteady Aerodynamic and Aeroelastic Analyses, NASA TM-2005-213908, November 2005.

[3] Le Balleur, J. C., Viscid-Inviscid Coupling Calculations for Two and Three Dimensional Flows," Von Karman Institute for Fluid Dynamics Lecture Series, March 29-April 2, 1982, pg. 24.

[4] Bartels, R. E., *An Interacting Boundary Layer Method for Unsteady Compressible Flows*, PhD Thesis, Iowa State University, 1994.

[5] Bartels, R. E., "Flow and Turbulence Modeling and Shock Buffet Onset for Conventional and Supercritical Airfoils," NASA TP-1998-206908.

[6] Bartels, R. E., Rothmayer, A. P., "An IBL Approach to Multi-scaled Shock Induced Oscillation," AIAA 95-2157, 26th AIAA Fluid Dynamics Conference, June 1995, San Diego.

[7] Epureanu, B. I., Dowell, E. H., Hall, K. C., "Reduced-Order Models of Unsteady Transonic Viscous Flows in Turbomachinery," *Journal of Fluids and Structures*, Vol 14, 2000, pp. 1215-1234.

[8] Van Dyke, M., *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford, CA, 1975.

[9] Van Dyke, M. D., "Higher Approximations in Boundary-Layer Theory, Part 1: General Analysis," *Journal of Fluid Mechanics*, Vol. 14, 1962, pp. 161-177.

[10] Davis, R. T., "A Procedure for Solving the Compressible Interacting Boundary Layer Equations for Subsonic and Supersonic Flows," *AIAA 17th Fluid Dynamics, Plasma Dynamics and Lasers Conference*, Snowmass, CO, June 25-27, 1984, AIAA Paper 1984-1614.

[11] Mellor, G. L., "The Large Reynolds Number, Asymptotic Theory of Turbulent Boundary Layers," *International Journal of Engineering Science,* Vol. 10, 1972, pp. 851-873.

[12] Yajnik, K. S., "Asymptotic theory of turbulent shear flows," *Journal of Fluid Mechanics*, Vol. 42, 1970, pp. 411-427.

[13] Davis, R. T., Werle, M. J., "Progress on Interacting Boundary-Layer Computations at High Reynolds Number," Chapter 12 in *Numerical and Physical Aspects of Aerodynamic Flows*, Springer-Verlag, 1982.

[14] Duck, P. W., "Triple-Deck Flow over Unsteady Surface Disturbances: Three-Dimensional Development of Tollmien-Schlichting Waves," *Computers and Fluids*, Vol. 18, 1990, No. 1, pp. 1-34.

[15] Duck, P. W., "Unsteady triple-deck flows leading to instabilities," Proc. IUTAM Symp. on Boundary Layer Separation," 1986, Springer, Berlin (1987).

[16] Tutty, O. R., Cowley, S. J., "On the stability and the numerical solution of the unsteady interactive boundary-layer equation," *Journal of Fluid Mechanics*, Vol. 168, 1986, pp. 431-456.

[17] Hoyle, J. M., Smith, F. T., "On finite-time break-up in three-dimensional unsteady interacting boundary layers," *Proceedings of the Royal Society,* Vol. 447, 1994, pp. 467-492.

[18] Smith, F. T., Finite-time breakup can occur in any unsteady interacting boundary layer, *Mathematica,* Vol. 35, pp. 256-273.

[19] Peridier, V., Smith, F. T., Walker, J. D. A., Vortex-induced boundary-layer separation. Part 1. The limit problem Re $\rightarrow \infty$, *Journal of Fluid Mechanics*, Vol. 232, pp. 99-131.

[20] Peridier, V. J., Smith, F. T., Walker, J. D. A., "Vortex-induced boundary layer separation. Part 2, Unsteady interacting boundary-layer theory," *Journal of Fluid Mechanics,* Vol. 232, 1991, pp. 133-165.

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This paper develops the asymptotic matching of an unsteady compressible boundary layer to an inviscid flow. Of particular importance is the velocity injection or transpiration boundary condition derived by this theory. It is found that in general the transpiration will contain a slope of the displacement thickness and a time derivative of a density integral.The conditions under which the second term may be neglected, and its consistency with the established results of interacting boundary layer are discussed.

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