Numerical Strip-Yield Calculation of CTOD and CTOA

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I. Abstract

A recently developed numerical method based on the strip yield analysis approach was used to calculate the Crack Tip Opening Displacement (CTOD) and Crack Tip Opening Angle (CTOA) for a number of complex crack configurations of practical interest. This method is an adaptation of the dislocation-density based boundary element method. In this Boundary Element approach, crack-face opening displacements are obtained at any point on a crack using a series of definite integrals evaluated exactly in closed form for a variety of crack geometries, including infinite or finite extent, with arbitrarily applied loading conditions.

Keywords: Crack Tip Opening Displacement; CTOD; Crack Tip Opening Angle; CTOA; Strip Yield model; Dugdale model; Boundary Element Method; BEM; NASGRO

II. Introduction

The Crack Tip Opening Displacement (CTOD) or, equivalently, the Crack Tip Opening Angle (CTOA), has become a reliable and convenient fracture criterion with which to assess crack extension and directionality in materials in which the effects of the plastic zone at the crack tip must be accounted for, *e.g.* in materials of considerable ductility and toughness. Many materials have been found[1] to exhibit a constant critical CTOD (or CTOA) from crack initiation through failure, making CTOD (or CTOA) a useful and reliable fracture criterion.

Linear Elastic Fracture Mechanics (LEFM) was extended to materials in which significant plastic yielding occurred by Wells[2], who postulated that CTOD was the variable which governs crack extension. Wells further proposed that the CTOD was proportional to the overall tensile strain even after general yield was reached. This proposition was later verified to be true, and became the basis of the widespread use of the CTOD in a variety of elastic-plastic assessments.

The need for an explicit expression for the CTOD was filled when Dugdale[3] developed the strip yield model. Dugdale proposed that for a thin sheet loaded in tension, yielding would be limited to a narrow strip lying on the extension of the crack line, and that the effect of the cohesive yield loading would be to neutralize the stress singularity arising from the remote loading. Using Muskhelishvili[4] methods of complex-variable analysis, Dugdale's method of strip yield analysis obtained an explicit expression for the CTOD, while circumventing the need to perform full-blown elastic-plastic analysis. It thus offers an efficient approach to modeling elastic-plastic behavior through the superimposition of two elastic solutions.

Dugdale's original development of the strip yield model involved a thin plate of infinite extent under plane stress conditions. While closed-form solutions have been obtained for other infinite geometries, these have necessarily been limited to a few particular cases; and efforts to extend the method to a general-purpose technique for arbitrary finite geometries have been hampered because the needed elastic solutions, in general, can be obtained only through painstaking numerical computations that have to be tailor-made for the problem at hand.

While methods like the Weight function, Green's function and Collocation methods[5-9] have been used to develop solutions for finite geometries, these have been techniques developed for particular configurations that also do not lend themselves to general-purpose use. Weight function methods, for instance, while capable of extension to arbitrary loading situations, require the calculation of two or more reference solutions, and impose thereby a heavy computational burden, often unacceptable to the practicing fracture analyst.

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Finite-element analysis, while certainly general, also exacts a severe computational toll from the user: a full strip yield analysis requires determination of stress intensity for the actual (arbitrary) loading and the cohesive yield stress loading near the crack tip, together with the determination of the crack-face opening displacement at the junction of the load-free and yield-loaded parts. In addition, one must choose from several locations behind the crack tip which have been proposed for measuring CTOD: the first node[10-12], the second node[12,13], or the 45° intercept[14]. Finally, finite element modeling, especially when done for a range of crack lengths, imposes a severe re-meshing burden because of the multitude of meshes required, and is thus not readily adaptable to general-purpose strip yield analysis.

To overcome the aforementioned shortcomings, a computational method has been developed[15] to obtain crack-face opening displacements at any point on a crack face, and thus has the capability to use the strip-yield model for general two-dimensional crack problems. This technique adapts the dislocation-density based boundary-element method (BEM) developed by Chang and Mear[16] in which the crack line integrals are treated in a special way to eliminate the difficulties associated with the application of the standard BEM to fracture problems. The resulting integral equations contain integrals associated with the fracture in terms of a distribution of point dislocations along the crack line. With this adaptation, presently implemented in the NASGRO® fracture mechanics analysis software[17], the CTOD can be easily and accurately computed for arbitrary plane geometries and loadings, and the modeling requirements are free of the onerous modeling burden of other methods.

This paper makes use of this novel approach to calculate the CTOD for several complex crack configurations of practical interest.

III. Theory

A direct application of conventional BEM to fracture problems leads to a mathematically degenerate formulation due to the geometric proximity of the crack surfaces, and information about the tractions on the crack is lost[18]. This can be circumvented by developing an integral equation for the crack face tractions, for example, by using the displacement integral equation for interior points (*i.e.* Somigliana's identity) and applying the strain-displacement relations, Hooke's Law, and a proper limiting process as the interior point approaches the crack surface. However, the resulting relation contains a hypersingular kernel which requires a special interpretation and is challenging to evaluate numerically. In addition, this formulation requires that the gradients of the relative crack face displacements be continuous, a difficult and computationally intensive requirement to implement[19]).

In an alternative approach proposed by Chang and Mear[16], the hypersingularity is avoided by eliminating the Cauchy-type kernel in the displacement integral equation before forming the displacement gradients. This is accomplished in by an appropriate choice of stress function for the stress fundamental solution due to a point load, and an integration by parts to obtain a "modified" displacement integral equation. Also, the integrals over the upper and lower crack surfaces can be written as integrals over a single crack surface, *e.g.* the upper surface. This results in the crack face displacement terms being replaced by the relative crack face displacements ΔD, where ΔD is the difference between the displacements of the upper and lower surfaces, *i.e.* the crack-opening displacement. A traction relation for points on the crack faces is then easily derived from this modified displacement equation in the usual manner.

It can readily be shown that the boundary integral equations upon which this BEM is based consist of the conventional ordinary boundary terms (*i.e.* conventional displacement and traction fundamental solutions) and two additional terms which can be identified as distributions of point loads and of point dislocations along the crack lines.

In this BEM the gradients of the relative crack face displacements are described by the dislocation density function $A(\zeta)$ via

$$A(\zeta) = \frac{i\mu}{\pi(\kappa+1)} \frac{\partial[\Delta D(\zeta)]}{\partial s} \tag{1}$$

where the complex-valued relative displacement function is defined as

$$\Delta D = \Delta u_x + i\,\Delta u_y \tag{2}$$

In (2), ΔD is the difference in displacements between the upper and lower crack surfaces, u_x and u_y are the x- and y-components, respectively, of displacement, κ equals $3\text{-}4v$ for plane strain and $(3\text{-}v)/(1+v)$ for plane stress, μ and v are the shear modulus and Poisson's ratio, respectively, and s is the distance along the crack from the tip to the point of interest.

Integrating (2) yields nominally

$$\Delta D(\zeta) = \frac{\pi(\kappa+1)}{i\mu} \int_0^s A[\zeta(s)]ds \tag{3}$$

where evaluation of the integral requires the summing up of the contribution from each element between the crack tip and the point of interest.

In Ref. 16, the geometric interpolation along the crack line is piecewise linear; the crack line geometry over the $(j+1)$th element is approximated by

$$\zeta(t) = \frac{1-t}{2}\zeta_j + \frac{1+t}{2}\zeta_{j+1}; \quad -1 \le t \le 1 \tag{4}$$

The dislocation density function on each element is approximated by functions that contain the requisite singularity multiplied by an interpolating part that depends on the natural element coordinate t,

$$A(t) = A[\zeta(t)] = \frac{1}{2\sqrt{\alpha_j}}\left(\frac{1-t}{\sqrt{\rho_j + t}}A_j + \frac{1+t}{\sqrt{\rho_j + t}}A_{j+1}\right); \quad 1 \le j \le N/2 \tag{5}$$

$$A(t) = A[\zeta(t)] = \frac{1}{2\sqrt{\alpha_j}}\left(\frac{1-t}{\sqrt{\rho_j - t}}A_j + \frac{1+t}{\sqrt{\rho_j - t}}A_{j+1}\right); \quad N/2 < j \le N \tag{6}$$

where it is assumed that the number of elements is even. In (5) and (6), A_j are nodal quantities, which are obtained from the vector of unknowns; α_j is the half-length of the jth element, and $\rho_j = \beta_j / \alpha_j$ in which β_j is the element-wise arc length from the center of the jth element to the nearest crack tip. To ensure continuity of the dislocation density, the total arc length of the "left" portion of the crack must be equal to the total arc length of the "right" portion:

$$\beta_j + \alpha_j = \beta_{j+1} + \alpha_{j+1} \qquad \text{for } j = \tfrac{1}{2}N \tag{7}$$

Substituting (4), (5) and (6) in (3) allows evaluation of ΔD as an element-wise sum of definite integrals. Most of the integrals thus obtained involve integrands that are non-singular and well behaved, and are generically of the form

$$I_1(s) = \int_{-1}^{s} (1 \mp t)/\sqrt{\rho + t}\; dt \tag{8}$$

and

$$I_2(s) = \int_{-1}^{s} (1 \mp t)/\sqrt{\rho - t}\; dt \tag{9}$$

These are integrated exactly in closed form using elementary methods. The significant point is that even those integrals which have the crack tips at the end-points of integration (and consequently integrands that are singular) have been integrated exactly in closed form, yielding results that are independent of the integration technique employed.

Furthermore, the crack tip stress intensity factor (SIF) is directly related to the limiting value of the dislocation density function at the crack tip:

$$K_I + iK_{II} = -(2\pi)^{3/2} e^{i\lambda} \lim_{s\to 0}\left\lfloor \sqrt{s}\,\overline{A}(s) \right\rfloor \tag{10}$$

where λ is the angle of inclination of the unit tangent vector at the crack tip. For the linear crack discretization described above, (10) simplifies to

$$K_I + iK_{II} = -(2\pi)^{3/2} e^{i\lambda} \lim_{s\to 0}\left\lfloor \sqrt{s}\,\overline{A}_1 \right\rfloor \tag{11}$$

for the left crack tip, where A_I is the nodal value of A(t) at the left crack tip. A similar expression can be written for the right crack tip.

This approach can be applied to any crack geometry, of either infinite or finite extent, with arbitrary applied loading conditions, including mixed mode crack problems.

IV. Procedure

Dugdale proposed that all plastic deformation is concentrated in a strip in front of the crack and that this problem can be solved by superposition of two elastic solutions. The problem of a body with a crack of physical length *2a* and plastic zone size ρ at each crack tip is equivalent to the same body with a crack of physical length $2(a+\rho)$ where the portion of the crack flanks corresponding the plastic zones (of length ρ) are subject to a closure stress equal to the yield stress, as shown in Figure 1[20].

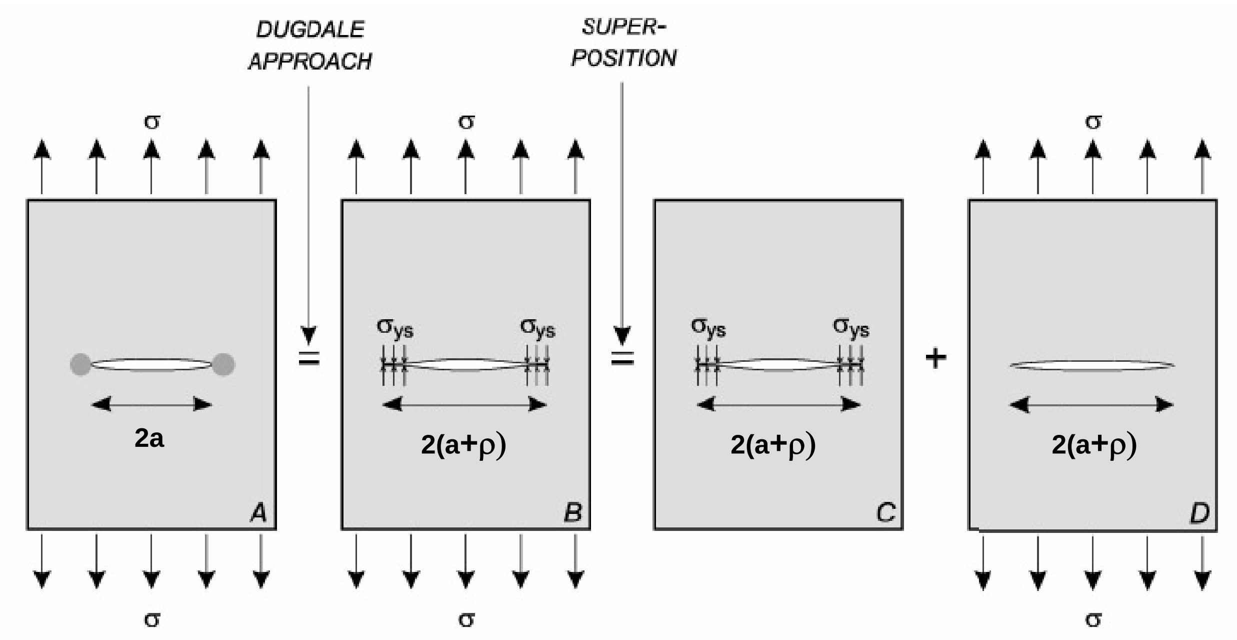


Figure 1. The Dugdale approach

The strip-yield model then is a superposition of the two elastic solutions in plates C and D wherein the cohesive yield loading on the crack flanks neutralizes the stress singularity arising from the remote loading.

The BEM analysis consists of constructing a model as shown in plate B with ρ chosen so that the SIF at the notional crack tip (*i.e.* at $a+\rho$) is 0. In practice, for a given yield stress, this can be achieved by either setting the plastic zone size and iterating on the remote stress, or setting the remote stress and iterating on the plastic zone size, until the SIF is zero. The former has the advantage of having to create a mesh for the crack and plastic zone just once, while the latter has the advantage of determining the CTOD for specific values of the remote stress chosen a priori. The CTOD value is then given by the crack face displacement at the intersection of the loaded and stress-free parts of the crack.

The BEM formulation used herein uses quadratic boundary elements and linear crack elements; an additional advantage of this formulation is that it generally requires far fewer elements for highly accurate stress intensity factor, stress, and displacement analysis than other BEM formulations. For example, for the crack configurations discussed in this paper, accurate results with errors of 3% or less can be obtained typically by using fewer than 20 elements each on the external and internal boundaries and crack line. The crack face loading discontinuity requires a finer mesh immediately to either side of the discontinuity than elsewhere on the crack, and it should be noted that in the absence of such a discontinuity even fewer crack elements would yield accurate results.

V. Results

In this section, several examples are presented to demonstrate the accuracy and versatility of this technique to calculate CTOD and CTOA. The examples include cases for finite and infinite domains, for center and edge cracks, and for complex configurations containing one or more holes. The verification cases are compared to well-known closed-form solutions as well as for experimental results for several common aerospace materials.

In each example, the crack was meshed with linear elements in several segments of differing element densities in order to adequately capture the discontinuity of the crack face loading. External and hole boundaries were meshed with quadratic elements, with comparatively sparse density required for accuracy, as discussed in the previous section.

A. Plastic zone size and CTOD for a crack in an infinite sheet

The first verification case is a simulation of the original Dugdale strip-yield model, a crack in an infinite sheet subject to remote uniform tension with cohesive loading equal to the yield stress on the portions of the crack faces of length ρ near each tip as shown in Figure 2.

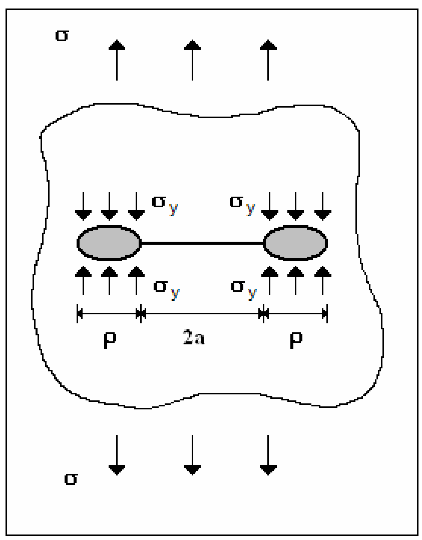


Figure 2. The Dugdale model - a crack in an infinite sheet with cohesive crack face loading near the crack tips

The physical crack is of length *2a* while the notional crack to be modeled is of length *2a+2ρ*. Results for a wide range of applied stress ratio are shown in Figure 3 for the CTOD (normalized by the yield stress σ_y, physical crack size *a*, and Young's modulus *E'*, where *E'* is *E* for plane stress and $E/(1-\nu^2)$ for plane strain) and for the relative crack size *c/a*, where *c= a+ρ*. In Dugdale's closed-form solution, the CTOD is given as

$$\frac{CTOD}{\sigma_y a / E'} = \frac{8}{\pi}\ln\left(\frac{c}{a}\right) \tag{12}$$

where the crack size ratio *c/a* is given by

$$\frac{c}{a} = \sec\left(\frac{\pi}{2}\frac{\sigma}{\sigma_y}\right) \tag{13}$$

Agreement between the BEM calculations and the Dugdale closed form solution is within 1%.

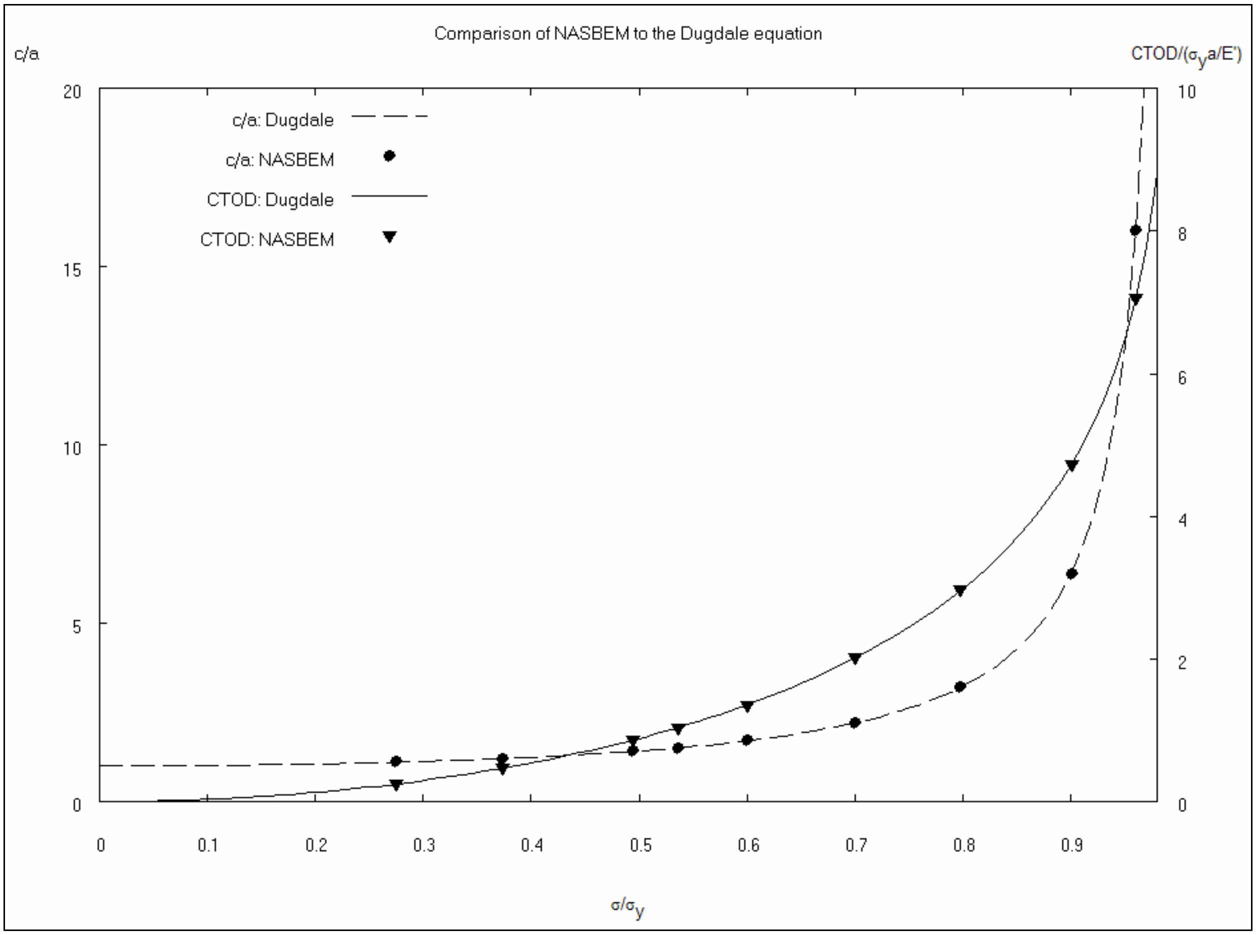


Figure 3. Comparison of the BEM to the Dugdale solution for relative crack size and normalized CTOD

B. Plastic zone size for a middle crack in a finite-width sheet

Figure 4 shows good agreement between NASBEM calculations and experimentally measured plastic zone sizes[21] for 0.020" thick AM350CRT steel sheet. As a comparison, the Dugdale and Irwin plastic zone sizes are also shown. For this combination of width and crack size the finite-width correction factor differs from the infinite sheet solution's value by less than 2%, so the small difference between the BEM calculations and the Dugdale infinite sheet solution is to be expected. Also, as is expected, the Dugdale and Irwin solutions bound the experimental results.

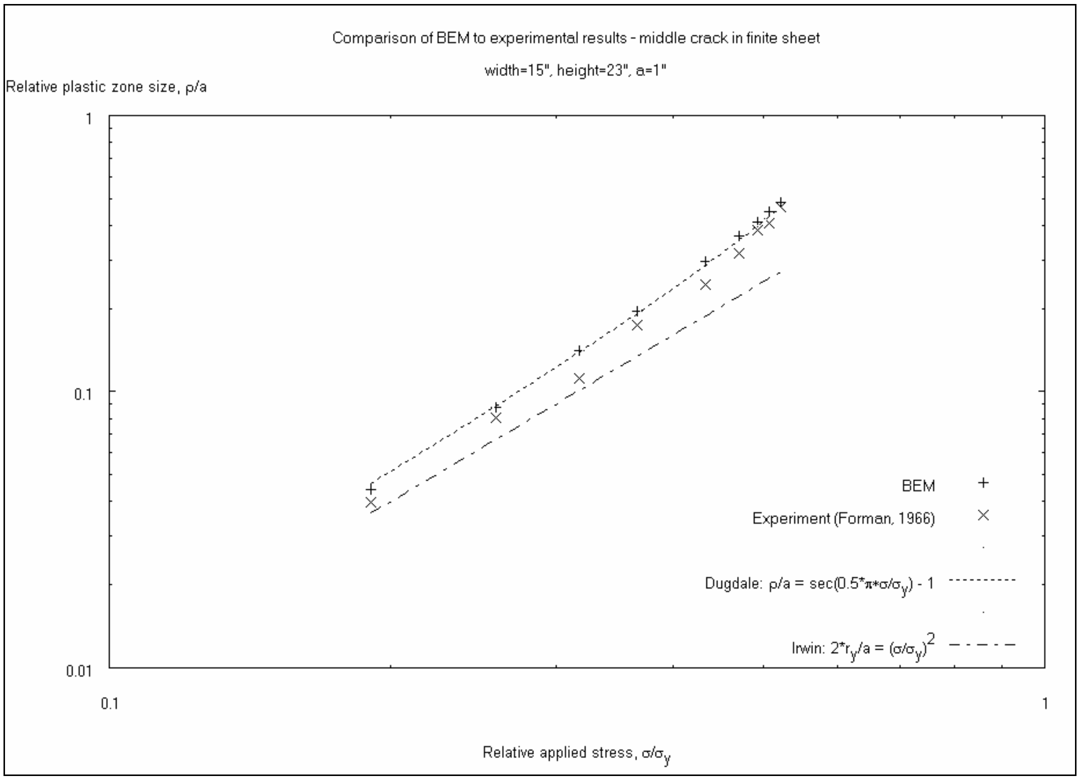


Figure 4. Comparison of the BEM to experimentally measured plastic zone sizes

C. Plastic zone size and CTOD for periodic cracks in an infinite sheet

One of the long-term goals of the current research is to demonstrate the viability of using the Dugdale technique as implemented in NASGRO® to calculate CTOD and CTOA for use in characterizing the fracture behavior and linkup of multiple cracks in very wide panels such as fuselage panels. To this end, Tada's solution for periodic crack in an infinite sheet[22] was chosen as a validation case.

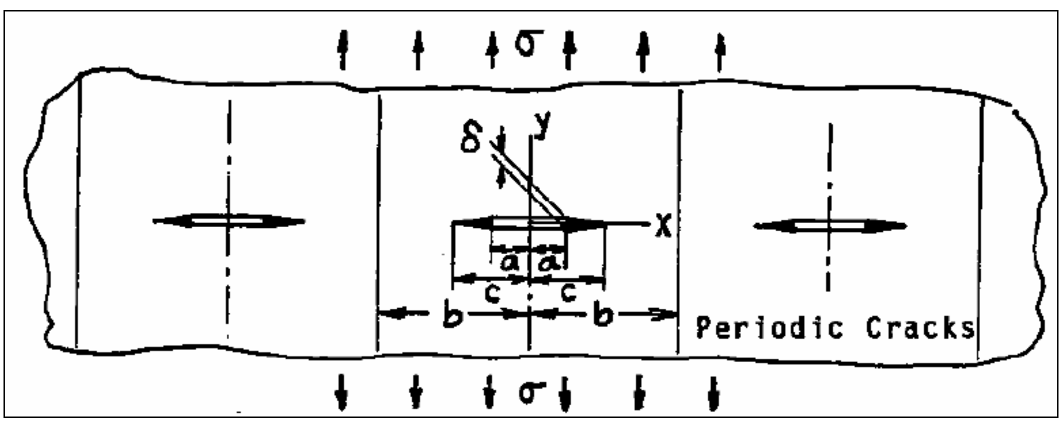


Figure 5. Configuration of the case of periodic cracks in an infinite sheet

This case presents some unique issues of practical concern to the analyst, namely, how to model an infinite number of cracks? Guidance was taken from literature on stress intensity factor analysis of stiffened fuselage panels[23] in which it was found that a periodic set of stiffeners in an infinite sheet could be reasonably approximated by seven stiffeners. In this vein it is postulated that a periodic set of cracks in an infinite sheet can be adequately represented by 7 cracks. As a practical matter, however, due to program size limitations the outer two cracks on each side were combined to form a single crack of double size, so that this case was in fact modeled as 5 collinear cracks. As can be seen in Figures 6 and 7, the computational results match the closed-form results very well for a wide range of crack size to crack spacing ratios, both for computed plastic zone sizes as well as for CTOD.

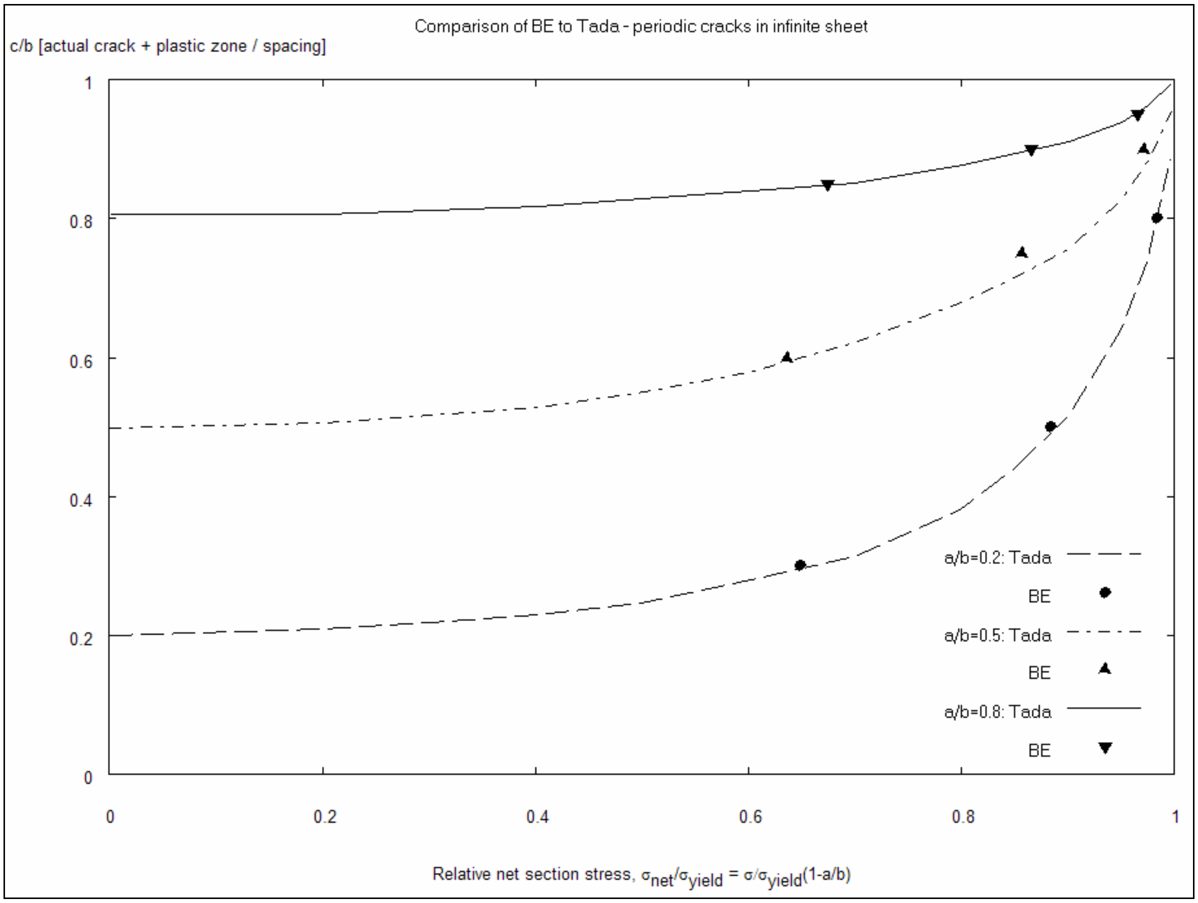


Figure 6. Comparison of the BEM to Tada's solution for periodic cracks in an infinite sheet for crack size to crack spacing ratios

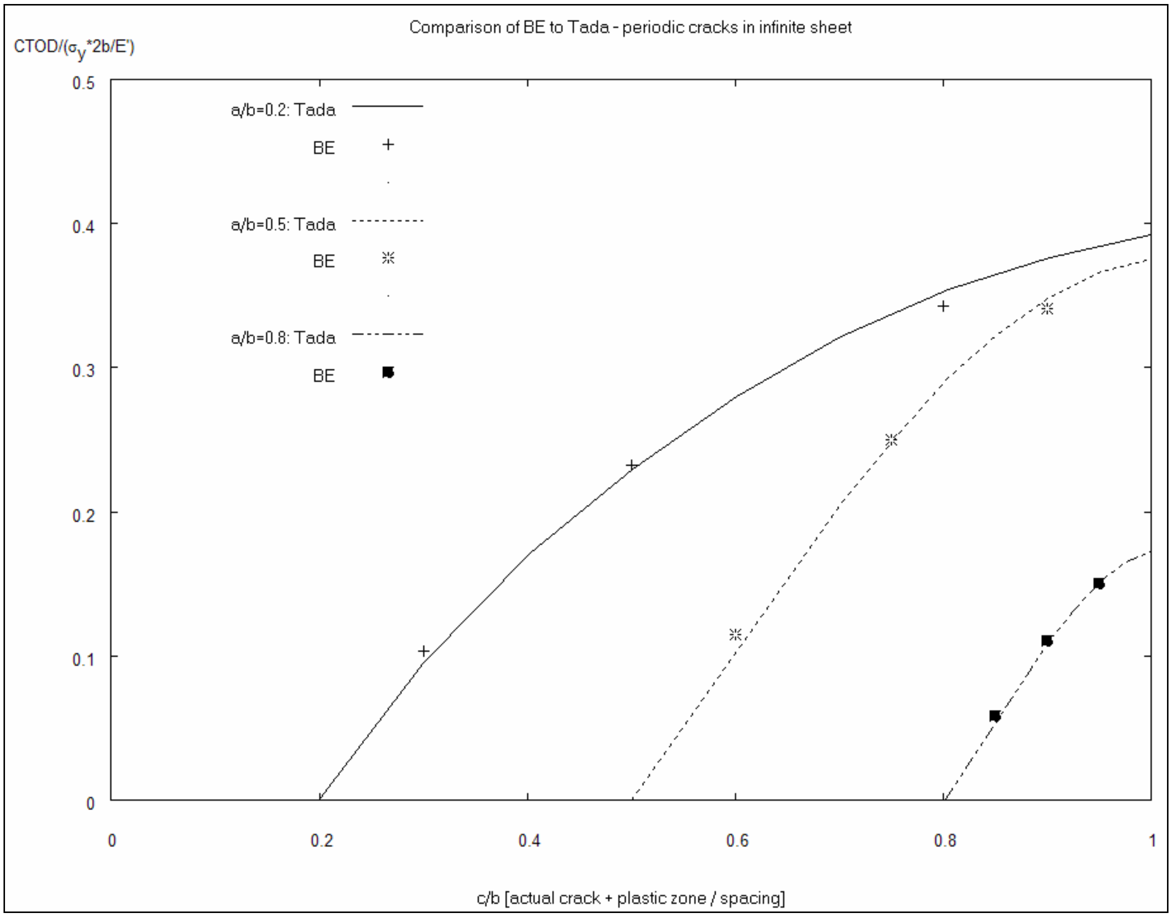


Figure 7. Comparison of the BEM to Tada's solution for periodic cracks in an infinite sheet for normalized CTOD

D. CTOA at onset of stable tearing for a middle crack in a finite-width sheet

The second long-term goal of the current research is to use the Dugdale technique to predict fracture when there is large stable tearing by using the critical CTOA. CTOA is often seen as a better fracture criterion due to its local nature than global measures such is the *J*-integral, which can be specimen and crack size dependent.

Figure 8 shows a typical distribution of CTOA values with respect to stable crack extension for Al 2024-T3 M(T) and C(T) specimens[24]. After an initial transient region (attributable to constraint and tunneling issues), the critical CTOA is constant over a large range of crack extension. It is this CTOA that is intended to be predicted in future research, and while one can calculate CTOA from CTOD, the Boundary Element code in NASGRO® in its present state is not yet able to simulate stable tearing.

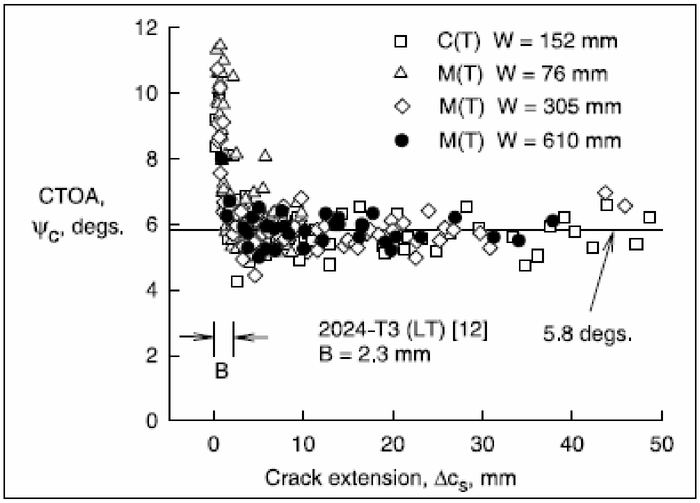


Figure 8. Typical distribution of experimentally measured CTOA as a function of stable crack extension

In the preceding comparisons CTOD was taken as the crack opening displacement at the tip of the physical crack, *i.e.* at the intersection of the traction-free and the yield-loaded portions of the crack. As a practical matter, however, CTOD and CTOA are typically measured at some distance behind the physical crack tip, usually 0.02-0.06" (0.5-1.5mm). CTOA is calculated from CTOD by the well-known formula

$$CTOA = 2\tan^{-1}\left(\frac{CTOD}{2d}\right) \tag{14}$$

where d is the distance behind the crack tip at which CTOD is measured.

Figure 9 shows that the CTOA calculated from the CTOD computed in the BEM at onset of stable tearing is reasonably constant for a range of initial crack sizes, and the values are similar to the data presented in Figure 8.

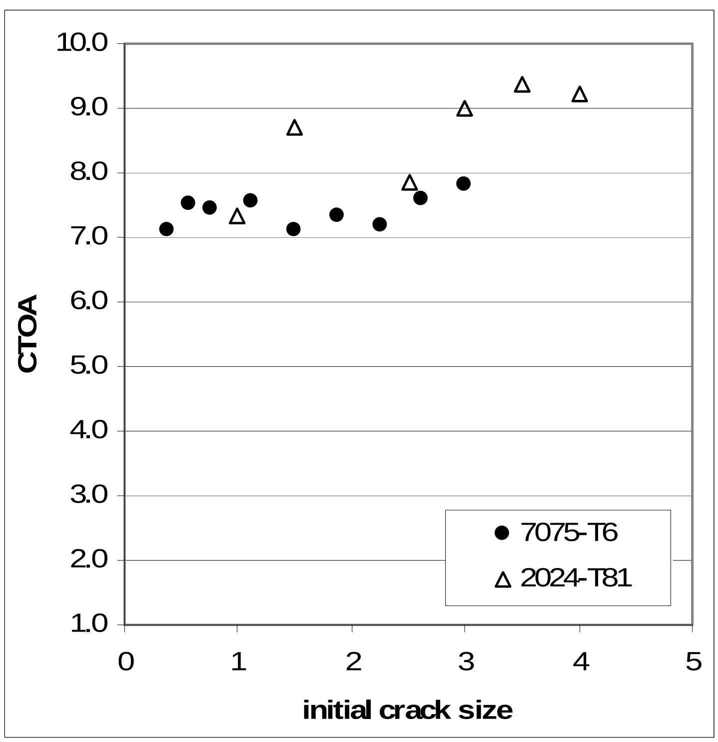
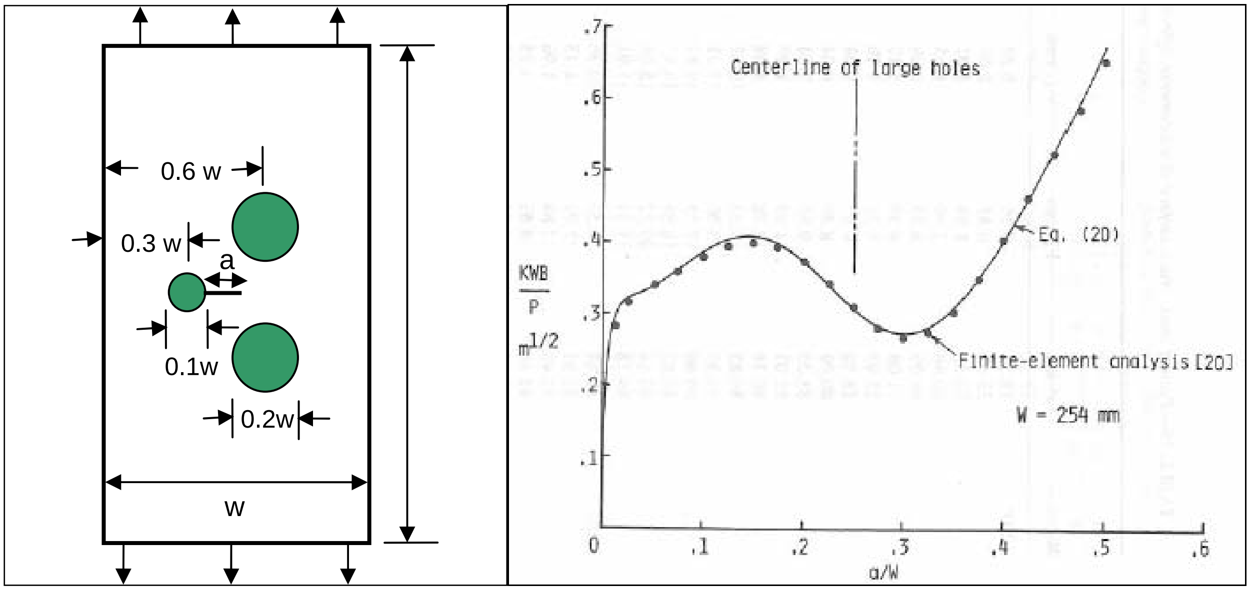


Figure 9. BEM calculation of CTOA at onset of stable growth for 0.060" M(T) specimens

E. CTOA at onset of stable tearing for a three-hole tension specimen

The final case studied is that of a three-hole tension specimen. This specimen, shown in Figure 10, was designed to simulate cracks in stiffened panels, and Figure 11 shows the complexity of the stress intensity factor as a function of crack length.



Figures 10 and 11. Configuration of the three-hole tension specimen and variation of the stress intensity factor with crack size[24]

Figure 12 shows the CTOA calculated by the BEM at onset of stable tearing for a number of initial crack sizes, and comparison to test results[24] in Figure 13 shows that the BEM values are within range. Also shown is the variation of failure load with initial crack size.

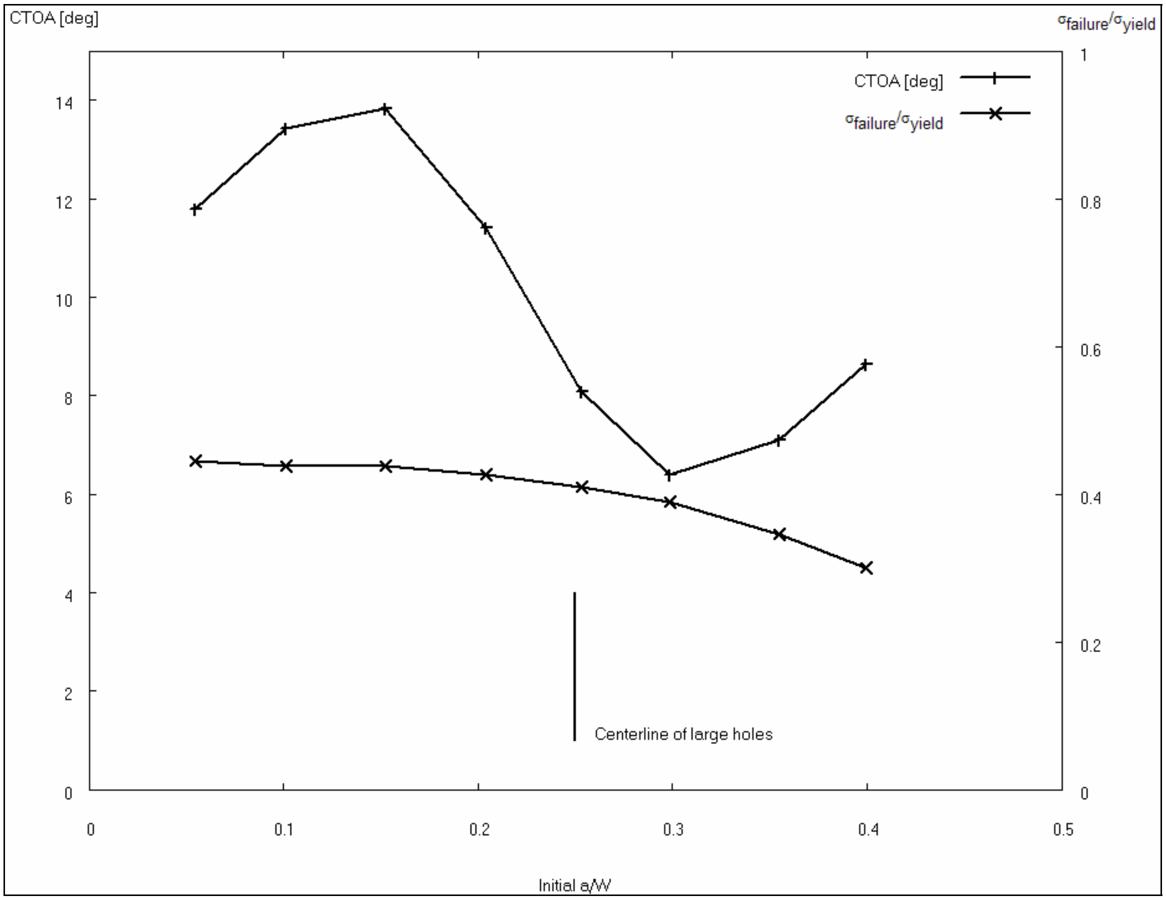


Figure 12. BEM calculation of CTOA at onset of stable growth
and corresponding failure stresses for three-hole tension specimen

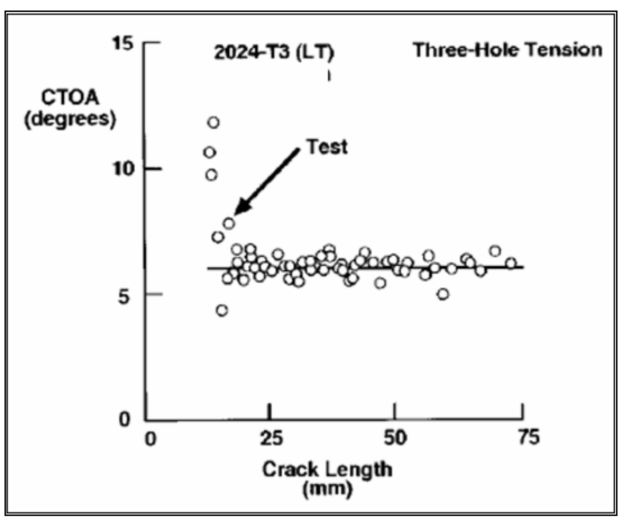


Figure 13. Experimentally measured CTOA as a function of stable crack extension
for a three-hole tension specimen[24]

VI. Summary

The feasibility of a new numerical method for calculating the Crack Tip Opening Displacement (CTOD) and Crack Tip Opening Angle (CTOA) was investigated. This new technique adapted the Boundary Element / Dislocation Density Method to obtain crack-face opening displacements at any point on the crack. Most of the methods in the literature used to calculate CTOD are complicated and resource intensive, and many of them apply only to specific geometries and loading conditions. The new approach developed in this paper was successfully applied to a number of different crack configurations including finite and infinite geometries containing complex geometrical features and arbitrary applied loading conditions. Results were obtained with reasonable accuracy, requiring a minimum of computational time and resources.

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