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Derivation of Formulations 1 and 1A of Farassat

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March 2007

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Derivation of Formulations 1 and 1A of Farassat

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Summary

Formulations 1 and 1A are the solutions of the Ffowcs Williams-Hawkings (FW-H) equation with surface sources only when the surface moves at subsonic speed. Both formulations have been successfully used for helicopter rotor and propeller noise prediction for many years although we now recommend using Formulation 1A for this purpose. Formulation 1 has an observer time derivative that is taken numerically, and thus, increasing execution time on a computer and reducing the accuracy of the results. After some discussion of the Green's function of the wave equation, we derive Formulation 1 which is the basis of deriving Formulation 1A. We will then show how to take this observer time derivative analytically to get Formulation 1A. We give here the most detailed derivation of these formulations. Once you see the whole derivation, you will ask yourself why you did not do it yourself!

1- Primary References

Formulation 1 was first published in:

1- F. Farassat: Theory of Noise Generation From Moving Bodies With an Application to Helicopter Rotors, NASA Technical Report R-451, 1975

However, the derivation of this formulation in this reference is very difficult. Later, I found a much simpler derivation which we will follow here. It was published in:

2- F. Farassat: Linear acoustic formulas for calculation of rotating blade noise, *AIAA Journal*, 19(9), 1981, 1122-1130

Formulation 1A was first published in:

3- F. Farassat, G. P. Succi, A review of propeller discrete frequency noise prediction technology with emphasis on two current methods for time domain calculations, 1980, *Journal of Sound and Vibration,* 71(3), 399-419

In deriving the formulation in the above reference, I assumed, as it was common then, that the blade surface did not move out of the disc plane. This assumption was removed by Kenneth S. Brentner and was published in the following excellent NASA Technical Memorandum:

4- Kenneth S. Brentner, Prediction of Helicopter Discrete Frequency Rotor Noise- A Computer Program Incorporating Realistic Blade Motions and Advanced Formulation, October 1986, NASA TM 87721

We refer to the result published in this paper as **Formulation 1A**. At the time of writing **Reference 3**, we at Langley had not started numbering and identifying our formulations. In hindsight, it was a good idea to identify each of our formulations because different codes use different results. Our numbering system has worked for identifying what formulation is used in a code. Formulation 1A has been used in helicopter rotor noise prediction codes **WOPWOP** and **PSU-WOPWOP**, and together with our supersonic **Formulation 3**, in our advanced propeller noise prediction code **ASSPIN** (Advanced Subsonic and Supersonic Propeller Induced Noise).

You will need much mathematical maturity to follow what we discuss here. To understand the mathematics behind our work, I strongly recommend reading **Reference 1**, above, and the following two NASA publications:

5- F. Farassat, Introduction to Generalized Functions With Applications in Aerodynamics and Aeroacoustics, May 1994 (Corrected April 1996), NASA Technical Paper 3428

6- F. Farassat: The Kirchhoff Formulas for Moving Surfaces in Aeroacoustics - The Subsonic and Supersonic Cases, NASA Technical Memorandum 110285, September 1996

Reference 6 was written to clarify and fill in some gaps in the mathematical analysis of **Reference 5**. It should be read together with the latter reference.

2- The Ffowcs Williams-Hawkings (FW-H) Equation

This equation was first published in:

7- J. E. Ffowcs Williams and D. L. Hawkings: Sound generated by turbulence and surfaces in arbitrary motion, *Philosophical Transactions of the Royal Society*, A264, 1969, 321-342

This paper is very difficult to read because of the high level of mathematics needed to follow the authors' reasoning. There are many original ideas in this paper such as the idea of imbedding a problem in a larger domain to use the available Green's function of the larger domain to solve the original problem. Another idea put forward by Ffowcs Williams and Hawkings is that conservation laws in differential form are also valid when all ordinary derivatives are viewed as generalized derivatives. This has important implications about the jump conditions across flow discontinuities. I have given the background mathematics needed to understand this paper in **References 1, 5** and **6**.

Ffowcs Williams-Hawkings (FW-H) Equation as originally proposed in **Reference 7** above:

$$\boxed{\square^2 p' = \frac{\partial}{\partial t}[\rho_0\, v_n\, \delta(f)] - \frac{\partial}{\partial x_i}[p\, n_i\, \delta(f)] + \frac{\partial^2}{\partial x_i\, \partial x_j}[H(f)\, T_{ij}]} \tag{1}$$

Here and elsewhere in the paper, we will use the summation convention on the repeated index. In this equation $\square^2$ is the wave or D'Alembertian operator in three dimensional space. The moving surface is described by $f(\boldsymbol{x}, t) = 0$ such that $\nabla f = \boldsymbol{n}$, $\boldsymbol{n}$ is the unit *outward* normal. This assumption implies that $f > 0$ outside the moving surface **(see Figure 1)**. Also $p' = c^2 \rho' = c^2(\rho - \rho_0)$, c and ρ_0 are the speed of sound and density in the undisturbed medium, respectively. Note that p' can only be interpreted as the acoustic pressure if $\rho' / \rho_0 << 1$. The symbols v_n, p and $T_{ij} = \rho\, u_i\, u_j - \sigma_{ij} + \left(p' - c^2 \rho'\right) \delta_{ij}$ are the local normal velocity of the surface, the local gage pressure on the surface (in fact $p - p_0$), and the Lighthill stress tensor, respectively. In the definition of the Lighthill stress tensor, σ_{ij} is the viscous stress tensor and δ_{ij} is the Kronecker delta. The Heaviside and the Dirac delta functions are denoted $H(f)$ and $\delta(f)$, respectively. In the second term on the right of eq. (1), we have neglected the viscous shear force over the blade surface acting on the fluid.

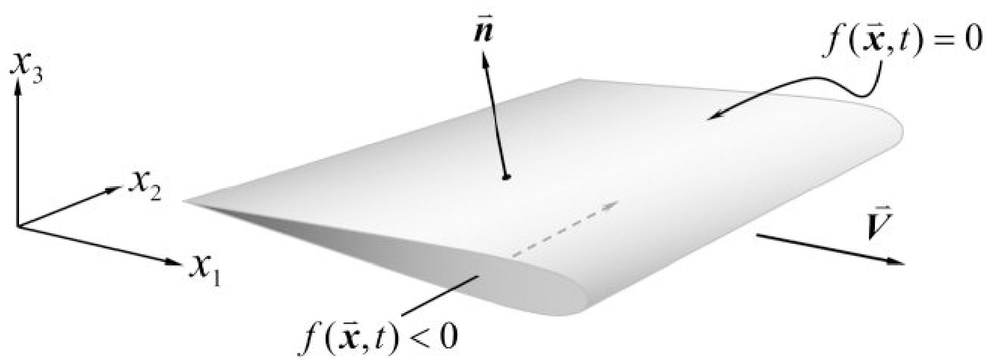

Figure 1- The definition of the moving surface implicitly as $f(\boldsymbol{x}, t) = 0$. Note that $\nabla f = \boldsymbol{n}$ where $\boldsymbol{n}$ is the unit outward normal to the surface.

We note that we have artificially converted a nonlinear problem of noise generation by a moving surface to a linear problem by using the acoustic analogy. All the nonlinearities are lumped into the Lighthill stress tensor which is assumed known from near field aerodynamic calculations. When we started working on helicopter and propeller noise in the early seventies, because of the limitations of digital computers, the most we could expect from aerodynamic calculations was the blade surface pressure. For this reason, using some physical reasoning, we neglected the quadrupole volume sources in FW-H equation and concentrated on development of formulations for the prediction of thickness and loading noise. Later on, as computers became more powerful, we included quadrupoles in our noise prediction. There was, however, another theoretical advance which led to the use of purely surface sources to which we will turn next.

It was Ffowcs Williams himself who proposed to use a penetrable (porous or permeable) data surface to account for nonlinearities in the vicinity of a moving surface. We again assume that the penetrable surface defined by $f(\boldsymbol{x}, t) = 0$ and the fluid velocity is denoted by $\boldsymbol{u}$. The **FW-H equation for penetrable (permeable, porous) data surface, FW-H$_{\text{pds}}$**, is:

$$\boxed{\square^2 c^2 \rho' \equiv \square^2 p' = \frac{\partial}{\partial t}[\rho_0 U_n]\,\delta(f) - \frac{\partial}{\partial x_i}[L_i\,\delta(f)] + \frac{\partial^2}{\partial x_i\,\partial x_j}[T_{ij}\,H(f)]} \tag{2}$$

We have used the following notations in the above equation:

$$U_n = \left(1 - \frac{\rho}{\rho_0}\right) v_n + \frac{\rho\, u_n}{\rho_0} \tag{3}$$

$$L_i = p\,\delta_{ij}\,n_j + \rho\,u_i\,(u_n - v_n) \tag{4}$$

where δ_{ij} is the Kronecker delta. As in the case of eq. (1), in the first term on the right of eq. (4), we have neglected the viscous shear force over the data surface acting on the fluid exterior to the surface. The philosophy behind using FW-H_{pds} is to locate the data surface $f = 0$ to enclose a moving surface, in such a way that all quadrupoles producing non-negligible noise are included within this surface. Therefore, no volume integration of the quadrupoles outside the data surface is necessary. High resolution CFD calculation is performed in the near field region (including turbulence simulation if broadband noise prediction is required). The data surface used for the acoustic calculation should be located within the region of high resolution CFD computation. The optimal location of the data surface, particularly when vortices cross the surface, is still the subject of research. One would like this surface to be as small as possible, because of the computer intensive nature of aerodynamic and turbulence simulation that require fine grid sizes and small time steps.

We will be concerned with the solution of the following two wave equations:

$$\boxed{\square^2 p'_T = \frac{\partial}{\partial t}[\rho_0\,v_n\,\delta(f)]} \qquad \textbf{Thickness Noise Equation} \tag{5}$$

$$\boxed{\square^2 p'_L = -\frac{\partial}{\partial x_i}[p\,n_i\,\delta(f)]} \qquad \textbf{Loading Noise Equation} \tag{6}$$

The source terms on the right of eqs. (5) and (6) are known also as **monopole** and **dipole** sources, respectively. This terminology is misleading for sources on a moving surface because the radiation patterns calculated from the above equations are not similar to those from stationary monopoles and dipoles. In fact, we can show mathematically that the thickness noise is equivalent to the loading noise from a uniform surface pressure distribution of magnitude $\rho_0\,c^2$. For this reason we avoid using the terminology of monopole and dipole sources here.

Remark 1- We will now give an example of a moving surface defined implicitly by $f = 0$. A sphere of radius a moving at the speed v along the x_1-axis is described by the equation:

$$f_1 = (x_1 - v\,t)^2 + x_2^2 + x_3^2 - a^2 = 0 \tag{7}$$

Note that $f_1 > 0$ outside the surface, and $f_1 < 0$ inside it. Now let us test the gradient of this function. We see that

$$\nabla f_1 = (2\,(x_1 - v\,t),\, 2\,x_2,\, 2\,x_3) \neq \boldsymbol{n} = \left(\frac{x_1 - v\,t}{a}, \frac{x_2}{a}, \frac{x_3}{a}\right) \tag{8}$$

We have $|\nabla f_1| = 2\sqrt{(x_1 - v\,t)^2 + x_2^2 + x_3^2}$. Therefore, we redefine the moving sphere by the equation:

$$f(\boldsymbol{x}, t) = \frac{f_1}{|\nabla f_1|} = \frac{1}{2}\sqrt{(x_1 - v\,t)^2 + x_2^2 + x_3^2} - \frac{a^2}{2\sqrt{(x_1 - v\,t)^2 + x_2^2 + x_3^2}} = 0 \tag{9}$$

We give the following general rule: if $\nabla f \neq \boldsymbol{n}$ on the surface $f = 0$, then redefine the *same* surface by the implicit equation $f/|\nabla f| = 0$. We can then show that $\nabla(f/|\nabla f|) = \boldsymbol{n}$ *on this surface*. The proof is simple.

Note that a surface can be defined implicitly in more than one way satisfying the condition $\nabla f = \boldsymbol{n}$. For example, for the above surface a much simpler representation in implicit form satisfying this condition is $f(\boldsymbol{x}, t) = \sqrt{(x_1 - v\,t)^2 + x_2^2 + x_3^2} - a = 0$.

We have not said anything about why we require the condition $|\nabla f| = 1$ on the surface. The FW-H equation as it was originally written in **Reference 7** had the term $|\nabla f|$ in both the thickness and loading terms. We have been assuming $|\nabla f| = 1$ in writing the FW-H equation in all of our publications for many years. To begin with, we notice that the term $|\nabla f|$ does not appear in both Formulations 1 and 1A. In the process of the derivation of some of our more advanced formulations where the source time and space derivatives of $|\nabla f|$ were required, I noticed that *all terms associated with these derivatives* canceled out exactly in the final result. After considerable search for the reason or reasons behind this cancellation, I discovered that $|\nabla f| = 1$ could be assumed to be true on any surface described implicitly from the beginning of the derivation of FW-H equation. This explained the reason for cancellation of the terms associated with the derivatives of $|\nabla f|$ in our advanced formulations. Although this assumption is of little value in the derivation of Formulation 1, it reduces the algebra somewhat in the derivation of Formulation 1A and enormously in our advanced formulations. (End of Remark 1)

3- Background Material and the Details of the Derivation

3.1- Green's Function of the Wave Equation in Unbounded Three Dimensional Space

The Green's function of the wave equation in the unbounded three dimensional space is:

$$\boxed{G(\boldsymbol{x}, t; \boldsymbol{y}, \tau) = \begin{cases} 0 & \tau > t \\ \delta(\tau - t + r/c)/4\pi r & \tau \leqslant t \end{cases}} \tag{10}$$

where $r = |\boldsymbol{x} - \boldsymbol{y}|$. Here $(\boldsymbol{x}, t)$ and $(\boldsymbol{y}, \tau)$ are the **observer** and the **source space-time variables**, respectively. In the above equation, the symbol $\delta(\cdot)$ stands for the Dirac delta function which is the most well-known generalized function. We usually use the following symbol: $g = \tau - t + r/c$. We will discuss in **Remark 3** below how we visualize and interpret the function $g = 0$. See also **References 1** and **5** above.

The Green's function given by eq. (10) is also known as the **free-space Green's function**.

Remark 2- It is a good idea to remember the following two simple results because they are frequently encountered in algebraic manipulations:

$$\frac{\partial r}{\partial x_i} = \hat{r}_i \quad \text{and} \quad \frac{\partial r}{\partial y_i} = -\hat{r}_i \tag{11}$$

where $\hat{r}_i$ is the component of the unit radiation vector $(\boldsymbol{x} - \boldsymbol{y})/r$. (End of Remark 2)

Remark 3- The surface $g = \tau - t + r/c = 0$ can be visualized as follows. Keep the observer space-time variables fixed. Then let us first rewrite the equation of this surface as $|x - y| = c(t - \tau)$. In the source space-time variables, this surface is simply a sphere with center at $\boldsymbol{x}$ and radius equal to $c(t - \tau)$. Note that the Green's function of the wave equation is nonzero when $\tau \leqslant t$, so that as the source time increases from $\tau = -\infty$ to $\tau = t$, the radius of this sphere shrinks from infinitely large value to zero. For this reason this sphere is known as the **collapsing sphere.** The *rate of contraction* of the radius of this sphere is the speed of sound c.

Mathematically, the collapsing sphere is the **characteristic cone** of the wave equation with the vertex at $(\boldsymbol{x}, t)$ and the cone pointing towards the past. Causality rules out using the part of the cone pointing towards the future. Can you think of a reason why this surface is called a cone? See **Reference 6**, above for the answer. (End of Remark 3)

3.2- Solution of the Wave Equation With Sources on a Moving Surface

We are interested in the solution of the equation:

$$\boxed{\Box^2 p' = Q(\boldsymbol{x}, t)\,\delta(f)} \tag{12}$$

We will derive the solution of this equation using *the free-space Green's function* given by eq. (10). This will show the interconnection between the variables that confuses novices in the field of aeroacoustics. The formal solution of eq. (12) is:

$$4\pi\, p'(\boldsymbol{x}, t) = \int Q(\boldsymbol{y}, \tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\boldsymbol{y}\,d\tau \tag{13}$$

The limits of the integral in this equation are given below:

$$\int \ldots .. \, d\boldsymbol{y}\, d\tau = \int_{-\infty}^{t}\int_{\mathbb{R}^3} \ldots ..\, d\boldsymbol{y}\, d\tau = \int_{-\infty}^{t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots ..\, dy_1\, dy_2\, dy_3\, d\tau \tag{14}$$

We emphasize at this point that the $\boldsymbol{x}-$frame and the $\boldsymbol{y}-$frame are fixed to the undisturbed medium. For the next analytic step, such a frame is not the most appropriate frame to interpret the solution of eq. (12) as we will show below. It is important to remember that from now on, the observer space-time variables $(\boldsymbol{x}, t)$ are kept fixed in all algebraic manipulations. Therefore, for all practical purposes, we are dealing with four variables $(\boldsymbol{y}, \tau)$.

For problems of interest in aeroacoustics, such as propeller and helicopter rotor noise prediction, one can always describe the surface (generally a blade) in a frame fixed *relative to the surface*. We will call this frame the $\boldsymbol{\eta}-$frame. Such a frame is used by the manufacturer to describe the design of the blade surface to the technicians who build the blade. We call the variable $\boldsymbol{\eta}$ the **Lagrangian variable** of a point on the moving surface. If we mark a point on the surface, say by a red dot, then we have essentially identified a point with the fixed variable $\boldsymbol{\eta}$ on the moving surface. In general, in the $\boldsymbol{\eta}$-frame the moving surface is time independent although this does not have to be so in the following mathematical manipulations. The thing to realize at this point is that once the motion of the blade is specified, then the **trajectory** of a point on the surface *described by a fixed* $\boldsymbol{\eta}$ is specified in space-time in the $\boldsymbol{y}$-frame, i.e., in space. This trajectory is given by the relation:

$$\boxed{\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{\eta}, \tau)} \tag{15}$$

The inverse transformation is given by:

$$\boxed{\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{y}, \tau)} \tag{16}$$

Note that the equation of the moving surface $f(\boldsymbol{y}, \tau) = 0$ in the $\boldsymbol{\eta}$-frame is $f(\boldsymbol{y}(\boldsymbol{\eta}, \tau), \tau) = \tilde{f}(\boldsymbol{\eta}, \tau) = 0$. In practice, $\tilde{f} = 0$ is independent of time, i.e., the surface is described as $\tilde{f}(\boldsymbol{\eta})$, and this is what we assume here. Later on we will see that it is customary to use f when we should properly use $\tilde{f}$ for designating the moving surface in an analytic expression (e.g., see eq. (29)). This is an *abuse* of notation which is acceptable for the reason that we will discuss in the paragraph following eq. (31). You should keep in mind that, in general, the analytic expressions for f and $\tilde{f}$ are different. For example, in the example of a moving (rigid) sphere in **Remark 1**, we have $f(\boldsymbol{y}, \tau) = \sqrt{(y_1 - v\,t)^2 + y_2^2 + y_3^2} - a = 0$ while $\tilde{f}(\boldsymbol{\eta}) = \sqrt{\eta_1^2 + \eta_2^2 + \eta_3^2} - a = 0$. Here, we are assuming that the $\boldsymbol{\eta}$-frame has its origin at the center of the sphere with its axes parallel to corresponding axes of the $\boldsymbol{y}$-frame. However, this distinction does not matter to us here because we are not able to integrate any of the acoustic integrals here in closed form for any nontrivial moving surface. We always evaluate our surface integrals numerically by finite difference scheme after we divide a data or a blade surface into panels.

For problems of interest to us in aeroacoustics, the transformations described by eqs. (15) and (16) are *isometric*, i.e., distance preserving, because they involve translations and rotations only. For this reason, the Jacobians of transformation are unity, that is, we have

$$\boxed{\det\left(\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{\eta}}\right) = 1} \text{ and } \boxed{\det\left(\frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{y}}\right) = 1} \tag{17}$$

If you have trouble accepting this, then use the *translation* transformation $\boldsymbol{y} = \boldsymbol{\eta} + \boldsymbol{v}\,\tau$ for a constant velocity $\boldsymbol{v}$ used in **Remark 1** (where $\boldsymbol{v} = v\,\boldsymbol{e}_1$ and $\boldsymbol{e}_1$ is the basis vector along $\eta_1 -$ axis) to test the validity of eq. (17). We now go back to the integration of the Dirac delta functions of eq. (13). We first use the transformation $\boldsymbol{y} \to \boldsymbol{\eta}$ in this equation to get

$$4\pi\, p'(\boldsymbol{x}, t) = \int Q(\boldsymbol{y}(\boldsymbol{\eta}, \tau), \tau)\, \delta(\tilde{f})\, \frac{\delta(g)}{r\,|\det(\partial \boldsymbol{\eta} / \partial \boldsymbol{y})|}\, d\boldsymbol{\eta}\, d\tau = \int_{-\infty}^{t} \int_{\mathbb{R}^3} \tilde{Q}(\boldsymbol{\eta}, \tau)\, \delta(\tilde{f})\, \frac{\delta(g)}{r}\, d\boldsymbol{\eta}\, d\tau \tag{18}$$

where we have defined $\tilde{Q}(\boldsymbol{\eta}, \tau) = Q(\boldsymbol{y}(\boldsymbol{\eta}, \tau), \tau)$. Next we use the transformation $\tau \to g$. The Jacobian of this transformation is $\partial g / \partial \tau$. Here the variable $\boldsymbol{\eta}$ is kept fixed in this partial differentiation. We will do the algebra in detail below. First we have

$$g = \tau - t + \frac{|\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{\eta}, \tau)|}{c} \tag{19}$$

The partial differentiation with respect to variable τ gives

$$\frac{\partial g}{\partial \tau} = 1 + \frac{1}{c}\frac{\partial r}{\partial y_i}\frac{\partial y_i}{\partial \tau} = 1 - \frac{\hat{r}_i v_i}{c} = 1 - M_r \tag{20}$$

where $M_r = \hat{r}_i v_i / c$ is the Mach number of the point $\boldsymbol{\eta}$ in the radiation direction at the time τ. Here $\hat{r}_i$ is the component of unit radiation vector $(\boldsymbol{x} - \boldsymbol{y})/r$ and $v_i = \partial y_i(\boldsymbol{\eta}, \tau)/\partial\tau$ is the component of the velocity $\boldsymbol{v}$ of the point $\boldsymbol{\eta}$ with respect to the $\boldsymbol{y}$ − frame fixed to the undisturbed medium.

Using the above results in eq. (18) and assuming that ε is a small positive number, we get

$$4\pi p'(\boldsymbol{x}, t) = \int \tilde{Q}(\boldsymbol{\eta}, \tau)\,\delta(\tilde{f})\,\frac{\delta(g)}{r\,|\partial g/\partial\tau|}\,d\boldsymbol{\eta}\,dg = \\ \int_{\mathbb{R}^3}\int_{-\varepsilon}^{\varepsilon} \tilde{Q}(\boldsymbol{\eta}, \tau)\,\delta(\tilde{f})\,\frac{\delta(g)}{r\,|1 - M_r|}\,dg\,d\boldsymbol{\eta} = \int_{\mathbb{R}^3}\left(\frac{\tilde{Q}(\boldsymbol{\eta}, \tau)}{r\,|1 - M_r|}\,\delta(\tilde{f})\right)_{g=0} d\boldsymbol{\eta} \tag{21}$$

Note that the limits of the inside integral (with respect to g) of the expression after the second equality sign are from $-\varepsilon$ to ε ($\varepsilon > 0$) because $\delta(g)$ could only contribute to the integral in this region. In what follows, we will drop the absolute value sign around $|1 - M_r|$ because for surfaces moving subsonically, we always have $1 - M_r > 0$. The expression $1 - M_r$ is known as the **Doppler factor**. Let us now interpret what exactly the condition $g = 0$ imposes on us. Remember that we have used the transformation $\tau \to g$. This means that, from eq. (19), the condition $g = 0$ makes the source time dependent on the other variables $(\boldsymbol{x}, t; \boldsymbol{\eta})$. This function is found analytically by solving for τ from the equation

$$g = \tau - t + \frac{|\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{\eta}, \tau)|}{c} = 0 \quad ((\boldsymbol{x}, t)\ \text{kept fixed}) \tag{22}$$

Before we go further, let us see what this source time signifies. Remember that when we fix $\boldsymbol{\eta}$, it means that we have marked a fixed point on the moving surface, say, with a red dot. Therefore, the trajectory of this red dot in space-time is known from the relation $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{\eta}, \tau)$. That is, given *any* source time τ, we know where the red dot is on its trajectory in space. But from eq. (22), we have

$$|\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{\eta}, \tau)| = c\,(t - \tau) \tag{23}$$

Let us write $\tau_e = \tau(\boldsymbol{x}, t; \boldsymbol{\eta})$ and $\boldsymbol{y}_e = \boldsymbol{y}(\boldsymbol{\eta}, \tau_e)$. We will shortly say what the subscript e stands for. With these notations, eq. (23) can be written as

$$r_e \equiv |\boldsymbol{x} - \boldsymbol{y}_e| = c\,(t - \tau_e) \tag{24}$$

Clearly, the source time τ_e is the **emission time**, $\boldsymbol{y}_e = \boldsymbol{y}(\boldsymbol{\eta}, \tau_e)$ is the **emission position**, and r_e is the **emission distance** of the source point $\boldsymbol{\eta}$ to the observer position $\boldsymbol{x}$. Both the emission position and the emission distance are viewed in the $\boldsymbol{y}$ − frame fixed to the undisturbed medium. See **Figure 2** for an illustration of some of the terms we use here. It can be shown that for a subsonically moving source, there is only one emission time for a given source point $\boldsymbol{\eta}$. In practical problems of rotating blade noise prediction, we cannot find a closed form analytic expression for $\tau = \tau_e(\boldsymbol{x}, t; \boldsymbol{\eta})$ because eq. (23) is a transcendental equation involving sines and cosines. However, we can find τ_e numerically easily by a shooting technique. Roughly speaking, here is what we do. At the observer time t, we know where the source (the red dot) position is and its trajectory in space. This position is called the **visual position** of the source. Walk in small time steps along the trajectory back in (source) time, i.e., for $\tau < t$, and each time check to see if eq. (23) is satisfied. You may be overshooting the emission position $\boldsymbol{y}(\boldsymbol{\eta}, \tau_e)$. In that case, walk along the trajectory towards the visual position of the source in smaller time steps till you find the emission time and position satisfying eq. (23). In practice, this method can be refined considerably to speed up the computation of the emission time.

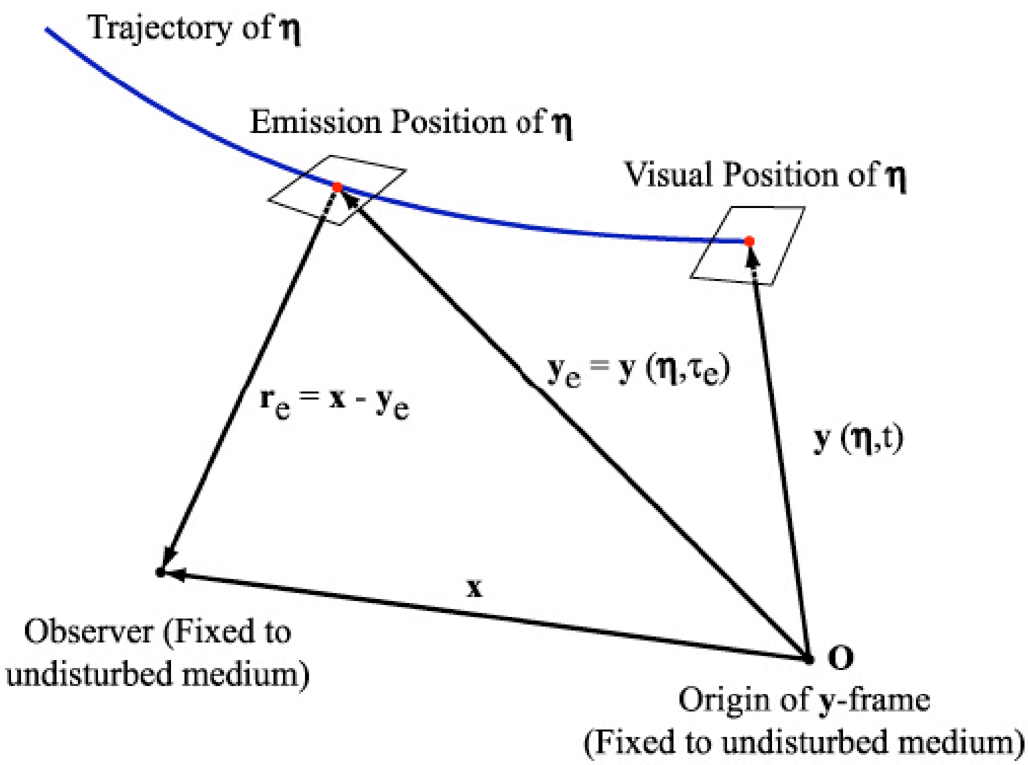

Figure 2- Sketch of the trajectory of a source point η (the red dot) as seen by an observer fixed to the medium (x- or y-frame) and the definitions of visual and emission positions of the source

With the above notations, eq. (21) can be written as:

$$4\pi p'(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \frac{\tilde{Q}(\boldsymbol{\eta}, \tau_e)}{r_e (1 - M_{r_e})} \delta\left[\tilde{f}(\boldsymbol{\eta})\right] d\boldsymbol{\eta} \tag{25}$$

where we have defined $M_{r_e} = \boldsymbol{M}(\boldsymbol{\eta}, \tau_e) \cdot \hat{\boldsymbol{r}}_e$ and $\boldsymbol{M}(\boldsymbol{\eta}, \tau_e) = \boldsymbol{v}(\boldsymbol{\eta}, \tau_e)/c$. Now note that

$$\frac{\tilde{Q}(\boldsymbol{\eta}, \tau_e)}{r_e (1 - M_{r_e})} = \psi(\boldsymbol{x}, t; \boldsymbol{\eta}) \tag{26}$$

where the function $\psi(\boldsymbol{x}, t; \boldsymbol{\eta})$ shows the dependence on variables of the expression on the left of eq. (26). We can show that for an arbitrary integrable function $q(\boldsymbol{y})$, we have the following easily remembered and beautiful result **(See References 5 and 6)**:

$$\boxed{\int_{\mathbb{R}^3} q(\boldsymbol{y})\,\delta(f)\,d\boldsymbol{y} = \int_{f=0} q(\boldsymbol{y})\,dS} \qquad \text{(assuming that } |\nabla f| = 1) \tag{27}$$

Note that $\boldsymbol{y}$ is a dummy variable here. Therefore, integrating the delta function in eq. (25) gives

$$\boxed{4\pi\, p'(\boldsymbol{x}, t) = \int_{\tilde{f}=0} \frac{\tilde{Q}(\boldsymbol{\eta}, \tau_e)}{r_e\,(1 - M_{r_e})}\, dS} \tag{28}$$

It is customary, *but confusing and not entirely correct*, to write the above result as:

$$\boxed{4\pi\, p'(\boldsymbol{x}, t) = \int_{f=0} \left[\frac{Q(\boldsymbol{y}, \tau)}{r\,(1 - M_r)}\right]_{ret} dS} \tag{29}$$

where the subscript *ret* stands for the **retarded time**. The reason that this result is not entirely correct is that the expression on the left of eq. (26) is not obtained simply by replacing the source time τ in the integrand of eq. (29) by $t - r/c$ which is what the subscript *ret* implies. Although it is always true that $\tau_e = t - r_e/c$, r_e itself is given by the expression $|\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{\eta}, \tau_e(\boldsymbol{x}, t; \boldsymbol{\eta}))|$ which shows that the retarded time notation of eq. (29), and even the use of the source variable $\boldsymbol{y}$ in its integrand, is not permissible. Also note that in this equation we have used the notation $f = 0$ instead of the correct notation $\tilde{f} = 0$. In summary, *for a moving surface*, we define:

$$\boxed{\left[\frac{Q(\boldsymbol{y}, \tau)}{r\,(1 - M_r)}\right]_{ret} = \left[\frac{Q(\boldsymbol{y}, \tau)}{r\,(1 - M_r)}(\boldsymbol{x}; \boldsymbol{\eta}, \tau)\right]_{\tau = \tau_e(\boldsymbol{x}, t; \boldsymbol{\eta})} \equiv \frac{\tilde{Q}(\boldsymbol{\eta}, \tau_e)}{r_e\,(1 - M_{r_e})}} \tag{30}$$

where in the second square brackets, Q and $\boldsymbol{M}$ are functions of $(\boldsymbol{\eta}, \tau)$, and r and $\hat{\boldsymbol{r}}$ are functions of $(\boldsymbol{x}; \boldsymbol{\eta}, \tau)$ and then the whole thing is evaluated at the emission time. Other symbols are defined as follows:

$$\tilde{Q}(\boldsymbol{\eta}, \tau_e) = Q(\boldsymbol{y}(\boldsymbol{\eta}, \tau_e), \tau_e),\; r_e = |\boldsymbol{x} - \boldsymbol{y}_e|,\; \boldsymbol{y}_e = \boldsymbol{y}(\boldsymbol{\eta}, \tau_e),\; M_{r_e} = \boldsymbol{M}(\boldsymbol{\eta}, \tau_e)\cdot\hat{\boldsymbol{r}}_e \tag{31}$$

This means that the left side of eq. (30) is **a shorthand notation** for its right side utilized to avoid the introduction of many new symbols, such as $\boldsymbol{\eta}$, $\tilde{f}$, $\tilde{Q}(\boldsymbol{\eta}, \tau_e)$, etc. However, as you will see below, when we seek the solution of FW-H equation, we always end up with integrals of the type in eq. (28). We will use the shorthand notation of eq. (29) repeatedly below without warning. Unfortunately, the notation of eq. (29) is *the source of considerable confusion* if one does not understand that the integrand of this equation is not to be interpreted by the conventional meaning of retarded time notation for stationary sources. For this reason, we have given here a detailed derivation of the solution of eq. (12).

We emphasize here that, in general, we cannot get analytic expressions for the quantities in eq. (31) for most problems of interest in aeroacoustics. However, all these quantities can be found numerically with arbitrary precision. Combining the analytic solution of FW-H equation with the power of modern digital computers allows us to use the exact geometry and kinematics of rotating blade machinery. This approach has been enormously successful in all areas of aeroacoustics.

Remark 4- When a source moves *rectilinearly at uniform speed*, then the source time can be found in closed form by solving eq. (22). Furthermore, we discover that we have one emission time for subsonic motion and two emission times for supersonic motion of the source. (End of Remark 4)

3.3- Derivation of Formulation 1

We will now derive Formulation 1. As will be seen, for thickness noise, we will have no trouble writing the solution. Based on what we know about the solution of eq. (12), the solution of eq. (5), **the thickness noise part of Formulation 1** can be written as:

$$\boxed{4\pi p'_T(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{\rho_0 v_n}{r(1-M_r)}\right]_{ret} dS} \tag{32}$$

We will say more about the mathematics behind keeping the time derivatives outside the integral sign later **(See Remark 5)**.

We will now derive the loading noise part of Formulation 1. We know that for an observer in the far field, we have the following approximation:

$$\frac{\partial}{\partial x_i} \simeq -\frac{\hat{r}_i}{c}\frac{\partial}{\partial t} \tag{33}$$

Can one derive the exact result? The affirmative answer was found by Farassat in **Reference 1**. We will give here the derivation in **Reference 2** which is shorter and more elegant. Let us start by writing the formal solution of eq. (6) using the free-space Green's function **(See Remark 6)**:

$$4\pi p'_L(\boldsymbol{x}, t) = -\frac{\partial}{\partial x_i}\int p\, n_i\, \delta(f)\, \frac{\delta(g)}{r}\, d\boldsymbol{y}\, d\tau = -\int p\, n_i\, \delta(f)\, \frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) d\boldsymbol{y}\, d\tau \tag{34}$$

Now we will use the following identity which can be proven by carrying out the differentiation on both sides:

$$\frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{r}_i\,\delta(g)}{r}\right) - \frac{\hat{r}_i\,\delta(g)}{r^2} \tag{35}$$

If we take the partial derivatives on both sides, we get

$$\frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) = \frac{1}{c}\frac{\hat{r}_i\,\delta'(g)}{r} - \frac{\hat{r}_i\,\delta(g)}{r^2}, \quad \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\hat{r}_i\,\delta(g)}{r}\right) = -\frac{1}{c}\frac{\hat{r}_i\,\delta'(g)}{r} \tag{36}$$

This proves the identity of eq. (35). Let us now use this identity on the right of eq. (34) and then take the observer time derivative out of the first integral. We obtain:

$$4\pi p'_L(\boldsymbol{x}, t) = \frac{1}{c}\int p\, n_i\, \delta(f)\, \frac{\partial}{\partial t}\left(\frac{\hat{r}_i\, \delta(g)}{r}\right) d\boldsymbol{y}\, d\tau + \int p\, n_i\, \hat{r}_i\, \delta(f)\, \frac{\delta(g)}{r^2}\, d\boldsymbol{y}\, d\tau = \frac{1}{c}\frac{\partial}{\partial t}\int p\, n_i\, \hat{r}_i\, \delta(f)\, \frac{\delta(g)}{r}\, d\boldsymbol{y}\, d\tau + \int p\, n_i\, \hat{r}_i\, \delta(f)\, \frac{\delta(g)}{r^2}\, d\boldsymbol{y}\, d\tau \tag{37}$$

See **Remark 6** about an important mathematical point in relation to this equation. Again using the solution of eq. (12), we write the solution of eq. (6) which is **the loading noise part of Formulation 1** as:

$$\boxed{4\pi p'_L(\boldsymbol{x}, t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{p\cos\theta}{r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]_{ret} dS} \tag{38}$$

where $\cos\theta = n_i\,\hat{r}_i$, i.e., θ is the local angle between normal to the surface and radiation direction $\hat{\boldsymbol{r}}$ *at the emission time*.

Formulation 1 of Farassat is the sum of eqs. (32) and (38):

$$\boxed{4\pi p'(\boldsymbol{x}, t) = 4\pi(p'_T(\boldsymbol{x}, t) + p'_L(\boldsymbol{x}, t)) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{\rho_0 v_n}{r(1-M_r)} + \frac{p\cos\theta}{c\, r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2(1-M_r)}\right]_{ret} dS} \tag{39}$$

The observer time differentiation of Formulation 1 is always taken <u>*numerically*</u> *in the applications of this result.*

Remark 5- The solution of the wave equation with derivatives acting on the inhomogeneous source term

Equations (5) and (6) are of the following types:

$$\Box^2\phi = \frac{\partial q}{\partial t} \quad \text{and} \quad \Box^2\phi_i = \frac{\partial q}{\partial x_i} \qquad \left(\boldsymbol{x} \in \mathbb{R}^3,\ q \text{ differentiable in } \boldsymbol{x} \text{ and } t\right) \tag{40}$$

Let us consider the first equation. Assume that we find the solution of the following wave equation:

$$\Box^2\psi = q \tag{41}$$

Then, by taking the time derivatives of both sides of eq. (41), we have the following result:

$$\frac{\partial}{\partial t}\Box^2\psi = \Box^2\frac{\partial\psi}{\partial t} = \frac{\partial q}{\partial t} \tag{42}$$

Comparing this equation with the first wave equation in eq. (40), we have

$$\phi(\boldsymbol{x}, t) = \frac{\partial\psi(\boldsymbol{x}, t)}{\partial t} \tag{43}$$

This is the result we used in writing down eq. (32). Similarly, we have

$$\frac{\partial}{\partial x_i} \Box^2 \psi = \Box^2 \frac{\partial \psi}{\partial x_i} = \frac{\partial q}{\partial x_i} \tag{44}$$

When this result is compared with the second wave equation in eq. (40), we conclude that

$$\phi_i (\boldsymbol{x}, t) = \frac{\partial \psi (\boldsymbol{x}, t)}{\partial x_i} \tag{45}$$

This is the result used in eq. (34).

We make two further important comments here. First deriving the results in eqs. (43) and (45) using the Green's function solution of the wave equations of eq. (40) *in the integral form* as it is often done in books and technical journals, is difficult and involves much more algebraic manipulations. Second, if we relax the differentiability of the source function $q(\boldsymbol{x}, t)$ in eq. (40), then the results expressed by eqs. (43) and (45) are valid if all the derivatives $\partial / \partial t$ and $\partial / \partial x_i$ on the right sides of eqs. (40), (43), and (45) are treated as *generalized derivatives* **(See References 5 and 6)**. In fact, it is always more convenient to work in the space of generalized functions because of their powerful operational properties (ease of exchanging limit processes without worrying about differentiability, etc.). As a matter of fact, in many practical problems, differentiability or even the continuity of the source term cannot be guaranteed. However, using generalized functions can overcome most, if not all, the mathematical ambiguities and difficulties. (End of Remark 5)

Remark 6- In eq. (37), we have taken the observer time derivative out of the integral sign after the second equality sign. But we note that the integral sign in this equation stands for a four dimensional integral as given by eq. (14). The upper limit of the integration with respect to the source time is t. A keen reader would recognize that the *Leibniz rule of differentiation under an integral sign* (see **Reference 8** below) must be used to establish the validity of the operation in eq. (37). We will now demonstrate how to do this.

Let us use the notation $Q = p\, n_i\, \hat{r}_i$. We start with the following integral appearing on the right of eq. (37):

$$I = \frac{\partial}{\partial t} \int Q(\boldsymbol{y}, \tau)\, \delta(f)\, \frac{\delta(g)}{r}\, d\boldsymbol{y}\, d\tau = \frac{\partial}{\partial t} \int_{-\infty}^{t} \int_{\mathbb{R}^3} Q(\boldsymbol{y}, \tau)\, \delta(f)\, \frac{\delta(g)}{r}\, d\boldsymbol{y}\, d\tau \tag{46}$$

Now take the time derivative inside the integral applying Leibniz rule:

$$I = \int_{-\infty}^{t} \int_{\mathbb{R}^3} \frac{Q(\boldsymbol{y}, \tau)}{r}\, \delta(f)\, \frac{\partial \delta(g)}{\partial t}\, d\boldsymbol{y}\, d\tau + \int_{\mathbb{R}^3} \frac{Q(\boldsymbol{y}, t)}{r}\, \delta[f(\boldsymbol{y}, t)]\, \delta\!\left(\frac{r}{c}\right) d\boldsymbol{y} \tag{47}$$

We will now show that the second integral is zero. First note that $\delta(r/c) = c\, \delta(r)$. So we must show that the following integral vanishes:

$$I_1 (\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{Q(\boldsymbol{y}, t)}{r}\, \delta[f(\boldsymbol{y}, t)]\, \delta(r)\, d\boldsymbol{y} \tag{48}$$

We use the spherical polar coordinates (r, φ, ϑ) *with center at the observer*. We have $d\boldsymbol{y} = r^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta$. Therefore, the above integral can be written as:

$$I_1(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \int_0^{\pi} \int_0^{2\pi} \int_0^{\infty} Q(\boldsymbol{y}, t)\, \delta[f(\boldsymbol{y}, t)]\, r \sin\vartheta\, \delta(r - \varepsilon)\, d r\, d\varphi\, d\vartheta = 0 \tag{49}$$

The reason is that $r\,\delta(r-\varepsilon) = \varepsilon\,\delta(r-\varepsilon)$ and, thus, the above triple integral is of the order of ε and goes to zero as $\varepsilon \to 0$. Therefore, we have shown that

$$\boxed{\frac{\partial}{\partial t} \int Q(\boldsymbol{y}, \tau)\, \delta(f)\, \frac{\delta(g)}{r}\, d\boldsymbol{y}\, d\tau = \int Q(\boldsymbol{y}, \tau)\, \delta(f)\, \frac{\partial}{\partial t} \left(\frac{\delta(g)}{r} \right) d\boldsymbol{y}\, d\tau} \tag{50}$$

Note that even though r is not a function of the observer time, we like to always associate r with $\delta(g)$ and write the observer time derivative operating on $\delta(g)/r$ in the integrand of eq. (50).

For *Leibniz rule of differentiation under an integral sign*, I recommend volume 1 of the following book:

8- R. Courant: Differential and Integral Calculus, 2 volumes, ***Interscience Publishers*, 1936.** (Published also more recently in Wiley Classic Library Series)

This book is considered by some mathematicians as perhaps the best calculus book of the twentieth century. A somewhat modernized version of this book is by R. Courant and F. John (in two volumes). Both versions are treasure-troves of the most useful calculus results for applications. Volume 2 of this book also includes all the fine points of the subjects of transformation of variables, n-dimensional volumes and surfaces that one needs to understand the present work. These are difficult subjects that are no longer emphasized in calculus courses. They are discussed by Courant without the use of excessive abstract language employed by some modern authors of books and articles on mathematics. One must pay careful attention to all of the footnotes in the above calculus book because of the many important examples and comments there. (End of Remark 6)

3.4- Derivation of Formulation 1A

Derivation of this result is based on Formulation 1. As will be seen, the discussions in Subsection 3.3 make the derivation of this formulation an exercise in partial differentiation, albeit a fairly complicated one algebraically.

Let us look again at eq. (30):

$$E(\boldsymbol{x}, t; \boldsymbol{\eta}) = \left[\frac{Q(\boldsymbol{y}, \tau)}{r(1 - M_r)} \right]_{ret} = \left[\frac{Q(\boldsymbol{y}, \tau)}{r(1 - M_r)} (\boldsymbol{x}; \boldsymbol{\eta}, \tau) \right]_{\tau = \tau_e} \tag{51}$$

We have shown that only the emission time τ_e is a function of the observer time. Therefore, if we need to take the observer time derivative of the above expression, we must use the chain rule as follows:

$$\left(\frac{\partial}{\partial t} \right)_{\boldsymbol{x}} = \left[\frac{\partial \tau(\boldsymbol{x}, t; \boldsymbol{\eta})}{\partial t} \frac{\partial}{\partial \tau} \right]_{\tau = \tau_e} \tag{52}$$

where $\tau(\boldsymbol{x}, t; \boldsymbol{\eta})$ is simply the solution of $\tau - t + |\boldsymbol{x} - \boldsymbol{y}(\boldsymbol{\eta}, \tau)|/c = 0$.

Remark 7- Whenever you see a partial derivative with respect to a variable, stop and ask yourself what *other* variables are kept fixed. The notation of partial differentiation can be very confusing when working with the wave equation with sources on a moving surface. We note that:

1- In deriving this equation, we assume that the moving surface $f = 0$ is defined in a **Lagrangian frame** $\boldsymbol{\eta}$ fixed to the surface. This frame is also called the *blade-fixed frame*. We have $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{\eta}, \tau)$, so that $r = r(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{\eta}, \tau))$. Also note that the surface integration is most conveniently carried out in the $\boldsymbol{\eta}$-frame, i.e., we are really integrating over the surface $\tilde{f}(\boldsymbol{\eta}) = 0$.

2- We have also shown that the emission time is analytically written as follows $\tau = \tau_e(\boldsymbol{x}, t; \boldsymbol{\eta})$ although we cannot find this function explicitly for rotating blades.

Therefore, in partial differentiation of the left side of eq. (52), we are keeping the variables $(\boldsymbol{x}, \boldsymbol{\eta})$ fixed. All these also mean that we need to find $\partial \tau_e / \partial t$ on the right side of eq. (52) keeping the variables $(\boldsymbol{x}, \boldsymbol{\eta})$ fixed. (End of Remark 7)

Now we find $\partial \tau / \partial t$. We have shown above that the emission time satisfies the following equation:

$$g = \tau - t + r(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{\eta}, \tau))/c = 0 \tag{53}$$

where $\boldsymbol{\eta}$ is a given **fixed** point on the moving surface. By taking partial derivative with respect to t of eq. (53), we get

$$\left(\frac{\partial \tau}{\partial t}\right)_{(\boldsymbol{x}, \eta)} - 1 + \frac{1}{c}\left(\frac{\partial r}{\partial t}\right)_{(\boldsymbol{x}, \eta)} = \left(\frac{\partial \tau_e}{\partial t}\right)_{(\boldsymbol{x}, \eta)} - 1 - M_r \left(\frac{\partial \tau}{\partial t}\right)_{(\boldsymbol{x}, \eta)} = 0 \tag{54}$$

where M_r is the Mach number of the point $\boldsymbol{\eta}$ in the radiation direction at the time τ. Here we have used the following relations:

$$\left(\frac{\partial r}{\partial t}\right)_{(\boldsymbol{x}, \eta)} = \left(\frac{\partial r}{\partial \tau}\right)_{(\boldsymbol{x}, \eta)} \left(\frac{\partial \tau}{\partial t}\right)_{(\boldsymbol{x}, \eta)} \tag{55}$$

$$\left(\frac{\partial r}{\partial \tau}\right)_{(\boldsymbol{x}, \eta)} = \frac{\partial r}{\partial y_i}\left(\frac{\partial y_i}{\partial \tau}\right)_{(\boldsymbol{x}, \eta)} = -\hat{r}_i v_i = -v_r \tag{56}$$

where $\hat{r}_i$ is the component of unit radiation vector and v_r is the velocity of the point $\boldsymbol{\eta}$ in the radiation direction.

From eqs. (54-56), we find

$$\boxed{\left(\frac{\partial \tau}{\partial t}\right)_{(x, \eta)} = \frac{1}{1 - M_r}} \tag{57}$$

This result leads to the following **important formula**:

$$\boxed{\frac{\partial}{\partial t}[q(\boldsymbol{x}, \boldsymbol{y}, \tau)]_{ret} = \frac{\partial}{\partial t}[q(\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{\eta}, \tau), \tau_e(\boldsymbol{x}, t; \boldsymbol{\eta}))] = \left[\frac{1}{1 - M_r} \frac{\partial q(\boldsymbol{x}, \boldsymbol{y}, \tau)}{\partial \tau}\right]_{\mathrm{ret}}} \tag{58}$$

We warn the readers once more that the right side of eq. (58) is a **shorthand notation** which must be interpreted as:

$$\left[\frac{1}{1-M_r}\,\frac{\partial q(\boldsymbol{x},\,\boldsymbol{y},\,\tau)}{\partial \tau}\right]_{\text{ret}} = \left[\frac{1}{1-M_r(\boldsymbol{x};\,\boldsymbol{\eta},\,\tau)}\,\frac{\partial q(\boldsymbol{x},\,\boldsymbol{y}\,(\boldsymbol{\eta},\,\tau),\,\tau)}{\partial \tau}\right]_{\tau=\tau_e} \tag{59}$$

This analytic expression seems complicated and intimidating. But the physical interpretation is simple if we work in the $\boldsymbol{\eta}$-frame but *continue using the notations we use in the* $\boldsymbol{y}$*-frame* to reduce confusion. Equation (59) can be simply rewritten as:

$$\left[\frac{1}{1-M_r}\,\frac{\partial q(\boldsymbol{x},\,\boldsymbol{y},\,\tau)}{\partial \tau}\right]_{\text{ret}} = \left[\frac{1}{1-M_r(\boldsymbol{x};\;\boldsymbol{\eta},\,\tau)}\,\frac{\partial q(\boldsymbol{x},\,\boldsymbol{\eta},\,\tau)}{\partial \tau}\right]_{\tau=\tau_e} \tag{60}$$

Since we perform noise prediction by dividing a blade surface into panels and compute the contributions of the sources on each panel separately, we are indeed working with a fixed $\boldsymbol{\eta}$ for each panel. Therefore, the physical meaning of every term in eq.(60) is quite clear and pertain to physical properties of a given panel at the emission time. In particular, when we deal with the loading source of Formulation 1A, $p\,(\boldsymbol{\eta},\,\tau)$ is precisely the unsteady blade surface gage pressure that a transducer fixed to the blade surface at position $\boldsymbol{\eta}$ measures. Therefore, in this formulation $\dot{p} = \partial\,p\,(\boldsymbol{\eta},\,\tau)/\partial\tau$. We discuss these ambiguities in notation because in some of our other formulations, the surface pressure p stands for $p\,(\boldsymbol{y},\,\tau)$ which is the unsteady gage pressure measured by a transduced *fixed to the undisturbed medium*. Note that $p\,(\boldsymbol{\eta},\,\tau) = p\,(\boldsymbol{y}\,(\boldsymbol{\eta},\,\tau),\,\tau)$ and, if we had not abused the notation, we should have used $\tilde{p}\,(\boldsymbol{\eta},\,\tau) = p\,(\boldsymbol{y}\,(\boldsymbol{\eta},\,\tau),\,\tau)$.

We are now just one short step away from Formulation 1A. We use Formulation 1, eq. (39), and take the observer time derivative inside the first integral to obtain:

$$4\,\pi\,p'\,(\boldsymbol{x},\,t) = \int_{f=0}\frac{\partial}{\partial t}\left[\frac{\rho_0\,v_n}{r\,(1-\,M_r)} + \frac{p\cos\theta}{c\,r\,(1-\,M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{p\cos\theta}{r^2\,(1-\,M_r)}\right]_{ret} dS \tag{61}$$

Next we use eq.(60) inside the first integral to get

$$\begin{aligned} 4\,\pi\,p'\,(\boldsymbol{x},\,t) &= 4\,\pi\,(p'_T\,(\boldsymbol{x},\,t) + p'_L\,(\boldsymbol{x},\,t)) = \\ &\int_{f=0}\left\{\frac{1}{1-M_r}\,\frac{\partial}{\partial\tau}\left[\frac{\rho_0\,v_n}{r\,(1-\,M_r)} + \frac{p\cos\theta}{c\,r\,(1-\,M_r)}\right]\right\}_{\tau_e} dS + \\ &\int_{f=0}\left[\frac{p\cos\theta}{r^2\,(1-\,M_r)}\right]_{\tau_e} dS \end{aligned} \tag{62}$$

Now we must remember that everything inside the integrand of the first integral, *before the variable* τ *is replaced with* τ_e, is a function of either $(\boldsymbol{\eta},\,\tau)$ or $(\boldsymbol{x};\,\boldsymbol{\eta},\,\tau)$. First note that $\cos\theta = \boldsymbol{n}\cdot\hat{\boldsymbol{r}}$ and $M_r = \boldsymbol{M}\cdot\hat{\boldsymbol{r}}$. The variables v_n, $\boldsymbol{M}$, p, and $\boldsymbol{n}$ are functions of $(\boldsymbol{\eta},\,\tau)$, and the variables r and $\hat{\boldsymbol{r}}$ are functions of $(\boldsymbol{x};\,\boldsymbol{\eta},\,\tau)$. To evaluate the integrand of the first integral of eq. (62), we need the source time derivatives of all these seven variables as we will see below.

We have already talked about the meaning of the time derivative of the surface pressure $\dot{p}$.The source time derivative of v_n will be derived next. We have the following algebra

$$\dot{v}_n \equiv \frac{\partial}{\partial \tau}(\boldsymbol{v}\cdot\boldsymbol{n}) = \dot{\boldsymbol{v}}\cdot\boldsymbol{n} + \boldsymbol{v}\cdot\dot{\boldsymbol{n}} = \boldsymbol{a_n} + \boldsymbol{v}\cdot\omega\times\boldsymbol{n} \tag{63}$$

where $\boldsymbol{a} = \dot{\boldsymbol{v}} = \partial\boldsymbol{v}/\partial\tau = c\,\dot{\boldsymbol{M}}$ is the acceleration of the point $\boldsymbol{\eta}$ and $\boldsymbol{\omega}$ is the angular velocity of the blade surface. Note that, in general, $\boldsymbol{v}\cdot\boldsymbol{\omega}\times\boldsymbol{n} \neq 0$ on a blade. We can now easily show the following results:

$$\frac{\partial r}{\partial \tau} = \frac{\partial r}{\partial y_i}\,\frac{\partial y_i(\boldsymbol{\eta},\tau)}{\partial \tau} = -\hat{\boldsymbol{r}}\cdot\boldsymbol{v} = v_r \tag{64}$$

$$\frac{\partial \hat{r}_i}{\partial \tau} = \frac{\hat{r}_i\,v_r - v_i}{r} \tag{65}$$

$$\frac{\partial M_r}{\partial \tau} = \frac{1}{c\,r}\left(r_i\,\dot{v}_i + v_r^2 - v^2\right) = \hat{r}_i\,\dot{M}_i + \frac{c\left(M_r^2 - M^2\right)}{r} \tag{66}$$

We can next perform the tedious source time differentiation in eq.(62) and utilize the above results in the algebra. Here we separate the thickness and loading noise results as in **Reference 4**. The **Thickness and Loading noise components of Formulation 1A** is

$$\boxed{\begin{aligned} &4\pi\,p'_T(\boldsymbol{x},t) = \\ &\int_{f=0}\left[\frac{\rho_0\,\dot{v}_n}{r(1-M_r)^2} + \frac{\rho_0\,v_n\,\hat{r}_i\,\dot{M}_i}{r(1-M_r)^3}\right]_{\mathrm{ret}} dS + \int_{f=0}\left[\frac{\rho_0\,c\,v_n\left(M_r - M^2\right)}{r^2(1-M_r)^3}\right]_{\mathrm{ret}} dS \end{aligned}} \tag{67}$$

$$\boxed{\begin{aligned} 4\pi\,p'_L(\boldsymbol{x},t) = &\int_{f=0}\left[\frac{\dot{p}\cos\theta}{c\,r(1-M_r)^2} + \frac{\hat{r}_i\,\dot{M}_i\,p\cos\theta}{c\,r(1-M_r)^3}\right]_{\mathrm{ret}} dS + \\ &\int_{f=0}\left[\frac{p(\cos\theta - M_i\,n_i)}{r^2(1-M_r)^2} + \frac{\left(M_r - M^2\right)p\cos\theta}{r^2(1-M_r)^3}\right]_{\mathrm{ret}} dS \end{aligned}} \tag{68}$$

We have written these equations differently from those in **Reference 4** by separating the **near field terms** (order $1/r^2$) from the **far field terms** (order $1/r$). We have left out much algebraic manipulations in obtaining the above results. The derivation of Formulation 1A has been checked many times by acoustic researchers. We mention here that although the combination of variables $M_r^2 - M^2$ appears in eq.(66), the appearance of the combination of variables $M_r - M^2$ in eqs.(67) and (68) is correct.

Remark 8- The loading noise part of Formulation 1A can be derived in other ways. For example, we have

$$
\begin{aligned}
4\pi p'_L(\boldsymbol{x}, t) = & -\frac{\partial}{\partial x_i}\int_{f=0}\left[\frac{p\cos\theta}{r(1-M_r)}(\boldsymbol{x};\boldsymbol{\eta},\tau)\right]_{\tau=\tau_e(\boldsymbol{x},t;\boldsymbol{\eta})} dS = \\
& -\int_{f=0}\frac{\partial}{\partial x_i}\left[\frac{p\cos\theta}{r(1-M_r)}(\boldsymbol{x};\boldsymbol{\eta},\tau)\right]_{\tau=\tau_e(\boldsymbol{x},t;\boldsymbol{\eta})} dS = \\
& -\int_{f=0}\left\{\left(\frac{\partial}{\partial x_i}\right)_{(\boldsymbol{\eta},\tau)}\left[\frac{p\cos\theta}{r(1-M_r)}(\boldsymbol{x};\boldsymbol{\eta},\tau)\right]+\right. \\
& \left.\qquad\frac{\partial\tau}{\partial x_i}\left(\frac{\partial}{\partial\tau}\right)_{(\boldsymbol{x};\boldsymbol{\eta})}\left[\frac{p\cos\theta}{r(1-M_r)}(\boldsymbol{x};\boldsymbol{\eta},\tau)\right]\right\}_{\tau=\tau_e(\boldsymbol{x},t;\boldsymbol{\eta})} dS
\end{aligned} \tag{69}
$$

Now, From eq. (53), we find

$$\frac{\partial\tau}{\partial x_i}+\frac{\hat{r}_i}{c}-M_r\frac{\partial\tau}{\partial x_i}=0 \tag{70}$$

From this, we find

$$\frac{\partial\tau}{\partial x_i}=-\frac{\hat{r}_i}{c(1-M_r)} \tag{71}$$

Use this result in eq. (69) and carry out all the differentiation with respect to x_i to get the loading part of Formulation 1A. Note that in this derivation, we do not need to use the Leibniz rule of differentiation under the integral sign when we take the observer space derivative inside of the integral in eq. (69).

We did not follow this procedure originally because we already had Formulation 1 in our possession and only the observer time differentiation had to be performed analytically for both thickness and loading noise results. There were some unexpected applications of the procedure of converting the observer space derivative to observer time derivative. Farassat and Brentner used this procedure to derive a formulation for quadrupole noise prediction similar to Formulations 1 which should properly have been called **Formulation Q1** **(Reference 9, eq. (14), Reference 10, eq. (4))**. Later, Brentner derived a second formulation for quadrupole noise prediction similar to Formulation 1A called **Formulation Q1A** **(Reference 10, eq. (17))** . This latter result has been successfully used in helicopter rotor noise prediction. The primary references to these works are:

9- F. Farassat and Kenneth S. Brentner: The uses and abuses of the acoustic analogy in helicopter rotor noise prediction, *Journal of the American Helicopter Society*, 1988, 33, 29-36

10- K. S. Brentner:An efficient and robust method for predicting helicopter high-speed impulsive noise, *Journal of Sound and Vibration, 1997,* 203(1), 87-100

Had we not developed the analytic technique of converting the observer space derivative exactly to the observer time derivative to derive the loading noise component of Formulation 1, we probably would not have been able to obtain our quadrupole noise formulations. (End of Remark 8)

3.5- Moving Observer

In most applications, we want to compute the noise in a frame moving with the aircraft, i.e., as measured by a microphone attached to the aircraft. All propeller and helicopter rotor noise prediction codes at Langley Research Center have this capability. It is important to realize that the observer variable $\boldsymbol{x}$ in Formulations 1 and 1A is in the frame fixed to the medium. Let us attach a frame fixed to the aircraft with axes parallel to the medium-fixed $\boldsymbol{x}$-frame. We will call this $\boldsymbol{X}$-frame. Assume that at the time $t = 0$ the two frames coincide. Let the velocity of the aircraft be $\boldsymbol{V}(t)$. Then the origin of the $\boldsymbol{X}$-frame will be at the point

$$\boldsymbol{X}_0(t) = \int_0^t \boldsymbol{V}(t')\, d t' \tag{72}$$

Now a point X in the $\boldsymbol{X}$-frame will be at the point $\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{X}_0(t)$ *at the time* t. Therefore, if we want to find the acoustic pressure $\tilde{p}'\,(\boldsymbol{X}, t)$ as a function of time in the $\boldsymbol{X}$-frame from our formulations, we use the following formula:

$$\boxed{\tilde{p}'\,(\boldsymbol{X}, t) = p'\,(\boldsymbol{X} + \boldsymbol{X}_0(t), t)} \tag{73}$$

The interpretation of this result is simple. To calculate the acoustic pressure $\tilde{p}'\,(\boldsymbol{X}, t)$ in the aircraft fixed frame, make sure you compute first where the observer (or the microphone) is in the $\boldsymbol{x}$-frame *at the time* t. Then use that observer position in the formulations given by eq. (73).

4- Concluding Remarks

Langley Research Center researchers have been involved in the prediction of the noise of rotating blades since the nineteen thirties. It was after the introduction of high speed digital computers in nineteen sixties that it was possible to use realistic geometry and kinematics of the rotors and propellers in the noise prediction process. The pace of aeroacoustic research increased since the early nineteen seventies because of the public pressure to reduce aircraft noise particularly around airports. Both the governments and the aircraft industry around the world realized that aeroacoustic research can lead to substantial aircraft noise reduction. The invention of high speed digital computers played a major role in this effort in the areas of aeroacoustic modeling, transducer design, data analysis, full scale flight and wind tunnel tests. Advanced mathematics have played a vital role in all these areas although this fact is not mentioned often by the researchers.

In developing noise prediction models at Langley Research Center for propellers and helicopter rotors, we used primarily the acoustic analogy based the FW-H equation. We developed *general solutions* of this equation which allowed using realistic blade geometry and kinematics. We also did not want to be limited to problems that generated periodic acoustic waveforms. Finally, we required that our formulations be valid in both near and far fields. These conditions could be met if we worked in the time domain and then applied a Fourier transform in time to obtain frequency domain results. This approach has been very fruitful so far and is being followed by many other researchers around the world.

The current trend in aircraft noise prediction appears to be toward using FW-H_{pds} for all helicopter rotor and propeller noise prediction. This would make Formulation 1A even more useful in the future for all rotating blade noise prediction problems. For example, it makes sense to use this formulation for predicting supersonic propeller noise when an advanced unsteady aerodynamic code becomes available. This is what we are proposing at present for this problem.

Now I would like to give some practical advice for developing a noise prediction code based on my own experience at Langley Research Center:

1- Always derive fully the acoustic formulation you want to use. This way you understand the exact meaning of all the terms in the formulation as well as its subtleties, e.g., the fluid velocities are defined relative to what frame. Also, it is possible that some misprints in the printed reference, such as a sign error, can cause serious errors in acoustic calculations.

2- Spend *a lot of time* designing the algorithms you want to use and studying the flow of information in the code. This is the most important step in the reduction of computation time and it should be given serious attention. We give some examples below:

– Use advanced surface integration techniques such as Gauss-Legendre integration .

– Emission time computation can be very time consuming on a computer. A well-designed algorithm here is essential for an efficient code.

– Sometimes a do loop in the wrong place in the code can increase the computation time considerably. This is where the experience of the code developer becomes important.

3- Since we have derived exact results here, the readers should realize that in the near field the Fourier transform of the computed acoustic pressure can have a large constant term. This term is negative on the suction side of the rotor or propeller disc and positive on the pressure side as the physics of the problem would dictate. This is confusing to a novice in the field of aeroacoustics because the measured results generally do not show this constant value (called DC shift by experimenters). Most commonly used microphones cannot measure this shift. These microphones are said to be AC coupled and remove the DC shift.

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14. ABSTRACT

Formulations 1 and 1A are the solutions of the Ffowcs Williams-Hawkings (FW-H) equation with surface sources only when the surface moves at subsonic speed. Both formulations have been successfully used for helicopter rotor and propeller noise prediction for many years although we now recommend using Formulation 1A for this purpose. Formulation 1 has an observer time derivative that is taken numerically, and thus, increasing execution time on a computer and reducing the accuracy of the results. After some discussion of the Green's function of the wave equation, we derive Formulation 1 which is the basis of deriving Formulation 1A. We will then show how to take this observer time derivative analytically to get Formulation 1A. We give here the most detailed derivation of these formulations. Once you see the whole derivation, you will ask yourself why you did not do it yourself!

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