The world problem: on the computability of the topology of 4-manifolds

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Abstract. Topological classification of the 4-manifolds bridges computation theory and physics. A proof of the undecidability of the homeomorphy problem for 4-manifolds is outlined here in a clarifying way. It is shown that an arbitrary Turing machine with an arbitrary input can be encoded into the topology of a 4-manifold, such that the 4-manifold is homeomorphic to a certain other 4-manifold if and only if the corresponding Turing machine halts on the associated input. Physical implications are briefly discussed.

1. Introduction

A theorem proved by Markov on the non-classifiability of the 4-manifolds implies that, given some comprehensive specification for the topology of a manifold (such as its triangulation, a la Regge calculus, or instructions for constructing it via cutting and gluing simpler spaces) there exists no general algorithm to decide whether the manifold is homeomorphic to some other manifold [1]. The impossibility of classifying the 4-manifolds is a well-known topological result, the proof of which, however, may not be well known in the physics community. It is potentially a result of profound physical implications, as the universe certainly appears to be a manifold of at least four dimensions. The burgeoning quest for the topology of the universe [2] is still in its infancy; Markov's theorem may ultimately bear upon what can be deduced about it. Already Markov's theorem impacts certain approaches to quantum gravity. On the basis of this theorem, and consideration of hypothetical quantum superpositions of manifolds, Penrose has heuristically argued that the universe is fundamentally non-computable [3]. As another example, in analogy with Feynman's sum over histories approach to quantum mechanics, the Euclidean path integral approach to quantum gravity requires a sum over all possible topologies, with appropriate weighting, in order to calculate expectation values. However, Markov's theorem implies inherent difficulties in computing such a summation, as it would be impossible to decide whether a particular topology had been counted more than once [4].

Owing to its theorized physical significance, the computability and tractability of this sum over topologies has received some attention in the literature. Although direct summation of the series is non-computable, it is unknown whether it might nonetheless be deducible by indirect means, perhaps as the computable limit of some sequence; failing in that, it has been implied that the sum can nevertheless be approximated to any desired order of accuracy [4]. However, without a systematic way to proceed, there is no guarantee that such an approximation could be carried through in finite time. To obviate such difficulty, it has been proposed to relax the condition of homeomorphy, when classifying the manifolds, and instead classify them according to a weaker condition, in terms of their triangulation [5, 6]. But such a classification scheme would keep infinite redundancy of physically distinct manifolds in the series and it is not clear how to interpret the resulting sum. More recently, partly sidestepping the issue of computability, deductions have been made about the density of topologies per "normalized volume" – a geometric quantity – in the context of a saddle-point approximation to the Euclidean path integral [7, 8, 9]. The above work was motivated by the tantalizing possibility that this sum over topologies might determine the value of the cosmological constant [7, 8, 9, 10].

Manifold non-classifiability represents a fascinating juxtaposition of theoretical computer science with physics. The intent here is to outline a proof that will establish a correspondence between Turing machines and 4-manifolds such that deciding whether a manifold is homeomorphic to a certain other manifold is tantamount to deciding whether

the corresponding Turing machine halts; to the author's knowledge this illuminating point has not been explicitly made elsewhere. It is further hoped that the proof sketched here will provide insight into the physical ~~implications~~ *reference* of Markov's theorem.

This paper is organized as follows. In Section 2, Turing machines, and the unsolvability of the halting problem, are reviewed. In Section 3 it is shown that if the group triviality problem could be solved then the halting problem could be solved. In Section 4 it is shown that if the 4-manifold homeomorphy problem could be solved then the group triviality problem could be solved. These results are discussed in Section 5.

2. Turing machines

A Turing machine is a formal idealization of a computer [11]. In its simplest formulation, a Turing machine consists of a linear tape divided into squares onto which symbols have been printed, and a movable head that scans each square one at a time. The sequence of symbols initially printed on the tape can be considered the input of the Turing machine. The head can overwrite the current scanned square, move one square to the right, or move one square to the left, depending on its internal state and its programmed instructions. Let the $h + 1$ possible states of the machine be denoted by $q_0, ...q_h$ and the $k + 1$ possible symbols printed on the tape be denoted by $s_0, ...s_k$. The instructions followed by the machine can be conceived as a list of if-then statements of the form: "if the current state is q_i and the current scanned symbol is s_j then [either move a square or print a symbol] and change to state q_k ". After updating its state and its current scanned symbol, the machine repeats the process, reviewing the list of if-then statements. This goes on forever or until the machine arrives at a (q_i, s_j) pair for which it has no instructions, at which point it halts. Note that, although more properly referred to as a program, by convention the term "Turing machine" is taken to be synonymous with its hardwired instructions.

Consider, as Turing did, machines designed to output a sequence of symbols, potentially never ending, as the digits of a real number. Its output can be printed on every other square of the tape, while the rest of the squares are reserved for "scratch paper". Rather than print the entire sequence continuously, these machines will print only j digits, given the integer j as an input (i.e., initially printed on some of the tape squares). All such machines, which input an integer and output a digit, can themselves be ordered and numbered by integers. Turing provided a specific way to encode the instructions which uniquely characterize each Turing machine into the digits of a (very large) integer; these integers can then be ordered and renumbered by consecutive integers – call them τ_i.

A Turing machine, which can examine another Turing machine by reviewing the latter's specifications on tape, cannot in general decide whether an arbitrary Turing machine will complete its computation and halt on a given input, or go into an infinite loop without ever printing any output. This assertion can be proven by contradiction. Assume the existence of a machine algorithm that decides, in a finite number of steps,

whether a given machine will halt on a given input. A machine δ can then be constructed which, given an input integer n, operates as follows. δ initializes a counter j to 1, checks to see whether τ_1 halts on input 1, and if so increments j by 1. δ then checks to see whether each subsequent machine τ_i halts on input j, in order, incrementing j for each halting machine. When τ_i is determined not to halt, j remains at the same value and the next machine τ_{i+1} is checked. Finally, δ checks to see whether τ_n halts on input j, where j now equals one plus the number of halting machines up through τ_{n-1}. If δ decides that τ_n halts, then δ prints the jth digit computed by τ_n and then halts itself. Otherwise, δ just halts. Note that, in the former case, as part of δ's assigned task, δ must effectively emulate machine τ_n. (Turing proved it is possible to design a machine such as δ to emulate any other arbitrary machine τ_i on command.) By assumption, δ can perform all of the above operations in a finite number of steps.

Since δ is essentially a machine that outputs a digit on being input an integer, δ itself ranks among the τ_i machines described previously. Now give δ input k, such that the kth halting machine is δ. δ will proceed by computing the first digit output from the first halting machine, the first two digits output from the second halting machine, and so forth, up to the first $k-1$ digits output from the $(k-1)$th halting machine. In so doing, δ will have computed the first $k-1$ digits of its own output sequence. Now δ must compute the first k digits of the kth halting machine, itself. According to the algorithm by which δ is defined, δ must recompute the first $k-1$ digits of its output sequence. Then to compute the kth digit, δ must recompute the first $k-1$ digits of its output sequence. And so forth, ad infinitum. We have arrived at a contradiction: the assumption that δ will halt on all input implies that δ will not halt on at least one input.

Alternatively, the unsolvability of the halting problem can be understood using Cantor's diagonal argument. If one attempts to enumerate all of the sequences computed by halting machines, i.e. put them on a one-to-one correspondence with the integers, one can always use a machine such as δ to construct a sequence not on the list - i.e., $1 - \delta(j)$, if the output digits are binary digits. This would imply that the computable sequences are uncountably infinite and, as there is at least one Turing machine for each such sequence, that Turing machines are also uncountable. However, since Turing machines are finitely specified, they must be countable: a contradiction, proving again that the halting problem is unsolvable.

3. Semigroups and groups

A few definitions are in order. A semigroup is a set of elements for which a binary operator has been defined so as to satisfy closure and associativity; equivalently, it is a group in which elements are not required to have inverses. A finitely generated semigroup or group, generally infinite, albeit discrete, and non-Abelian, has a finite alphabet of generators. Its elements can be represented as "words", i.e. strings "spelled out" by products of generators. A finitely presented semigroup or group is specified

by a finite number of generators and a finite number of relations, where relations are equations between words. The word problem for semigroups or groups is the problem of finding a general algorithm which, by successive application of the relations, can decide whether two arbitrary words are equal (in a finite number of steps).

The following proof of the unsolvability of the semigroup word problem proceeds very much like that of Post [12] but has been modified to connect it more directly with the halting problem. Consider a semigroup Γ_τ with generators $q_0, q_1, ...q_h$, $s_0, s_1, ...s_k$, and l. Each q_i will represent a state of a Turing machine, each s_j will represent a symbol on the tape, s_0 will represent a blank, and l will represent the left and right bounds of the string of symbols input to the machine.

All of the operations of a Turing machine τ can then be represented by relations in Γ_τ. The action of printing over symbol s_b with symbol s_d can be represented by the following relation,

$$q_a s_b = q_c s_d \tag{1}$$

where a and c have some specific values between 1 and h, and likewise b and d between 0 and k. In accordance with Turing's convention, all machine actions will be accompanied by a simultaneous change of state. Similarly, the action of moving to the left one space can be represented by the following $h + 2$ relations.

$$s_i q_a s_b = q_c s_i s_b, \quad i = 0, 1, ..., h \tag{2}$$
$$l q_a s_b = l q_c s_0 s_b \tag{3}$$

And the action of moving to the right one space can be represented by the following $h + 2$ relations.

$$q_a s_b s_i = s_b q_c s_i, \quad i = 0, 1, ..., h \tag{4}$$
$$q_a s_b l = s_b q_c s_0 l \tag{5}$$

This completes the semigroup "emulation" of a Turing machine.

For the purpose of investigating the halting problem, I'm going to introduce two new generators with the unconventional notation $\rangle$ and $\langle$, for reasons that will soon become clear. For every $q_a s_b$ pair that does not appear in the left hand side of equations (1-5), add the relation:

$$q_a s_b = \rangle s_b \tag{6}$$

Now add the following $2h + 3$ relations:

$$s_i \rangle = \rangle, \quad i = 0, 1, ..., h \tag{7}$$
$$l \rangle = l \langle \tag{8}$$
$$\langle s_i = \langle, \quad i = 0, 1, ..., h \tag{9}$$

In effect, $\rangle$ devours all symbols to its left. If it comes to the end-marker l, it mutates into $\langle$. $\langle$ devours all symbols to its right.

The outcome is that if any word ω_ι corresponds to an input ι on which the associated Turing machine halts, then it can be shown to be equivalent, by repeated application of

the above relations (7-9), to the word $l \langle l$. If a word does not correspond to an input on which the associated Turing machine halts, then it is not equivalent to the word $l \langle l$. By convention, q_0 is reserved for the halting state, so the relation $l \langle l = q_0$ might be added - then $\omega_\iota = q_0$ in Γ_τ if and only if τ halts on input ι. An algorithm that could solve the word problem for semigroups, therefore, could solve the halting problem for Turing machines.

The above result for semigroups has direct implications for groups. For each finitely presented semigroup Γ_τ described above, there is a prescription for constructing a finitely presented group G'_τ such that for every generator and relation in Γ_τ there is a corresponding generator and relation in G'_τ, and the following theorem holds: There exist words u_ι and v_ι in the finitely presented group G'_τ that are equal if and only if $\omega_\iota = q_0$ in the finitely presented semigroup Γ_τ [13]. Equivalently, $w_\iota \equiv u_\iota v_\iota^{-1} = 1$ in G'_τ if and only if $\omega_\iota = q_0$ in Γ_τ. Further, for each finitely presented group G'_τ and each word w_ι in G'_τ there is a prescription for constructing a finitely presented group $G_\tau(w_\iota)$ such that for every generator and relation in G'_τ there is a corresponding generator and relation in $G_\tau(w_\iota)$ and the following theorem holds: $G_\tau(w_\iota)$ is trivial, i.e. contains only the identity element, if and only if $w_\iota = 1$ in G'_τ [14]. It follows that the triviality of finitely presented groups is algorithmically undecidable.

4. Manifolds

Each element of the fundamental group of a manifold represents an equivalence class of closed paths in the manifold that can be continuously deformed into one another, i.e., a homotopy class of closed paths. As an example, a trivial element in a fundamental group represents a class of paths that can be contracted to a point, and a trivial fundamental group implies a simply connected manifold. As another example, the infinite cyclic group, which can be finitely presented by one generator and no relations, is the fundamental group of a hypersphere with one arcwise connected handle: each element of the group, equal to the generator raised to some power p, corresponds with the homotopy class of paths that wind about the handle p times (and negative powers will be said to correspond to counterwindings, described below). It will be shown that for any given finitely presented group, a manifold can always be constructed for which the given group is fundamental. The prescription can be summarized as attaching to a hypersphere a handle for each generator of the group, followed by further surgery to accommodate each relation.

The following construction is homeomorphic to that of Markov, but the method of construction has been streamlined for pedagogical purposes. Consider an arbitrary finitely presented group of the form

$$G = \{g_1, ..., g_m | r_1, ..., r_n\} \tag{10}$$

where each r_i is a word representing a relation of the form $r_i = 1$ and is called a relator. Beginning with the 4-sphere, S^4, for each generator g_i attach a handle of the

form $H_i = S^3 \times [-1, +1]$. Each such attachment is performed by removing from S^4 two non-intersecting, open 4-balls and identifying the resulting 3-spherical boundaries with the ends of H_i. Calling the former S^4 region A, the attachments are subject to the conditions that no two handles intersect, and the intersection of each handle with A is a union of two 3-spheres: $H_i \cap H_j = 0$, $i \neq j$, $A \cap H_i = S^3 \times \{-1, +1\}$. In this manner a manifold can be handily constructed for each free fundamental group of the form $\{g_1, ..., g_n|\}$. To understand this, note that the construction thus far is homeomorphic to the connected sum of m copies of $S^3 \times S^1$, then use the fact that the fundamental group of the cross product of manifolds is the free product of the fundamental groups of the manifolds, while the fundamental group of the connected sum of manifolds is the direct product of the fundamental groups of the manifolds.

An arbitrary word can be represented by a closed path in the above construction as follows. Consider a path that begins at some point inside A. Reading the word from left to right, represent each generator g_i of positive power p by a path that enters its associated handle H_i at $S^2 \times \{-1\}$, then exits H_i at $S^2 \times \{+1\}$, then circles back around and repeats $p-1$ times. Represent negative powers $-p$ the same way but switch $S^2 \times \{-1\}$ and $S^2 \times \{+1\}$ (hence negative powers "unwind" positive powers). After exiting the handle for the pth time, continue the path to the handle associated with the next generator in the word, and repeat the winding process, continuing in this way until the last generator in the word has been represented. Finally, join the end of the path with its starting point to close the loop.

A relator of the finitely presented fundamental group, being a word equated with the identity, corresponds to paths that can be continuously deformed to a point. Obviously such deformation of a path through a handle is obstructed; some topological surgery will be necessary to bypass the obstruction. For each relator r_j, gouge out a region from the above constructed manifold (call the manifold M) along the vicinity of a path representative of r_j such that the gouged-out region is homeomorphic to $U^3 \times S^2$, where U^3 is the open 3-ball. Simultaneously, in a copy of S^4, gouge out a similar $U^3 \times S^1$ region; call this manifold O_j. Finally, identify the $S^2 \times S^1$ boundary of the gouged-out region in M (call this boundary T_j) with the $S^2 \times S^1$ boundary of the gouged-out region in O_j. Note that O_j is simply connected. (To see this, consider that the only conceivably non-trivial closed path in O_j is one that interlocks with the loop formed by the gouged-out region. But the former can be continuously deformed to the boundary of the latter, whereupon it can be made to encircle a cross-section homeomorphic to S^2, and thereon contracted to a point.) Any path in the homotopy class of paths associated with the relator r_j can now be continuously deformed to the surface of T_j, then contracted to a point in O_j. Repeat this surgery for each relator, in this way gluing to M, n copies of O_j. This completes the construction. It can be verified, by considering the fundamental groups of the subspaces that cover M [15], that the fundamental group of M is the given group G as advertised.

If two manifolds are homeomorphic, their fundamental groups are isomorphic. But the converse is not necessarily true, thus the non-classifiability of the manifolds does not

immediately follow from the non-classifiability of their fundamental groups. Fortunately for the purposes of this proof, the manifolds constructed above have the following critical property. First consider another manifold formed by gouging out from S^4, m non-intersecting regions homeomorphic to $U^3 \times S^1$, and gluing the remaining boundaries to those of an identical copy; call the resulting manifold N_m. Given one of the previously constructed manifolds M such that its fundamental group G has m generators, if G is trivial then, it turns out, M must be homeomorphic to N_m [1].

To come full circle, let the fundamental group of the manifold M represent a Turing machine: let $M = M_\tau(w_\iota)$ such that its fundamental group is $G_\tau(w_\iota)$, as described in Section 2. Call $M_\tau(w_\iota)$ a Turing manifold. Call $N_{m(\tau,\iota)}$, where $m(\tau,\iota)$ is the number of generators required to represent the Turing machine τ with input ι by $G_\tau(w_\iota)$, a halting manifold. It follows that the Turing manifold $M_\tau(w_\iota)$ is homeomorphic to the halting manifold $N_{m(\tau,\iota)}$ if and only if Turing machine τ halts on input ι .

5. Discussion

A sketch of a proof has been given for the non-classifiability of the 4-manifolds, by way of a topological construction whereby a 4-manifold represents a Turing machine. More precisely, a Turing machine has been encoded into a finitely presented semigroup, which has been encoded into a finitely presented group, which along with a particular Turing input has been encoded into another finitely presented group, which has been encoded into a 4-manifold. The chain of encodings is such that solving the homeomorphy problem for 4-manifolds would solve the halting problem for Turing machines, which is unsolvable. Expressed more intuitively, the essence of the problem is that the topology of a 4-manifold is potentially so rich that its complexity can rival that of any computer program intended to analyze it. Inputting the specifications of a 4-manifold to such a computer program can, in a sense, be equivocated with inputting a computer program to a computer program – an enterprise subject to logical paradoxes and limitations of the kind brought to light by Turing.

Regarding the physical applicability of Markov's theorem, while the constructions considered above are compact 4-manifolds, spacetime is often considered to be non-compact, and is sometimes speculated to have hidden extra dimensions. Markov's proof applies equally well to higher dimensional manifolds - consider $M \times S^{d-4}$, where $d > 4$ - as well as non-compact manifolds - consider $M \# R^4$. Granted Markov's theorem only applies to manifolds that are permitted to be non-simply connected, but there is a strong possibility that the universe lives in this category. On the cosmic scale, the universe may be multiply-connected [2]; on the stellar scale, black hole interiors may be topologically nontrivial, though such nontriviality might be rendered undetectable by event horizons [16] (on the other hand, traversable worm holes might exist [17]); on the subatomic scale, particles are sometimes speculated to be topological geons [18, 19]; and on the Planck scale, spacetime foam is conjectured to perturb the local topology to no end [7, 8, 9, 20].

It is conceivable that some physical criteria could be found which would restrict permissible 4-manifolds to classifiable manifolds. For example, if a strict interpretation of causality is imposed, in the form of the conditions of isochrony and the exclusion of closed timelike curves, then it can be shown that the allowed 4-manifolds are constrained to those of the form $C \times [0,1]$, $C \times [0,\infty)$, and $C \times (-\infty,\infty)$, where C is a 3-manifold [21]. These manifolds are classifiable if the 3-manifolds are classifiable; although whether the 3-manifolds are classifiable is still an open question. Note that the proof of Markov's theorem, as sketched above, is not applicable to 3-manifolds; for example, the three-dimensional analog of O_j is not simply connected, as required. In a sense, there is not enough "room" in a 3-manifold to topologically encode a Turing machine, and so there is hope that 3-manifolds might be classifiable. However, whether the universe obeys the previously mentioned interpretation of causality is unknown. These particular conditions may be too restrictive; they would preclude Wheeler's spacetime foam, as well as other exotic but physically motivated topological proposals. In summary, on the basis of current physical knowledge, the non-classifiability of the 4-manifolds remains relevant.

Acknowledgments

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