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ON THE ROLE OF INVOLUTIONS IN THE DISCONTINUOUS GALERKIN DISCRETIZATION OF MAXWELL AND MAGNETOHYDRODYNAMIC SYSTEMS

TIMOTHY BARTH*

Abstract. The role of involutions in energy stability of the discontinuous Galerkin (DG) discretization of Maxwell and magnetohydrodynamic (MHD) systems is examined. Important differences are identified in the symmetrization of the Maxwell and MHD systems that impact the construction of energy stable discretizations using the DG method. Specifically, general sufficient conditions to be imposed on the DG numerical flux and approximation space are given so that energy stability is retained. These sufficient conditions reveal the favorable energy consequence of imposing continuity in the normal component of the magnetic induction field at interelement boundaries for MHD discretizations. Counterintuitively, this condition is not required for stability of Maxwell discretizations using the discontinuous Galerkin method.

Key words. Nonlinear conservation laws, energy stability, Maxwell equations, magnetohydrodynamics, symmetrization, discontinuous Galerkin finite element method

AMS(MOS) subject classifications. 35L02, 65M02, 65K02, 76N02

1. Overview. Various mathematical models such the Maxwell equations governing electrodynamics and the magnetohydrodynamic (MHD) equations modeling fluid plasmas have the added complexity of possessing *involutions*. An involution in the sense of conservation law systems is an additional equation that if satisfied at some initial time is satisfied for all future time for both classical and weak solutions [Boi88, Daf86]. Involutions should not be confused with *constraints* that are needed for closure of the system. An example of such a constraint is the continuity equation in incompressible flow. In this note, the role of involutions in obtaining energy stable discretizations using the discontinuous Galerkin method [RH73, LR74, JP86, CLS89, CHS90] is briefly examined. Specifically, the surprisingly different role played by involutions in the discontinuous Galerkin (DG) discretization of Maxwell and ideal compressible MHD systems is contrasted. Although both systems possess solenoidal involutions, it is the interplay between involutions and symmetrization of the Maxwell and MHD systems that enters fundamentally into the construction of stable discretizations. In this regard, the two systems are vastly different. The Maxwell equations are naturally expressed in essentially symmetric form. Consequently, the analysis given in Sects. 2.1 and 3.1 shows that "standard" DG discretizations can then be used. In contrast, symmetrization of the MHD system utilizes the solenoidal involution as a necessary ingredient in the symmetrization process. Details of this symmetrization process are given in Sect. 2.2. Thus, the precise sense in which involutions are sat-

*NASA Ames Research Center, Exploration Technology Directorate, Moffett Field, California, 94035-1000 USA (Timothy.J.Barth@nasa.gov)

isfied in element interiors and across interelement boundaries enters prominently into the MHD discrete energy analysis. The analysis of Sect. 3.2 gives general sufficient conditions to be imposed on the DG numerical flux and approximation space in the presence of involutions so that energy stability is retained. These sufficient conditions reveal the favorable consequences of imposing continuity in the normal component of the magnetic induction field at interelement boundaries for MHD discretizations. This is a condition that is not required for stability of Maxwell discretizations using the discontinuous Galerkin method but is often a requirement of other methods that build satisfaction of solenoidal conditions into the discretization. Techniques for achieving this include staggered mesh and specialized differencing techniques [Yee66] as well as edge, face, and volume finite element formulations [Ned80, Bos98, BR02] or the discrete mimetic approximations as given in [HS99]. The present analysis for MHD also provides alternatives to the "divergence cleaning" procedures designed to exactly or approximately satisfy the solenoidal condition, see [BB80, TÓ0, DKK+02, BK04] and references therein. Since the DG method reduces to the simplest finite volume method in the special case of piecewise constant basis approximation, the results given here impact finite volume discretization as well.

2. Symmetrization of Conservation Laws without Involution. Consider the Cauchy initial value problem for a system of m coupled first-order differential equations in d space coordinates and time which represents a conservation law process. Let $\mathbf{u}(x,t) : \mathbb{R}^d \times \mathbb{R}^+ \mapsto \mathbb{R}^m$ denote the dependent solution variables and $\mathbf{f}(\mathbf{u}) : \mathbb{R}^m \mapsto \mathbb{R}^{m\times d}$ the flux vector. The model Cauchy problem is then given by

$$\left\{ \begin{array}{l} \mathbf{u}_{,t} + \mathbf{f}_{i,x_i} = 0 \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{array} \right. \tag{2.1}$$

with implied summation on the index $i = 1,\ldots,d$. Additionally, the system is assumed to possess a convex scalar entropy extension. Let $U(\mathbf{u}) : \mathbb{R}^m \mapsto \mathbb{R}$ and $F(\mathbf{u}) : \mathbb{R}^m \mapsto \mathbb{R}^d$ denote an entropy-entropy flux pair for the system such that in addition to (2.1) the following inequality holds

$$U_{,t} + F_{i,x_i} \leq 0 \tag{2.2}$$

with equality for classical (smooth) solutions. In the symmetrization theory for first-order conservation laws without involution [God61, Moc80], one seeks a mapping $\mathbf{u}(\mathbf{v}) : \mathbb{R}^m \mapsto \mathbb{R}^m$ applied to (2.1) so that when transformed

$$\mathbf{u}_{,\mathbf{v}}\mathbf{v}_{,t} + \mathbf{f}_{i,\mathbf{v}}\,\mathbf{v}_{,x_i} = 0 \tag{2.3}$$

the matrix $\mathbf{u}_{,\mathbf{v}}$ is symmetric positive definite (SPD) and the matrices $\mathbf{f}_{i,\mathbf{v}}$ are symmetric. Clearly, if twice differentiable functions $\mathcal{U}(\mathbf{v}) : \mathbb{R}^m \mapsto \mathbb{R}$ and $\mathcal{F}_i(\mathbf{v}) : \mathbb{R}^m \mapsto \mathbb{R}$ can be found so that

$$\mathbf{u} = \mathcal{U}^T_{,\mathbf{v}}, \quad \mathbf{f}_i = \mathcal{F}^T_{i,\mathbf{v}} \tag{2.4}$$

then the matrices

$$\mathbf{u}_{,\mathbf{v}} = \mathcal{U}_{,\mathbf{vv}}, \quad \mathbf{f}_{i,\mathbf{v}} = \mathcal{F}_{i,\mathbf{vv}}$$

are symmetric. Further, we shall require that $\mathcal{U}(\mathbf{v})$ be a convex function such that

$$\lim_{\mathbf{v}\to\infty} \frac{\mathcal{U}(\mathbf{v})}{|\mathbf{v}|} = +\infty \tag{2.5}$$

so that $U(\mathbf{u})$ can be interpreted as a Legendre transform of $\mathcal{U}(\mathbf{v})$

$$U(\mathbf{u}) = \sup_{\mathbf{v}} \{\mathbf{v}\cdot\mathbf{u} - \mathcal{U}(\mathbf{v})\} \ .$$

From (2.5), it follows that $\exists\, \mathbf{v}^* \in \mathbb{R}^m$ such that $\mathbf{v}\cdot\mathbf{u} - \mathcal{U}(\mathbf{v})$ achieves a maximum at $\mathbf{v}^*$

$$U(\mathbf{u}) = \mathbf{v}^*\cdot\mathbf{u} - \mathcal{U}(\mathbf{v}^*) \ . \tag{2.6}$$

At this maximum $\mathbf{u} = \mathcal{U}_{,\mathbf{v}}(\mathbf{v}^*)$ which can be locally inverted to the form $\mathbf{v}^* = \mathbf{v}(\mathbf{u})$. Elimination of $\mathbf{v}^*$ in (2.6) yields the simplified duality relationship

$$U(\mathbf{u}) = \mathbf{v}(\mathbf{u})\cdot\mathbf{u} - \mathcal{U}(\mathbf{v}(\mathbf{u})) \ .$$

Differentiation of this expression

$$U^T_{,\mathbf{u}} = \mathbf{v} + \mathbf{v}_{,\mathbf{u}}\mathbf{u} - \mathbf{v}_{,\mathbf{u}}\mathcal{U}^T_{,\mathbf{v}} = \mathbf{v} \tag{2.7}$$

gives an explicit formula for the entropy variables $\mathbf{v}$ in terms of derivatives of the entropy function $U(\mathbf{u})$. Using the mapping relation $\mathbf{v}(\mathbf{u})$, a duality pairing for entropy flux components is defined

$$F_i(\mathbf{u}) = \mathbf{v}(\mathbf{u})\cdot\mathbf{f}_i(\mathbf{u}) - \mathcal{F}_i(\mathbf{v}(\mathbf{u})) \ .$$

Differentiation then yields the flux relation

$$F_{i,\mathbf{u}} = \mathbf{v}\cdot\mathbf{f}_{i,\mathbf{u}} + \mathbf{v}_{,\mathbf{u}}\mathbf{f}_i - \mathbf{v}_{,\mathbf{u}}\mathcal{F}^T_{i,\mathbf{v}} = \mathbf{v}\cdot\mathbf{f}_{i,\mathbf{u}}$$

and the fundamental relationship for classical solutions

$$\mathbf{v}\cdot(\mathbf{u}_{,t} + \mathbf{f}_{i,x_i}) = U_{,t} + F_{i,x_i} = 0 \ .$$

These relationships are used extensively in the discrete energy analysis of the discontinuous Galerkin method.

2.1. Maxwell Equations in Symmetric Form. The time-dependent Maxwell equations are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} + \nabla\times\begin{pmatrix}-c^2\mathbf{B}\\ \mathbf{E}\end{pmatrix} = \begin{pmatrix}-\mathbf{j}/\epsilon_0\\ 0\end{pmatrix} \quad \text{(Maxwell equations)}$$

where $\mathbf{E} \in \mathbb{R}^d$, $\mathbf{B} \in \mathbb{R}^d$, $\rho_c \in \mathbb{R}$, and $\mathbf{j} \in \mathbb{R}^d$ denote the electric field, magnetic induction, charge and current density with ϵ_0 and c the free-space permittivity and speed of light, respectively. If the charge conservation equation

$$(\rho_c)_{,t} + \nabla \cdot \mathbf{j} = 0 \tag{2.8}$$

is satisfied for all time then the Maxwell system possesses the following involutions

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho_c/\epsilon_0 \\ \nabla \cdot \mathbf{B} &= 0 \,. \end{aligned}$$

Writing the Maxwell system in matrix coefficient form reveals that the above system is essentially already in symmetric form using the variables $\mathbf{u} \equiv (\mathbf{E}, \mathbf{B})^T$

$$\mathbf{u}_{,t} + A_i\, \mathbf{u}_{,x_i} = \mathbf{q}(\mathbf{u}) \ , \quad A_i = \begin{bmatrix} 0 & c^2 M_i \\ M_i^T & 0 \end{bmatrix}$$

where in three space dimensions

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} , \quad M_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \quad M_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consequently, a suitable entropy-entropy flux pair for the Maxwell system are given by the scaled "square entropy" and square entropy flux

$$U(\mathbf{u}) = \frac{1}{2}(|\mathbf{E}|^2 + c^2\,|\mathbf{B}|^2) \ , \quad F(\mathbf{u}) = c^2\,(\mathbf{E} \times \mathbf{B}) \,.$$

Using this entropy function, the symmetrization variables and right symmetrizer are then obtained

$$\mathbf{v} = U^T_{,\mathbf{u}} = \begin{pmatrix} \mathbf{E} \\ c^2\mathbf{B} \end{pmatrix} \ , \quad \mathbf{u}_{,\mathbf{v}} = \begin{bmatrix} I_{d\times d} & \\ & c^{-2} I_{d\times d} \end{bmatrix}$$

thus rendering the coefficient matrices symmetric as expected

$$A_i \mathbf{u}_{,\mathbf{v}} = \begin{bmatrix} 0 & M_i \\ M_i^T & 0 \end{bmatrix} \,.$$

Observe that the Maxwell system has been successfully symmetrized without utilizing the involutions. Consequently, the energy analysis for Maxwell's equations in a vacuum domain is identical to the energy analysis for conservation law systems without involution as also observed in [CLS04].

2.2. Ideal MHD in Symmetric Form. The equations of ideal compressible MHD are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{V}\\ E\\ \mathbf{B}\end{pmatrix}+\nabla\cdot\begin{pmatrix}\rho\mathbf{V}\\ \rho\mathbf{V}\mathbf{V}+I_{d\times d}\,(p+|\mathbf{B}|^2/2)-\mathbf{B}\,\mathbf{B}\\ (E+p+|\mathbf{B}|^2/2)\,\mathbf{V}-(\mathbf{V}\cdot\mathbf{B})\,\mathbf{B}\\ \mathbf{V}\mathbf{B}-\mathbf{B}\mathbf{V}\end{pmatrix}=0 \quad \text{(Ideal MHD)}$$

where $\rho \in \mathbb{R}$, $\mathbf{V} \in \mathbb{R}^d$, $\mathbf{B} \in \mathbb{R}^d$, and $p \in \mathbb{R}$ denote the fluid density, velocity, magnetic induction, and pressure with $E \in \mathbb{R}$ the total specific energy given by

$$E = \frac{p}{\gamma-1} + \rho|\mathbf{V}|^2/2 + |\mathbf{B}|^2/2$$

and γ the ratio of specific heats. In addition, the MHD system possesses the solenoidal involution

$$\nabla\cdot\mathbf{B} = 0$$

which in consistent with the absence of experimentally observed magnetic monopoles.

It is well known that thermodynamic entropy s is transported along velocity induced particle paths for ideal MHD. Recall that $s = \log(p\rho^{-\gamma})$ for MHD so that a differential of s is given by

$$ds = -\frac{\gamma}{\rho}d\rho + \frac{1}{p}dp\,.$$

Inserting equations derived from the MHD system (2.2) yields

$$s_{,t} + \mathbf{V}\cdot\nabla s + (\gamma-1)\frac{\mathbf{V}\cdot\mathbf{B}}{p}\nabla\cdot\mathbf{B} = 0$$

or after combining with the continuity equation

$$(\rho s)_{,t} + \mathrm{div}(\rho\mathbf{V}s) + (\gamma-1)\frac{\rho\mathbf{V}\cdot\mathbf{B}}{p}\nabla\cdot\mathbf{B} = 0$$

suggesting that $U(\mathbf{u}) = -\rho s$ may be a suitable entropy function only if the involution $\nabla\cdot\mathbf{B} = 0$ is satisfied. Indeed, a straightforward calculation for ideal MHD shows that this entropy function does *not* symmetrize the system under the change of variable $\mathbf{u} \mapsto \mathbf{v}$ with $\mathbf{v} = U_{,\mathbf{u}}^T$ (see for example Barth [Bar98])

$$\mathbf{f}_{,\mathbf{v}} \neq \mathbf{f}_{,\mathbf{v}}^T$$

since the involution equation has not been used. Godunov [God72] observed this phenomenon as well which lead to his development of a symmetrization technique for ideal MHD. The basic technique is reviewed here using a modified presentation from that originally given. The model MHD system with solenoidal involution is given by

$$\left\{\begin{array}{r} \mathbf{u}_{,t} + \mathbf{f}_{i,x_i} = 0 \\ \mathbf{B}_{i,x_i} = 0 \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) \end{array}\right. \tag{2.9}$$

with convex entropy extension

$$U_{,t} + F_{i,x_i} \leq 0\ . \tag{2.10}$$

To analyze this system, Godunov considered augmenting the MHD system by adding multiples of the involution where the multipliers are themselves the gradient of a scalar function $\phi(\mathbf{v}) : \mathbb{R}^m \mapsto \mathbb{R}$ with respect to the symmetrization variables $\mathbf{v}$

$$\mathbf{u}_{,t} + \mathbf{f}_{i,x_i} + \phi^T_{,\mathbf{v}}\, \mathbf{B}_{i,x_i} = 0\ .$$

Consider the following ansatz for the dependent variables $\mathbf{u}$ and flux components $\mathbf{f}_i$

$$\begin{aligned} \mathbf{u} &= \mathcal{U}^T_{,\mathbf{v}} \\ \mathbf{f}_i &= \mathcal{F}^T_{i,\mathbf{v}} - \mathbf{r}(\mathbf{v})\,\mathbf{B}_i \end{aligned}$$

with $\mathcal{U}$ a convex scalar function and $\mathbf{r}(\mathbf{v}) : \mathbb{R}^m \mapsto \mathbb{R}^m$ an unknown vector-valued function. Observe that the augmented MHD system

$$(\mathcal{U}_{,\mathbf{v}})_{,t} + (\mathcal{F}_{i,\mathbf{v}} - \mathbf{r}(\mathbf{v})\,\mathbf{B}_i)^T_{,x_i} + \phi^T_{,\mathbf{v}}\,\mathbf{B}_{i,x_i} = 0 \tag{2.11}$$

possesses a symmetric quasilinear form in $\mathbf{v}$ variables whenever $\mathbf{r}(\mathbf{v}) = \phi^T_{,\mathbf{v}}$ since the system (2.11) then reduces to

$$\underbrace{\mathcal{U}_{,\mathbf{vv}}}_{\text{SPD}}\,\mathbf{v}_{,t} + \underbrace{(\mathcal{F}_{i,\mathbf{vv}} - \phi_{,\mathbf{vv}}\mathbf{B}_i)}_{\text{SYMM}}\,\mathbf{v}_{,x_i} = 0$$

so that the final flux relationship is obtained

$$\mathbf{f}_i = \mathcal{F}^T_{i,\mathbf{v}} - \phi^T_{,\mathbf{v}}\,\mathbf{B}_i\ .$$

The entropy function $U(\mathbf{u})$ for MHD can be interpreted as a Legendre transform of $\mathcal{U}(\mathbf{v})$

$$U(\mathbf{u}) = \sup_{\mathbf{v}} \{\mathbf{v}\cdot\mathbf{u} - \mathcal{U}(\mathbf{v})\}$$

eventually producing the generalized duality relationships

$$\begin{aligned} U(\mathbf{u}) &= \mathbf{v}(\mathbf{u})\cdot\mathcal{U}_{,\mathbf{v}}(\mathbf{v}(\mathbf{u})) - \mathcal{U}(\mathbf{v}(\mathbf{u})) \\ F_i(\mathbf{u}) &= \mathbf{v}(\mathbf{u})\cdot\mathcal{F}_{i,\mathbf{v}}(\mathbf{v}(\mathbf{u})) - \mathcal{F}_i(\mathbf{v}(\mathbf{u})) \end{aligned} \tag{2.12}$$

so that for classical MHD solutions

$$\mathbf{v} \cdot (\mathbf{u}_{,t} + \mathbf{f}_{i,x_i} + \phi_{,\mathbf{v}}^T \mathbf{B}_{i,x_i}) = U_{,t} + F_{i,x_i} = 0 \, .$$

This relationship will be used heavily in later analysis of the discontinuous Galerkin method.

Choosing the entropy function $U(\mathbf{u}) = -\rho s$ yields $\phi(\mathbf{v}) = (\gamma - 1)\, \rho \, \mathbf{V} \cdot \mathbf{B}/p$, a homogeneous function of degree one in $\mathbf{v}$ so that $\phi = \phi_{,\mathbf{v}} \, \mathbf{v}$. The resulting involution multipliers $\phi_{,\mathbf{v}}$ are identical to those derived by Powell [Pow94] using a completely different argument motivated by (in part) the lack of Galilean invariance of the original MHD system and the subsequent addition of a divergence wave family into the local Riemann problem solution to restore Galilean invariance.

REMARK 2.1. *Observe that MHD provides one particular example of a symmetrizable system with a given entropy-entropy flux pair $\{U, F_i\}$ for which the flux is* not *expressed as the gradient of a primative function $\mathcal{F}_i$ but rather*

$$\mathbf{f}_i = \mathcal{F}_{i,\mathbf{v}}^T - \phi_{,\mathbf{v}}^T \, \mathbf{B}_i \, .$$

In fact, for the specific MHD entropy function $U(\mathbf{u}) = -\rho s$, it is possible to show that there cannot exist a function $\widetilde{\mathcal{F}}_i$ such that

$$\mathbf{f}_i = \widetilde{\mathcal{F}}_{i,\mathbf{v}}^T \, .$$

Thus, the DG energy analysis of MHD systems is fundamentally different from the energy analysis of systems not possessing involutions.

3. The DG Finite Element Method. Let Ω denote a spatial domain composed of stationary nonoverlapping elements K_i, $\Omega = \cup K_i$, $K_i \cap K_j = \emptyset$, $i \neq j$ and time slab intervals $I^n \equiv [t_+^n, t_-^{n+1}]$, $n = 0, \ldots, N-1$. Both continuous in time approximation and full space-time approximation on tensor space-time elements $K_i \times I^n$ will be considered in the analysis. It is useful to also define the element set $\mathcal{T} = \{K_1, K_2, \ldots\}$ and the interface set $\mathcal{E} = \{e_1, e_2, \ldots\}$ with interface members $\overline{K}_i \cap \overline{K}_j, i \neq j$ of measure $d-1$ corresponding to edges in 2-D and faces in 3-D. Let $\mathcal{P}_k(Q)$ denote the set of polynomials of degree at most k in a domain $Q \subset \mathbb{R}^d$. In the discontinuous Galerkin method, the approximating functions are discontinuous polynomials in both space and time

$$\mathcal{V}^h = \left\{ \mathbf{w} \,|\, \mathbf{w}_{|_{K \times I^n}} \in \left(\mathcal{P}_k(K \times I^n) \right)^m, \forall K \in \mathcal{T}, n = 0, \ldots, N-1 \right\} \, .$$

Alternatively, [CLS89, CHS90, Shu99] utilize a semi-discrete formulation of the DG method together with Runge-Kutta time integration. In this case, the set of approximating functions are discontinuous polynomials in space and continuous functions in time denoted by $\mathcal{V}_c^h$.

For ease of exposition, the spatial domain Ω is assumed either periodic in all space dimensions or nonperiodic with compactly supported initial data. In this domain, we first consider the standard first-order Cauchy initial value problem (without involution)

$$\begin{cases} \mathbf{u}_{,t} + \mathbf{f}_{i,x_i} = 0 \\ \mathbf{u}(x, t^0_-) = \mathbf{u}_0(x) \end{cases} \tag{3.1}$$

with convex entropy extension

$$U_{,t} + F_{i,x_i} \le 0 \ . \tag{3.2}$$

The DG method for the time interval $[t^0_+, t^N_-]$ with weakly imposed initial data $\mathbf{v}_h(x, t^0_-)$ obtained from a suitable projection of the initial data $\mathbf{v}(\mathbf{u}_0(x))$ is given by the following statement:

<u>DG FEM:</u> Find $\mathbf{v}_h \in \mathcal{V}^h$ such that

$$B_{\rm DG}(\mathbf{v}_h, \mathbf{w}_h) = 0 \ , \ \forall \mathbf{w}_h \in \mathcal{V}^h \tag{3.3}$$

with

$$\begin{aligned} B_{\rm DG}(\mathbf{v},\mathbf{w}) = \sum_{n=0}^{N-1} \Bigg(& \sum_{K\in\mathcal{T}} \int_{I^n}\int_K -(\mathbf{u}(\mathbf{v})\cdot\mathbf{w}_{,t} + \mathbf{f}_i(\mathbf{v})\cdot\mathbf{w}_{,x_i})\, dx\, dt \\ & + \sum_{K\in\mathcal{T}} \int_{I^n}\int_{\partial K} \mathbf{w}(x_-)\cdot\mathbf{h}(\mathbf{v}(x_-),\mathbf{v}(x_+);\mathbf{n})\, ds\, dt \\ & + \sum_{K\in\mathcal{T}} \int_K \left(\mathbf{w}(t^{n+1}_-)\cdot\mathbf{u}(\mathbf{v}(t^{n+1}_-)) - \mathbf{w}(t^n_+)\cdot\mathbf{u}(\mathbf{v}(t^n_-))\right) dx \Bigg) \end{aligned} \tag{3.4}$$

with suitable modifications when source terms are present. In this statement $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^d \mapsto \mathbb{R}^m$ denotes a numerical flux function, a vector-valued function of two interface states $\mathbf{v}_\pm$ and an oriented interface normal $\mathbf{n}$ with the following consistency and conservation properties:

- Consistency with the true flux, $\mathbf{h}(\mathbf{v}, \mathbf{v}; \mathbf{n}) = \mathbf{f}(\mathbf{v}) \cdot \mathbf{n}$
- Discrete cell conservation, $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\mathbf{h}(\mathbf{v}_+, \mathbf{v}_-; -\mathbf{n})$.

For a given symmetrizable system with entropy function $U(\mathbf{u})$, the DG method is uniquely specified once $\mathcal{V}^h$, the entropy function $U(\mathbf{u})$, and the numerical flux function $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ are chosen. In this formulation, the finite-dimensional space of symmetrization variables $\mathbf{v}_h$ are the basic unknowns in the trial space $\mathcal{V}^h$ and the dependent variables are then derived via $\mathbf{u}(\mathbf{v}_h)$. When not needed for clarity, this mapping is sometimes explicitly omitted, e.g. $U(\mathbf{v}_h)$ is written rather than $U(\mathbf{u}(\mathbf{v}_h))$. An important product of the DG energy analysis given below are sufficient conditions to be imposed on the numerical flux so that discrete entropy inequalities and total entropy bounds of the following form are obtained for the discretization of the Cauchy initial value problem:

- A local cell entropy inequality assuming continuous in time approximation, $\mathbf{v}_h \in \mathcal{V}^h_c$

$$\frac{d}{dt}\int_K U(\mathbf{v}_h)\, dx + \int_{\partial K} \overline{F}(\mathbf{v}_{-,h}, \mathbf{v}_{+,h}; \mathbf{n})\, ds \le 0 \ , \quad \text{for each } K \in \mathcal{T} \tag{3.5}$$

where $\overline{F}(\mathbf{v}_{-,h}, \mathbf{v}_{+,h}; \mathbf{n})$ denotes a conservative numerical entropy flux. Summing over all elements then yields the global inequality

$$\frac{d}{dt}\int_\Omega U(\mathbf{v}_h)\,dx \le 0 \ . \tag{3.6}$$

- A total entropy bound assuming full space-time approximation, $\mathbf{v}_h \in \mathcal{V}^h$

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^N)))\,dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^0))\,dx \tag{3.7}$$

where $\mathbf{u}^*(t_-^0)$ denotes the minimum total entropy state of the projected initial data

$$\mathbf{u}^*(t_-^0) \equiv \frac{1}{\text{meas}(\Omega)}\int_\Omega \mathbf{u}(\mathbf{v}_h(x,t_-^0))\,dx \ .$$

Under the assumption that the symmetrizer $\mathbf{u}_{,\mathbf{v}}$ remains spectrally bounded in space-time, i.e. there exist positive constants c_0 and C_0 independent of $\mathbf{v}_h$ such that

$$0 < c_0\,\|\mathbf{z}\|^2 \le \mathbf{z}\cdot\mathbf{u}_{,\mathbf{v}}(\mathbf{v}_h(x,t))\,\mathbf{z} \le C_0\,\|\mathbf{z}\|^2$$

for all $\mathbf{z} \neq 0$, the following L_2 stability result is then readily obtained for the Cauchy problem

$$\|\mathbf{u}(\mathbf{v}_h(\cdot,t_-^N)) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)} \le \left(\frac{C_0}{c_0}\right)^{1/2}\|\mathbf{u}(\mathbf{v}_h(\cdot,t_-^0)) - \mathbf{u}^*(t_-^0)\|_{L_2(\Omega)}$$

3.1. DG Energy Analysis for Systems without Involution. In this section, the DG energy analysis for systems of conservation laws without involution is reviewed. From Sect. 2.1 it was shown that this analysis is also the relevant analysis for the Maxwell system since this system can be symmetrized without using the Maxwell system involutions. Consequently, consider the DG method applied to the nonlinear system (3.1). For brevity, we avoid the introduction of trace operators and instead use the shorthand notation for interface quantities $f_\pm \equiv f(\mathbf{v}(x_\pm))$, $\langle f\rangle_-^+ \equiv (f_- + f_+)/2$ and $[f]_-^+ = f_+ - f_-$. An energy analysis assuming continuous in time functions, $\mathbf{v}_h \in \mathcal{V}_c^h$, yields the following cell-wise local entropy inequality which build upon previous scalar conservation law analysis for DG by [JJS95, JS94] and further related DG analysis for systems in [CS97] and [Bar98, Bar99].

THEOREM 3.1 (**DG Semi-Discrete Cell Entropy Inequality**). *Let* $\mathbf{v}_h \in \mathcal{V}_c^h$ *denote a numerical solution obtained using the discontinuous Galerkin method (3.4) assuming a continuous in time approximation for the Cauchy initial value problem (3.1) with convex entropy extension (3.2). Assume the numerical flux* $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ *satisfies the system E-flux condition*

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta))\cdot\mathbf{n}) \le 0 \ , \quad \forall \theta \in [0,1] \tag{3.8}$$

where $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\, [\mathbf{v}]_-^+$. *The numerical solution* $\mathbf{v}_h$ *then satisfies the local semi-discrete cell entropy inequality*

$$\frac{d}{dt}\int_K U(\mathbf{v}_h)\, dx + \int_{\partial K} \overline{F}(\mathbf{v}_{-,h}, \mathbf{v}_{+,h}; \mathbf{n})\, ds \le 0 \ , \quad \textit{for each } K \in \mathcal{T} \tag{3.9}$$

with

$$\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) \equiv \langle \mathbf{v} \rangle_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \langle \mathcal{F} \cdot \mathbf{n} \rangle_-^+ \tag{3.10}$$

as well as the global semi-discrete entropy inequality

$$\frac{d}{dt}\int_\Omega U(\mathbf{v}_h)\, dx \le 0\ . \tag{3.11}$$

Proof. Evaluate the energy, $B_{\rm DG}(\mathbf{v}_h, \mathbf{v}_h)$, for a single stationary element K assuming continuous in time functions

$$\begin{aligned}
\int_K \mathbf{v}\cdot \mathbf{u}_{,t}\, dx &= \frac{d}{dt}\int_K U\, dx \\
&= -\left(\int_K -\mathbf{v}_{,x_i}\cdot \mathbf{f}_i\, dx + \int_{\partial K} \mathbf{v}_- \cdot \mathbf{h}\, ds\right) \\
&= -\left(\int_K -\mathcal{F}_{i,x_i}\, dx + \int_{\partial K} \mathbf{v}_- \cdot \mathbf{h}\, ds\right) \\
&= -\int_{\partial K} (-\mathcal{F}_- \cdot \mathbf{n} + \mathbf{v}_- \cdot \mathbf{h})\, ds \\
&= -\int_{\partial K} (\ \underbrace{\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})}_{\text{Conservative Flux}} \ + \ \underbrace{D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})}_{\text{Entropy Dissipation}}\ )\, ds
\end{aligned}$$

for carefully chosen conservative entropy flux and entropy dissipation functions

$$\begin{aligned}
\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) &\equiv \langle \mathbf{v} \rangle_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \langle \mathcal{F} \cdot \mathbf{n} \rangle_-^+ \\
D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) &\equiv -\frac{1}{2}([\mathbf{v}]_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - [\mathcal{F} \cdot \mathbf{n}]_-^+)\ .
\end{aligned}$$

Observe that the chosen form of $\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ is a consistent and conservative approximation to the true entropy flux $F(\mathbf{v})$

- $\overline{F}(\mathbf{v}, \mathbf{v}; \mathbf{n}) = (\mathbf{v}\cdot \mathbf{f} - \mathcal{F})\cdot \mathbf{n} = F \cdot \mathbf{n}$ (consistency)
- $\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\overline{F}(\mathbf{v}_+, \mathbf{v}_-; -\mathbf{n})$ (conservation) .

The only remaining task is to determine sufficient conditions in the design of the numerical flux $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ so that $D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) \ge 0$. Rewriting the jump term appearing in the entropy dissipation term as a path integration in state space

$$D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\frac{1}{2}([\mathbf{v}]_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - [\mathcal{F} \cdot \mathbf{n}]_-^+)$$

$$
\begin{aligned}
&= -\frac{1}{2}[\mathbf{v}]_-^+ \cdot \left(\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \int_0^1 \mathcal{F}^T_{,\mathbf{v}}(\mathbf{v}(\theta)) \cdot \mathbf{n}\, d\theta\right) \\
&= -\frac{1}{2}[\mathbf{v}]_-^+ \cdot \left(\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \int_0^1 \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n}\, d\theta\right) \\
&= -\frac{1}{2}\int_0^1 [\mathbf{v}]_-^+ \cdot \left(\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n}\right)\, d\theta\,.
\end{aligned}
$$

A sufficient condition for nonnegativity of $D(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n})$ and the local cell entropy inequality (3.9) when applied to finite-dimensional subspaces is that the integrand be nonpositive. This yields a system generalization of Osher's famous E-flux condition for scalar conservation laws given in [Osh84]

$$
[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n}) \le 0\ ,\quad \forall\theta \in [0,1]\,. \tag{3.12}
$$

Summation of (3.9) over all elements in the mesh together with the conservative telescoping property of $\overline{F}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n})$ yields the global entropy inequality (3.11). □

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ denote ordered eigenvalues of $\mathbf{f}_{,\mathbf{u}}$. Some specific examples of system E-fluxes (proofs omitted here) include

- Symmetric variable variant of the local Lax-Friedrichs flux

$$
\mathbf{h}_{\mathrm{SLF}}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) = \langle \mathbf{f}\cdot\mathbf{n}\rangle_-^+ - \frac{1}{2}\lambda_{\max}\,[\mathbf{u}(\mathbf{v})]_{x-}^{x+} \tag{3.13}
$$

  with

$$
\lambda_{\max} \equiv \sup_{0\le\xi\le1}\ \max_{1\le i\le m} |\lambda_i(\mathbf{v}(\xi))|
$$

  where $\mathbf{v}(\xi) = \mathbf{v}_- + \xi\,[\mathbf{v}]_-^+$.

- Symmetric variable variant of the Harten-Lax-van Leer-Einfeldt flux [HLvL83, EMRS92]

$$
\mathbf{h}_{\mathrm{SHLLE}}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) = \langle \mathbf{f}\cdot\mathbf{n}\rangle_-^+ - \frac{1}{2}\,\mathbf{h}^d_{\mathrm{SHLLE}}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) \tag{3.14}
$$

  with

$$
\mathbf{h}^d_{\mathrm{SHLLE}}(\mathbf{v}_-,\mathbf{v}_+;\mathbf{n}) = \frac{\lambda_{\max}+\lambda_{\min}}{\lambda_{\max}-\lambda_{\min}}[\mathbf{f}(\mathbf{v};\mathbf{n})]_-^+ - \frac{2\lambda_{\max}\lambda_{\min}}{\lambda_{\max}-\lambda_{\min}}[\mathbf{u}(\mathbf{v})]_-^+
$$

  and

$$
\lambda_{\max} \equiv \sup_{0\le\xi\le1} \max(0,\lambda_m(\mathbf{v}(\xi)))\ ,\quad \lambda_{\min} \equiv \inf_{0\le\xi\le1} \min(0,\lambda_1(\mathbf{v}(\xi)))
$$

  where $\mathbf{v}(\xi) = \mathbf{v}_- + \xi\,[\mathbf{v}]_-^+$.

Fully discrete entropy bounds are readily derived assuming DG finite element discretization in time.

THEOREM 3.2 (**DG Fully-discrete Total Entropy Bounds**). *Let* $\mathbf{v}_h \in \mathcal{V}^h$ *denote the space-time numerical solution obtained using the discontinuous Galerkin method (3.4) for the Cauchy initial value problem (3.1) with convex entropy extension (3.2). Assume the numerical flux* $\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ *satisfies the system E-flux condition*

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n}) \leq 0 \ , \quad \forall \theta \in [0,1]$$

where $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\, [\mathbf{v}]_-^+$. *The numerical solution* $\mathbf{v}_h$ *then satisfies the total entropy bound*

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\, dx \leq \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x, t_-^N)))\, dx \leq \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x, t_-^0))\, dx \tag{3.15}$$

where $\mathbf{u}^*(t_-^0)$ *denotes the minimum total entropy state of the initial projected data*

$$\mathbf{u}^*(t_-^0) \equiv \frac{1}{\text{meas}(\Omega)} \int_\Omega \mathbf{u}(\mathbf{v}_h(x, t_-^0))\, dx \ .$$

Proof. Analysis of the spatial terms follows the same path taken in Theorem 3.1 (omitted here) with an additional integration performed in the time coordinate. Consider the energy of the remaining time evolution terms in (3.4) after integration-by-parts for a single time slab interval I^n

$$\begin{aligned}\int_{I^n}\int_\Omega \mathbf{v}\cdot \mathbf{u}_{,t}\, dx\, dt + \int_\Omega \mathbf{v}(t_+^n)\cdot [\mathbf{u}]_{t_-^n}^{t_+^n}\, dx &= \int_\Omega \int_{I^n} U_{,t}\, dt\, dx + \int_\Omega \mathbf{v}(t_+^n)\cdot [\mathbf{u}]_{t_-^n}^{t_+^n} dx \\ &= \int_\Omega \left([U]_{t_-^n}^{t_-^{n+1}} - [U]_{t_-^n}^{t_+^n} + \mathbf{v}(t_+^n)\cdot [\mathbf{u}]_{t_-^n}^{t_+^n}\right) dx\end{aligned}$$

Taylor series with integral remainder together with the duality relationship (2) yields

$$[U]_{t_-^n}^{t_+^n} - \mathbf{v}(t_-^n)\cdot[\mathbf{u}]_{t_-^n}^{t_+^n} + R^n = 0 \ , \ R^n \equiv \int_0^1 (1-\theta)\, [\mathbf{v}]_{t_-^n}^{t_+^n}\cdot \mathbf{u}_{,\mathbf{v}}(\mathbf{v}(\theta))\, [\mathbf{v}]_{t_-^n}^{t_+^n}\, d\theta \geq 0$$

where $\mathbf{v}(\theta) = \mathbf{v}(t_-^n) + \theta\, [\mathbf{v}]_{t_-^n}^{t_+^n}$. Inserting into the time evolution terms

$$\int_{I^n}\int_\Omega \mathbf{v}\cdot \mathbf{u}_{,t}\, dx\, dt + \int_\Omega \mathbf{v}(t_+^n)\cdot [\mathbf{u}]_{t_-^n}^{t_+^n}\, dx = \int_\Omega \left([U]_{t_-^n}^{t_-^{n+1}} + R^n\right) dx \ .$$

Summing over all time slabs, the first term on the right-hand side of this equation vanishes except for initial and final time slab contributions. Utilizing nonnegativity of the remainder terms R^n then yields the following inequality for the time evolution terms

$$\sum_{n=0}^{N-1}\left(\int_{I^n}\int_\Omega \mathbf{v}\cdot \mathbf{u}_{,t}\, dx\, dt + \int_\Omega \mathbf{v}(t_+^n)\cdot [\mathbf{u}]_{t_-^n}^{t_+^n}\, dx\right) \geq \int_\Omega \left(U(t_-^N) - U(t_-^0)\right)\, dx \ .$$

Assume satisfaction of the system E-flux condition, the spatial term analysis used in the proof of Theorem 3.1 reduces to the inequality

$$\sum_{n=0}^{N-1} \sum_{K\in T} \sum_{K} \int_{I^n} \left(\int_K -\mathbf{v}_{,x_i} \cdot \mathbf{f}_i \, dx + \int_{\partial K} \mathbf{v}_- \cdot \mathbf{h} \, ds \right) dt \geq 0 \, .$$

Combining temporal and spatial results yields

$$0 = B_{\rm DG}(\mathbf{v},\mathbf{v}) \geq \int_\Omega \left(U(t_-^N) - U(t_-^0) \right) \, dx \, .$$

Hence, the desired upper bound in (3.15) is established when applied to finite-dimensional subspaces

$$\int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^N)))\, dx \leq \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^0))\, dx \, . \tag{3.16}$$

To obtain the lower bound in (3.15), we exploit the well-known thermodynamic concept of a *minimum total entropy state* (see for example [Mer88]). Define the integral average state $\mathbf{u}^*$ at time slab boundaries

$$\mathbf{u}^*(t_-^n) \equiv \frac{1}{\text{meas}(\Omega)} \int_\Omega \mathbf{u}(\mathbf{v}_h(x,t_-^n))\, dx \quad , \quad n = 0,\ldots,N \, .$$

For the DG space-time discretization of the Cauchy initial value problem, $\mathbf{u}^*$ is invariant when evaluated at time slab boundaries, i.e.

$$\mathbf{u}^*(t_-^n) = \mathbf{u}^*(t_-^{n-1}) = \ldots = \mathbf{u}^*(t_-^0) \tag{3.17}$$

owing to discrete conservation in both space and time. A Taylor series with integral remainder expansion of the entropy function given two states $\mathbf{u}^*(t_-^n)$ and $\mathbf{u}(\mathbf{v}_h(x,t_-^n))$ for a fixed n yields

$$U(\mathbf{u}) = U(\mathbf{u}^*) + \mathbf{v}(\mathbf{u}^*)\cdot(\mathbf{u}-\mathbf{u}^*) + \int_0^1 (1-\theta))(\mathbf{u}-\mathbf{u}^*)\cdot U_{,\mathbf{uu}}(\theta)(\mathbf{u}-\mathbf{u}^*)\, d\theta \, .$$

When integrated over Ω, the second right-hand side term vanishes identically by the definition of $\mathbf{u}^*$

$$\int_\Omega U(\mathbf{u})\, dx = \int_\Omega U(\mathbf{u}^*)\, dx + \int_\Omega \int_0^1 (1-\theta))(\mathbf{u}-\mathbf{u}^*)\cdot U_{,\mathbf{uu}}(\theta)(\mathbf{u}-\mathbf{u}^*)\, d\theta\, dx \, .$$

From strict convexity of the entropy function, it follows that $\mathbf{u}^*$ is a minimum total entropy state since $\int_\Omega U\, dx$ is minimized when $\mathbf{u} = \mathbf{u}^*$. Finally, since $\mathbf{u}^*(t_-^n)$ is constant for $n = 0,\ldots,N$, then

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\, dx = \int_\Omega U(\mathbf{u}^*(t_-^N))\, dx \leq \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x,t_-^N)))\, dx \, .$$

This establishes the lower bound in (3.15). □

3.2. DG Stability Analysis for Systems with Solenoidal Involution. Our attention shifts to the MHD system with solenoidal involution

$$\left\{\begin{aligned} \mathbf{u}_{,t} + \mathbf{f}_{i,x_i} &= 0 \\ \mathbf{B}_{i,x_i} &= 0 \\ \mathbf{u}(x, t^0_-) &= \mathbf{u}_0(x) \end{aligned}\right. \tag{3.18}$$

with convex entropy extension

$$U_{,t} + F_{i,x_i} \leq 0\,. \tag{3.19}$$

The goal is to derive sufficient conditions for MHD system discretizations so that the cell entropy inequality (3.5), the global semi-discrete bound (3.6), and the global space-time bound (3.7) are obtained. Motivated by the Godunov MHD symmetrization theory, we consider an implementation of the DG method using the Godunov augmented MHD system.

DG FEM for MHD: Find $\mathbf{v}_h \in \mathcal{V}^h$ such that

$$B_{\text{DG-MHD}}(\mathbf{v}_h, \mathbf{w}_h) = 0 \ \ , \forall \mathbf{w}_h \in \mathcal{V}^h \tag{3.20}$$

with

$$\begin{aligned} B_{\text{DG-MHD}}(\mathbf{v}, \mathbf{w}) = \sum_{n=0}^{N-1} \Bigg(& \sum_{K\in\mathcal{T}} \int_{I^n}\int_K -(\mathbf{u}(\mathbf{v}) \cdot \mathbf{w}_{,t} + \mathbf{f}_i(\mathbf{v}) \cdot \mathbf{w}_{,x_i})\, dx\, dt \\ & - \sum_{K\in\mathcal{T}} \int_{I^n}\int_K \sigma_K\, (\mathbf{w} \cdot \phi^T_{,\mathbf{v}})\, \nabla \cdot \mathbf{B}(\mathbf{v})\, dx\, dt \\ & + \sum_{K\in\mathcal{T}} \int_{I^n}\int_{\partial K} \mathbf{w}(x_-) \cdot \mathbf{h}(\mathbf{v}(x_-), \mathbf{v}(x_+); \mathbf{n})\, ds\, dt \\ & + \sum_{K\in\mathcal{T}} \int_K \left(\mathbf{w}(t^{n+1}_-) \cdot \mathbf{u}(\mathbf{v}(t^{n+1}_-)) - \mathbf{w}(t^n_+) \cdot \mathbf{u}(\mathbf{v}(t^n_-))\right) dx \Bigg)\,. \end{aligned} \tag{3.21}$$

Observe the added $\nabla \cdot \mathbf{B}$ term with adjustable coefficient σ_K is motivated by the theory given in Sect. 2.2. The value of σ_K will be determined from the discrete energy analysis. This term is identical to that proposed by Powell [Pow94] using a different motivating argument. Unfortunately, without placing further constraints on the discrete $\mathbf{B}$ field, the Powell term is only valid for classical (smooth) solutions since this term cannot be written in divergence form. Consequently incorrect Rankine-Hugonoit jump conditions are observed for computed weak (discontinuous) solutions [Csi02]. Note that this term vanishes identically and correct weak solutions are obtained when a locally divergence-free basis is employed.

A DG analysis similar to that used in Theorem 3.1 yields the following conditions for a discrete cell entropy inequality for the MHD formulation.

THEOREM 3.3 (**DG Semi-Discrete MHD Cell Entropy Inequality**). *Let $\mathbf{v}_h \in \mathcal{V}^h_c$ denote a numerical solution obtained using the discontinuous Galerkin method (3.21) assuming continuous in time approximation for the MHD Cauchy initial value problem (3.18) with convex entropy extension (3.19). Assume the following conditions are satisfied:*

1. *Either* $\sigma_K = 1$ *or the local solenoidal condition holds pointwise*
$$\nabla \cdot \mathbf{B}(\mathbf{v}_h)|_K = 0\,.$$
2. *The MHD system E-flux condition*
$$[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n} + \phi(\mathbf{v}(\theta))\,(\mathbf{B}(\mathbf{v}(\theta)) \cdot \mathbf{n})_{,\mathbf{v}}^T) \le 0\,,$$
$\forall \theta \in [0,1]$ *where* $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\,[\mathbf{v}]_-^+$.

The numerical solution $\mathbf{v}_h$ *then satisfies the local semi-discrete cell entropy inequality*

$$\frac{d}{dt}\int_K U(\mathbf{v}_h)\,dx + \int_{\partial K} \overline{F}(\mathbf{v}_{-,h}, \mathbf{v}_{+,h}; \mathbf{n})\,ds \le 0 \;, \quad \textit{for each } K \in \mathcal{T} \tag{3.22}$$

with

$$\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) \equiv \langle \mathbf{v} \rangle_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \langle \mathcal{F} \cdot \mathbf{n} - \phi\,\mathbf{B} \cdot \mathbf{n} \rangle_-^+ \tag{3.23}$$

as well as the global semi-discrete entropy inequality

$$\frac{d}{dt}\int_\Omega U(\mathbf{v}_h)\,dx \le 0\,. \tag{3.24}$$

Proof. Evaluate the energy, $B_{\mathrm{DG}}(\mathbf{v}_h, \mathbf{v}_h)$, for a single stationary element K in the DG discretization of the MHD system assuming continuous in time approximation

$$\begin{aligned}
\frac{d}{dt}\int_K U\,dx &= -\int_K (-\mathbf{v}_{,x_i} \cdot \mathbf{f}_i)\,dx + \int_{\partial K} \mathbf{v}_- \cdot \mathbf{h}\,ds \\
&= -\int_{\partial K} (-\mathcal{F}_- \cdot \mathbf{n} + \phi_-\,(\mathbf{B}_- \cdot \mathbf{n}) + \mathbf{v}_- \cdot \mathbf{h})\,ds \\
&\quad - \int_K (1 - \sigma_K)\,\phi\,\nabla \cdot \mathbf{B}\,dx \\
&= -\int_{\partial K} (\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) + D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}))\,ds \\
&\quad - \int_K (1 - \sigma_K)\,\phi\,\nabla \cdot \mathbf{B}\,dx\,.
\end{aligned}$$

The remaining element interior term vanishes identically by either imposing $\sigma_K = 1$ or the local solenoidal condition on the magnetic induction field, $\nabla \cdot \mathbf{B}|_K = 0$. Suitable definitions for the conservative entropy flux and entropy dissipation are given by

$$\begin{aligned}
\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) &\equiv \langle \mathbf{v} \rangle_-^+ \cdot \mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) + \langle -\mathcal{F} \cdot \mathbf{n} + \phi\,\mathbf{B} \cdot \mathbf{n} \rangle_-^+ \\
D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) &\equiv -\frac{1}{2}([\mathbf{v}]_-^+ \cdot \mathbf{h} + [-\mathcal{F} \cdot \mathbf{n} + \phi\,\mathbf{B} \cdot \mathbf{n}]_-^+)\,.
\end{aligned}$$

This choice of numerical entropy flux satisfies conservation and consistency properties

- $\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\overline{F}(\mathbf{v}_+, \mathbf{v}_-; -\mathbf{n})$ (conservation)
- $\overline{F}(\mathbf{v}, \mathbf{v}; \mathbf{n}) = (\mathbf{v} \cdot \mathbf{f} - \mathcal{F} + \phi\, \mathbf{B}) \cdot \mathbf{n} = F \cdot \mathbf{n}$ (consistency) .

Rewriting the jump term appearing in the entropy dissipation term as a path integration assuming a parameterized state space $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\, [\mathbf{v}]_-^+$

$$D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) = -\frac{1}{2}([\mathbf{v}]_-^+ \cdot \mathbf{h} + [-\mathcal{F} \cdot \mathbf{n} + \phi\, \mathbf{B} \cdot \mathbf{n}]_-^+)$$
$$= -\frac{1}{2}[\mathbf{v}]_-^+ \cdot \left(\mathbf{h} - \int_0^1 (\mathcal{F}_{,\mathbf{v}}^T(\mathbf{v}(\theta)) \cdot \mathbf{n} - (\phi\, \mathbf{B} \cdot \mathbf{n})_{,\mathbf{v}}^T(\mathbf{v}(\theta)))\, d\theta\right)$$
$$= -\frac{1}{2}\int_0^1 [\mathbf{v}]_-^+ \cdot (\mathbf{h} - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n} + \phi(\mathbf{v}(\theta))\, (\mathbf{B}(\mathbf{v}(\theta)) \cdot \mathbf{n})_{,\mathbf{v}}^T)\, d\theta\, .$$

A sufficient condition for nonnegativity of $D(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ is that the integrand be nonpositive. This yields the MHD E-flux condition

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n} + \phi(\mathbf{v}(\theta))\, (\mathbf{B}(\mathbf{v}(\theta)) \cdot \mathbf{n})_{,\mathbf{v}}^T) \le 0\ , \quad \forall \theta \in [0, 1]\, .$$

This establishes the semi-discrete cell entropy inequality for MHD. Summation of (3.22) over all elements in the mesh together with the conservative telescoping property of $\overline{F}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n})$ yields the global semi-discrete entropy inequality (3.24). □

The conditions set forth in Theorem 3.3 are also sufficient to establish two-sided bounds on the total entropy.

THEOREM 3.4 (**DG Fully-discrete MHD Total Entropy Bounds**). *Let $\mathbf{v}_h \in \mathcal{V}^h$ denote the space-time numerical solution obtained using the discontinuous Galerkin method (3.21) for the MHD Cauchy initial value problem (3.18) with convex entropy extension (3.19). Assume the following conditions are satisfied:*

1. *Either $\sigma_K = 1$ or the local solenoidal condition holds pointwise*

$$\nabla \cdot \mathbf{B}(\mathbf{v}_h)|_K = 0\, .$$

2. *The MHD system E-flux condition*

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h}(\mathbf{v}_-, \mathbf{v}_+; \mathbf{n}) - \mathbf{f}(\mathbf{v}(\theta)) \cdot \mathbf{n} + \phi(\mathbf{v}(\theta))\, (\mathbf{B}(\mathbf{v}(\theta)) \cdot \mathbf{n})_{,\mathbf{v}}^T) \le 0\ ,$$

$\forall \theta \in [0, 1]$ *where* $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\, [\mathbf{v}]_-^+$.

The numerical solution $\mathbf{v}_h$ then satisfies the total entropy bound

$$\int_\Omega U(\mathbf{u}^*(t_-^0))\, dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x, t_-^N)))\, dx \le \int_\Omega U(\mathbf{u}(\mathbf{v}_h(x, t_-^0))\, dx \tag{3.25}$$

where $\mathbf{u}^*(t_-^0)$ denotes the minimum total entropy state of the initial projected data

$$\mathbf{u}^*(t_-^0) \equiv \frac{1}{\text{meas}(\Omega)} \int_\Omega \mathbf{u}(\mathbf{v}_h(x, t_-^0))\, dx\, .$$

Proof. Omitted, see Theorem 3.2. □

3.2.1. A Compatible B Field Representation. Unfortunately, conventional system E-fluxes do not satisfy the MHD system E-flux condition. Furthermore, calculation of the actual symmetrization variables for the MHD system (2.2) associated with the entropy function, $U(\mathbf{u}) = -\rho\, s$, reveals that $\mathbf{B}$ is not a vector component of $\mathbf{v}$, viz.

$$\mathbf{v} = U_{\mathbf{u}}^T = (\gamma - 1)\begin{pmatrix} \frac{\gamma - s}{\gamma - 1} + \frac{\rho \mathbf{V}^2}{2p} \\ \frac{\rho \mathbf{V}}{p} \\ -\frac{\rho}{p} \\ \frac{\rho \mathbf{B}}{p} \end{pmatrix} \tag{3.26}$$

Observe, however, that the last vector component $\rho\mathbf{B}/p$ is a $-\mathbf{B}$ multiple of the preceding component $-\rho/p$. Hence, it is possible to parameterize $\mathbf{v}$ on a line, $\mathbf{v}(\theta) = \mathbf{v}_- + \theta\,[\mathbf{v}]_-^+$, *and* constrain $\mathbf{B}\cdot\mathbf{n}$ independent of θ so that $[\mathbf{B}\cdot\mathbf{n}]_-^+ = 0$. The following lemma states that under this constraint, the MHD system E-flux condition reduces to a constrained variant of the system E-flux condition (3.8).

LEMMA 3.1 (**B Field Compatibility**). *Assume the MHD system E-flux condition as given in Theorems 3.3 and 3.4. In addition, assume that* $\mathbf{B}(\mathbf{v})\cdot\mathbf{n}$ *is constrained to be continuous at interelement interfaces, i.e.* $[\mathbf{B}(\mathbf{v})\cdot\mathbf{n}]_-^+ = 0$. *Then, under this assumption, the results of Theorems 3.3 and 3.4 are identically obtained with the MHD system E-flux condition*

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h} - \mathbf{f}(\mathbf{v}(\theta))\cdot\mathbf{n} + \phi(\mathbf{v}(\theta))(\mathbf{B}(\mathbf{v}(\theta))\cdot\mathbf{n})_{,\mathbf{v}}^T) \le 0 \quad , \ \forall\,\theta \in [0,1]$$

replaced by the constrained system E-flux condition

$$[\mathbf{v}]_-^+ \cdot (\mathbf{h} - \mathbf{f}(\mathbf{v}(\theta))\cdot\mathbf{n})|_{\mathbf{B}\cdot\mathbf{n}\,\mathrm{const}} \le 0 \quad , \ \forall\,\theta \in [0,1]\,.$$

Proof. The result follows immediately since

$$[\mathbf{v}]_-^+ \cdot (\mathbf{B}(\mathbf{v}(\theta))\cdot\mathbf{n})_{,\mathbf{v}}^T = \frac{d\mathbf{B}(\mathbf{v}(\theta))\cdot\mathbf{n}}{d\theta} = 0 \tag{3.27}$$

due to the θ independence of $\mathbf{B}\cdot\mathbf{n}$ at element interfaces. □

This result indicates the underlying intrinsic compatibility requirement of continuity in the normal component of the magnetic induction field for DG discretizations of MHD. Precise implementational details are given in a separate work [Bar04]. In that same work, several other DG discretization formulations and simplified flux functions are given which satisfy the sufficient conditions given in Theorems 3.3 and 3.4

- Transformed variable formulations
- Constrained formulations
- Penalty formulations

4. Conclusions. The energy analysis presented herein reveals the subtle interplay of involutions in the nonlinear stability of the DG method. Sufficient conditions for energy stability of DG discretizations of Maxwell and MHD systems have been obtained. From the viewpoint of discrete energy stability, analysis indicates that "standard" DG discretization Maxwell's equations are energy stable without modification. Surprisingly, sufficient conditions for MHD discretization stability place more demanding requirements as set forth in Theorems 3.3 and 3.4. More complete details and DG formulations for MHD can be found in [Bar04].

REFERENCES

[Bar98] T.J. Barth. Numerical methods for gasdynamic systems on unstructured meshes. In Kröner, Ohlberger, and Rohde, editors, *An Introduction to Recent Developments in Theory and Numerics for Conservation Laws*, volume 5 of *Lecture Notes in Computational Science and Engineering*, pages 195–285. Springer-Verlag, Heidelberg, 1998.

[Bar99] T.J. Barth. Simplified discontinuous Galerkin methods for systems of conservation laws with convex extension. In Cockburn, Karniadakis, and Shu, editors, *Discontinuous Galerkin Methods*, volume 11 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Heidelberg, 1999.

[Bar04] T.J. Barth. On the discontinuous Galerkin approximation of compressible ideal magnetohydrodynamics I: Energy stable discretizations. *In preparation*, 2004.

[BB80] J.U. Brackbill and D.C. Barnes. The effect of nonzero $\nabla \cdot \mathbf{B}$ on the numerical solution of the magnetohydrodynamic equations. *J. Comp. Phys.*, 35:426–430, 1980.

[BK04] N. Besse and D. Kröner. Convergence of locally divergence-free discontinuous Galerkin methods for the induction equations of the MHD system. Technical Report Submitted to M2AN, Wolfgang Pauli Institute, Austria, 2004.

[Boi88] G. Boillat. Involutions des systéms conservatifs. *C. R. Acad. Sci. Paris, Sére I*, 307:891–894, 1988.

[Bos98] A. Bossavit. *Computational Electromagnetism, Variational Formulations, Complementarity, Edge Elements*. Academic Press, San Diego, 1998.

[BR02] P.B. Bochev and A.C. Robinson. Matching algorithms and physics: Exact sequences of finite element spaces. In D. Estep and S. Tavener, editors, **Collected Lectures on the Preservation of Stability Under Discretization**, Philadephia, 2002. SIAM.

[CHS90] B. Cockburn, S. Hou, and C.W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case. *Math. Comp.*, 54:545–581, 1990.

[CLS89] B. Cockburn, S.Y. Lin, and C.W. Shu. TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: One dimensional systems. *J. Comp. Phys.*, 84:90–113, 1989.

[CLS04] B. Cockburn, F. Li, and C.W. Shu. Locally divergence-free discontinuous Galerkin methods for Maxwell equations. *J. Comp. Phys.*, 194:588–610, 2004.

[CS97] B. Cockburn and C.W. Shu. The Runge-Kutta discontinuous Galerkin method for conservation laws V: Multidimensional systems. Technical Report 201737, Institite for Computer Applications in Science and Engineering (ICASE), NASA Langley R.C., 1997.

[Csi02] A. Csik. *Upwind Residual Distribution Schemes for General Hyperbolic Conservation Laws with Application to Ideal Magnetohydrodynamics*. PhD thesis, University of Leuven, Belgium, 2002.

[Daf86] C. Dafermos. Quasilinear hyperbolic systems with involutions. *Arch. Rational Mech. Anal.*, 106:373–389, 1986.

[DKK+02] A. Dedner, F. Kemm, D. Kröner, C.-D. Munz, T. Schnitner, and M. Wesenberg. Hyperbolic divergence cleaning for the MHD equations. *J. Comp. Phys.*, 175:645–673, 2002.

[EMRS92] B. Einfeldt, C. Munz, P. Roe, and B. Sjögreen. On Godunov-type methods near low densities. *J. Comp. Phys.*, 92:273–295, 1992.

[God61] S. K. Godunov. An interesting class of quasilinear systems. *Dokl. Akad. Nauk. SSSR*, 139:521–523, 1961.

[God72] S. K. Godunov. The symmetric form of magnetohydrodynamics equation. *Num. Meth. Mech. Cont. Media*, 1:26–34, 1972.

[HLvL83] A. Harten, P. D. Lax, and B. van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Rev.*, 25:35–61, 1983.

[HS99] J.M. Hyman and M. Shashkov. Mimetic discretizations for Maxwell's equations. *J. Comp. Phys.*, 151:881–909, 1999.

[JJS95] J. Jaffre, C. Johnson, and A. Szepessy. Convergence of the discontinuous Galerkin finite element method for hyperbolic conservation laws. *Math. Models and Methods in Appl. Sci.*, 5(3):367–386, 1995.

[JP86] C. Johnson and J. Pitkäranta. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Math. Comp.*, 46:1–26, 1986.

[JS94] G. Jiang and C.-W. Shu. On a cell entropy inequality for discontinuous galerkin methods. *Math. Comp.*, 62:531–538, 1994.

[LR74] P. LeSaint and P.A. Raviart. On a finite element method for solving the neuton transport equation. In C. de Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 89–145. Academic Press, 1974.

[Mer88] M. L. Merriam. *An Entropy-Based Approach to Nonlinear Stability*. PhD thesis, Stanford University, 1988.

[Moc80] M. S. Mock. Systems of conservation laws of mixed type. *J. Diff. Eqns.*, 37:70–88, 1980.

[Ned80] J.C. Nedelec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35:315–341, 1980.

[Osh84] S. Osher. Riemann solvers, the entropy condition, and difference approximations. *SIAM J. Numer. Anal.*, 21(2):217–235, 1984.

[Pow94] K. G. Powell. An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension). Technical Report 94-24, Institite for Computer Applications in Science and Engineering (ICASE), NASA Langley R.C., 1994.

[RH73] W. H. Reed and T. R. Hill. Triangular mesh methods for the neutron transport equation. Technical Report LA-UR-73-479, Los Alamos National Laboratory, Los Alamos, New Mexico, 1973.

[Shu99] C.-W. Shu. Discontinuous Galerkin methods for convection-dominated problems. In Barth and Deconinck, editors, *High-Order Discretization Methods in Computational Physics*, volume 9 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Heidelberg, 1999.

[TÓ0] G. Tóth. The $\nabla \cdot \mathbf{B} = 0$ constraint in shock-capturing magnetohydrodynamics codes. *J. Comp. Phys.*, 161:605–652, 2000.

[Yee66] K.S. Yee. Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media. *IEEE Trans. Ant. Prop.*, AP-14:302–307, 1966.