ROBUST INTEGRATION SCHEMES FOR GENERALIZED VISCOPLASTICITY WITH INTERNAL-STATE VARIABLES *

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Introduction

The scope of the work in this presentation focuses on the development of algorithms for the integration of rate dependent constitutive equations. In view of their robustness; i.e., their superior stability and convergence properties for isotropic and anisotropic coupled viscoplastic-damage models, implicit integration schemes have been selected. This is the simplest in its class and is one of the most widely used implicit integrators at present.

Viscoplastic Models

Several viscoplastic models have been proposed and developed to treat the complex time dependent viscoplastic behavior of metals, alloys and composites at high temperature. The deformation behavior of materials at high temperature involves energy dissipation and material stiffness variations due to physical changes in the material's microstructure. Consequently, thermodynamic arguments have often been utilized as a foundation on which phenomenological constitutive laws may be formulated. The complete potential-based class of inelastic constitutive models exhibit a number of unique advantages from both a theoretical and a computational standpoint, for example, the symmetry of the resulting consistent tangent stiffness matrix, and possesses a form which is convenient for further development of new deformation and damage models. The Generalized Viscoplastic with Potential Structure (GVIPS) [1] model possesses both the thermodynamic potential (Gibb's function) and the dissipation function (Ω form).

Another class of constitutive models are the Non-Associative Viscoplastic (NAV) models. the NAV models refer to those that have a partially (e.g. Ω form only) or totally incomplete potential form. An example of a NAV model is that of Freed [2]. Recent work

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has demonstrated that the above models may be modified to restore the complete potential structure.

For these two different classes of models, a general computational framework suitable for implementation of both is needed.

Integration Schemes

Computational algorithms for the integration of constitutive relations play a key role in the inelastic finite element analysis of engineering structures. Consequently, much research effort has been devoted over the years to the development and critical assessment of integration schemes for the rate equations in these material models.

In early applications the explicit integration schemes, (i.e., forward Euler method) were predominate because of their ease of implementation, and because they do not require evaluating and inverting a Jacobian matrix. However, explicit integrators may not be efficient. That is, too many iteration steps may be required and convergence (stability) cannot be guaranteed. As a result, several alternative approaches have been used, for example, Gear's multi-step method [3] and Walker's asymptotic method [4]. Note that every integration scheme has its own particular application domain and is problem dependent.

The majority of recent work has emphasized the use of implicit integration methods in view of their stability and convergence properties. Based on the fully implicit, backward Euler scheme, the corresponding algorithmic (consistent) tangent stiffness arrays are derived from the integration rule, which are important for finite element solutions using (global) Newton-Raphson iterative methods.

Line Search

Although the implicit scheme is unconditionally stable, its successful application may still require proper selection of the size of the steps utilized. In this regard two factors are important: (i) accuracy, and (ii) convergence of the local iterations. A simple time subincrementing strategy was found to be effective in obtaining accurate results especially when dealing with regions of discontinuity in the state space. However this was found to be insufficient to obtain a computationally efficient solution for a highly nonlinear problem such as viscoplasticity. When a large time-step size is chosen, too many subincrementing are needed, which leads to inefficiency. Thus a more sophisticated solution procedure, namely, a line search algorithm, is required to produce an effective robust solution algorithm.

It is well know that classical Newton-Raphson is fast and stable only when the trial solution is close to the converged value. Thus, the purpose of the line search algorithm is to guide the solution towards convergence by searching for a scalar multiplier that adjusts the amount of the increment vector to be updated within each iteration [5]. The concept of line search may be applied at either the global (structural) iteration level or at the local (constitutive) iteration level. At the global level, the concept of the line search algorithm pertains to minimizing the total potential energy, that is, the work done by the residual force due to the iterative displacement. It has been suggested that the line search be incorporated with a consistent tangent stiffness and that the use of the line search is essential for robust performance of Newton's method [6,7]. It also demonstrated that in elasto-plastic analysis convergence is not guaranteed unless the global line search is used [8]. On the local (constitutive) level, line search is used to adjust the suitable increment of stress and internal variables to guarantee the convergence of the local iterations.

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MOTIVATION

Two major obstacles for fully utilizing recent time-dependent/hereditary constitutive models in practical engineering analysis

- **Lack of efficient and robust integration algorithms**
  - Coupled system of tensorial rate (differential) equations or general kernal convolution integration
  - Increased mathematical complexity and associated numerical stiffness
- Difficulties associated with characterizing large number of required material parameters

Fig. 1

OBJECTIVE

- Develop a robust and efficient integration algorithm for viscoplastic constitutive equations

Fig. 2

BACKGROUND

- Computational algorithms for integrating the constitutive models are a key component for an efficient inelastic finite element analysis
- Two classes of integration schemes are: iterative and non-iterative
- Iterative
  - Fully implicit scheme
  - Requires local iterations
- Non-iterative
  - Semi-implicit
  - Fully explicit
  - No local iterations
  - Usually less "overhead"
- History dependent integral representation
  - Full history data storage

Fig. 3

INTEGRATORS

- General differential form: $\varepsilon^I_{n+1} = \varepsilon^I_n + \Delta t\left[(1-\alpha)\dot{\varepsilon}^I_n + \alpha\dot{\varepsilon}^I_{n+1}\right]$
- Fully explicit: ($\alpha = 0$) Forward Euler
- Fully implicit: ($\alpha = 1$) Backward Euler, ($\alpha = 1/2$) Midpoint rule
  - $\dot{\varepsilon}^I$ must be evaluated at $n+\alpha$, which requires a local iterative procedure
- Semi-implicit: ($0 \le \alpha \le 1$)
  - to avoid iterations, some methods employ an approximation for $\dot{\varepsilon}^I_{n+1}$
  - these are referred to as foward gradient methods, i.e., $\dot{\varepsilon}^I_{n+1}$ is approximated in terms of known quantities at time n using a Taylor series expansion,

$$\dot{\varepsilon}^I_{n+1} = \varepsilon^I_n + \left(\frac{\partial\dot{\varepsilon}^I}{\partial\sigma}\right)_n \Delta\sigma_n + \left(\frac{\partial\dot{\varepsilon}^I}{\partial\bar{\varepsilon}^I}\right)_n \Delta\bar{\varepsilon}^I_n$$

Fig. 4

INTEGRATORS

General Integral Form:

$$A = \int_0^t K(t, \tau)\dot{\varepsilon}(\tau)d\tau$$

A = Typical state variable

$K(t, \tau)$ = Kernal function, e.g.,

- exponential
- power form
- functional derivative

Fig. 5

MODEL CLASSES INVESTIGATED

- Differential/Internal variable type
  - nonassociative/dynamic recovery (NAV)
  - fully associative/nonlinear kinematic hardening (GVIPS)

Fig. 6

NON-ASSOCIATIVE MODELS (NAV)

FREED/WALKER VISCOPLASTIC MODEL (1993)
(DYNAMIC RECOVERY)

Flow Law: $\dot{\underset{\sim}{\varepsilon}}^{I} = f(J,D)\underset{\sim}{\Gamma} \qquad \underset{\sim}{\Gamma} = \underset{\sim}{M}(\underset{\sim}{\sigma} - \underset{\sim}{\alpha}) \qquad \underset{\sim}{\alpha} = \underset{\sim}{\alpha}_s + \underset{\sim}{\alpha}_l$

$$f(J,D) = \theta A \frac{1}{2\sqrt{J}} \sinh\left(\frac{\sqrt{J}}{D}\right)^n$$

$\underset{\sim}{M}$ = isotropic/deviatoric tensor operator

Evolution Laws:

$$\dot{\underset{\sim}{\alpha}}_s = 2\underset{\sim}{Z}\left[H_s \dot{\underset{\sim}{\varepsilon}}^{I} - g_s \underset{\sim}{\pi}_s\right] \qquad \dot{\underset{\sim}{\alpha}}_l = 2\underset{\sim}{Z}\left[H_l \dot{\underset{\sim}{\varepsilon}}^{I} - g_l \underset{\sim}{\pi}_l\right]$$

$$\underset{\sim}{\pi}_s = \underset{\sim}{M}\underset{\sim}{\alpha}_s \qquad \underset{\sim}{\pi}_l = \underset{\sim}{M}\underset{\sim}{\alpha}_l \quad \dot{D} = q_J - q_D$$

$\underset{\sim}{\alpha}_s$ = “short-term” back stress; $\underset{\sim}{\alpha}_l$ = “long-term” back stress

Fig. 7

POTENTIAL BASED MODELS

GENERALIZED VISCOPLASTICITY WITH POTENTIAL STRUCTURE, GVIPS
(NONLINEAR KINEMATIC HARDENING)

Gibb's Potential: $\Psi(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) = \Psi^e(\underset{\sim}{\sigma}) + \Psi^i(\underset{\sim}{\alpha})$ Dissipation Potential: $\Omega = \Omega(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$

Flow Law: $\dot{\underset{\sim}{\varepsilon}}^{I} = \dfrac{\partial\Omega}{\partial\underset{\sim}{\sigma}} = f(F)\underset{\sim}{\Gamma} \qquad \underset{\sim}{\Gamma} = \underset{\sim}{M}(\underset{\sim}{\sigma} - \underset{\sim}{\alpha})$

$\underset{\sim}{M}$ = isotropic/anisotropic tensor operator

$$f(F) = \frac{F^n}{2\mu}$$

Evolution Law: $\dot{\underset{\sim}{\alpha}} = \underset{\sim}{L}^{-1}\dfrac{\partial\Omega}{\partial\underset{\sim}{\alpha}} = \dfrac{\partial^2\Psi}{\partial\underset{\sim}{\alpha}\partial\underset{\sim}{\alpha}}\dfrac{\partial\Omega}{\partial\underset{\sim}{\alpha}} = \underset{\sim}{L}^{-1}\left(\dot{\underset{\sim}{\varepsilon}}^{I} - \dfrac{\gamma}{h}\underset{\sim}{\pi}\right) \qquad \underset{\sim}{\pi} = \underset{\sim}{M}\underset{\sim}{\alpha}$

$$\underset{\sim}{L}^{-1} = \left[\frac{\partial^2\Psi}{\partial\underset{\sim}{\alpha}\partial\underset{\sim}{\alpha}}\right]^{-1} \text{ = hardening stiffness operator}$$

Fig. 8

IMPLICIT INTEGRATOR

$$\underset{\sim}{\Sigma}_{n+1} = \underset{\sim}{\Sigma}_n + \eta d\underset{\sim}{\Sigma}_{n+1}$$

- $\underset{\sim}{\Sigma}_{n+1}$ is the vector of state variables

For the NAV model: $\underset{\sim}{\Sigma}_n = \begin{pmatrix} \underset{\sim}{\sigma}_n \\ \underset{\sim}{\alpha}_{sn} \\ \underset{\sim}{\alpha}_{ln} \\ D \end{pmatrix}$ and for GVIPS: $\underset{\sim}{\Sigma}_n = \begin{pmatrix} \underset{\sim}{\sigma}_n \\ \underset{\sim}{\alpha}_n \end{pmatrix}$

$\underset{\sim}{\sigma}_n$ stress

$\underset{\sim}{\alpha}_n$ back stress (s and l denote short and long term)

D drag stress

Fig. 9

IMPLICIT INTEGRATOR

- $d\underset{\sim}{\Sigma}_{n+1}$ is the increment in state variables and may be expressed as:

$$d\underset{\sim}{\Sigma}_{n+1} = (\underset{\sim}{K}_{\Sigma})^{-1}\underset{\sim}{R}_{n+1}$$

- where $\underset{\sim}{K}_{\Sigma}$ is the iterative Jacobian matrix of state variables
- $\underset{\sim}{R}_{n+1}$ is the residual function of state variables, e.g., $\underset{\sim}{R}_{n+1} = \begin{pmatrix} \underset{\sim}{R}_{\sigma} \\ \underset{\sim}{R}_{\alpha_s} \\ \underset{\sim}{R}_{\alpha_l} \\ R_D \end{pmatrix}$
- and the residual functions for NAV are

$$\underset{\sim}{R}_{\sigma} = \underset{\sim}{\sigma}_{n+1} - \underset{\sim}{\sigma}_n - \underset{\sim}{C}^e(\Delta\underset{\sim}{\varepsilon} + \Delta t f \underset{\sim}{\Gamma}_{n+1})$$

$$\underset{\sim}{R}_{\alpha_{s,l}} = \underset{\sim}{\alpha}_{s,l_{n+1}} - \underset{\sim}{\alpha}_{s,l_n} - 2\Delta t H_{s,l} f \underset{\sim}{Z}\underset{\sim}{\Gamma}_{n+1} + 2\Delta t g_{s,l} \underset{\sim}{Z}\underset{\sim}{\pi}_{s,l_{n+1}}$$

$$\underset{\sim}{R}_{\alpha} = \underset{\sim}{R}_{\alpha_s} + \underset{\sim}{R}_{\alpha_l} \qquad R_D = D_{n+1} - D_n - \Delta t(q_J - q_D)$$

Fig. 10

LINE SEARCH

- The factor η is a scalar $(0 \le \eta \le 1)$ that adjusts the step size to optimize the iterative solution
  - η is obtained by a line search algorithm
  - the objective is to minimize the dot product $s_i = \left| \underset{\sim}{R}^{k}_{n+1} \bullet d\underset{\sim}{\Sigma}^{k}_{n+1} \right|$

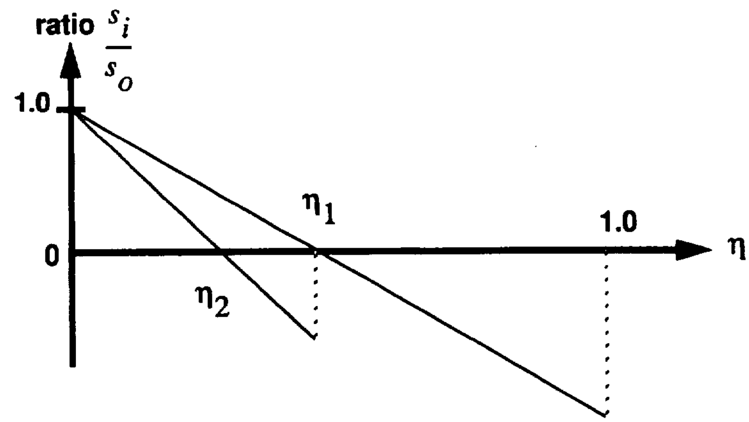
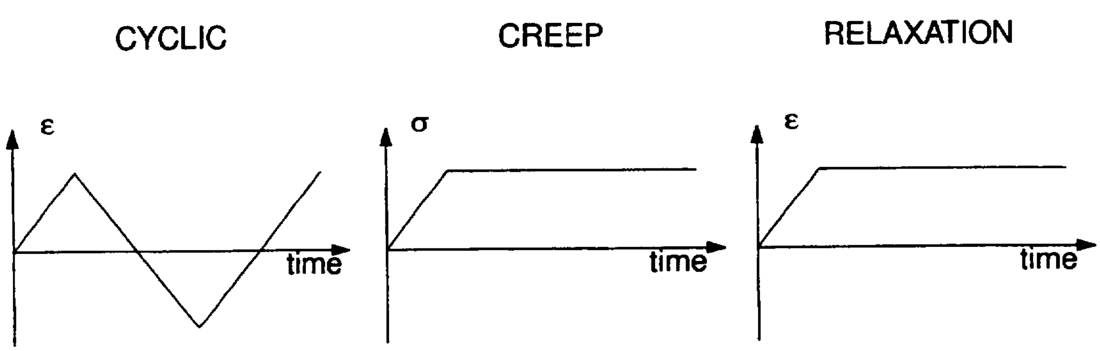


Fig. 11

VALIDATION TESTS



CYCLIC	CREEP	RELAXATION
$\dot{\varepsilon} = 2x10^{-3}\ s^{-1}$	$\dot{\sigma} = 1$ ksi/s	$\dot{\varepsilon} = 1x10^{-3}\ s^{-1}$
$\varepsilon_{max} = 0.0144$	$\sigma_{max} = 10$ ksi	$\varepsilon_{max} = 0.01$
	hold time = 1000 s	hold time = 1000 s

Fig. 12

CYCLIC TEST RESULTS

NAV MODEL

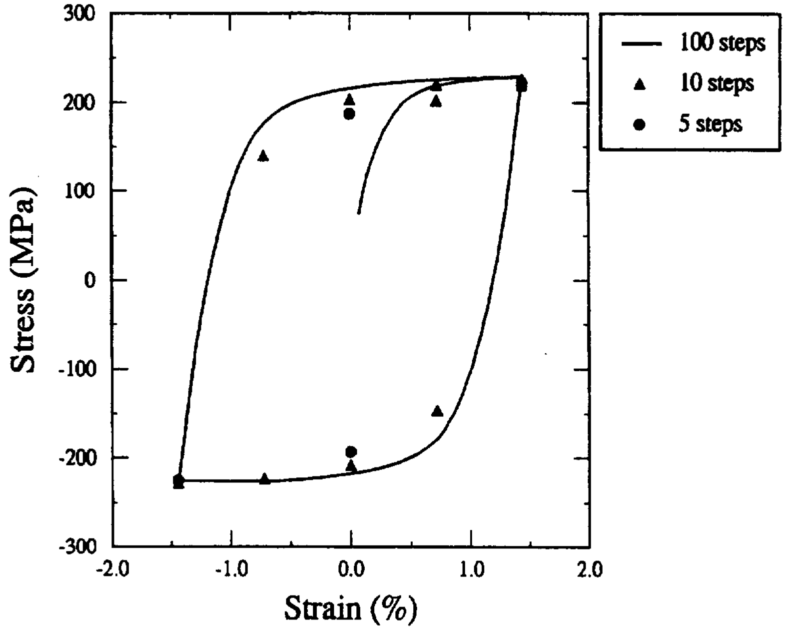


Fig. 13

CREEP AND RELAXATION RESULTS

GVIPS MODEL

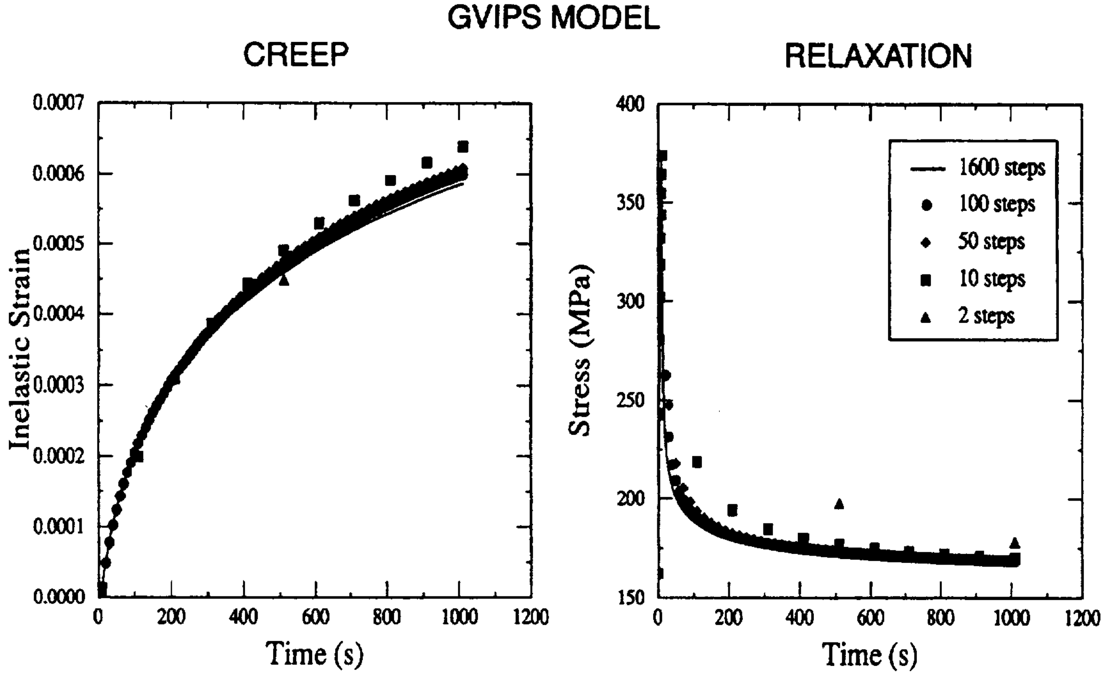


Fig. 14

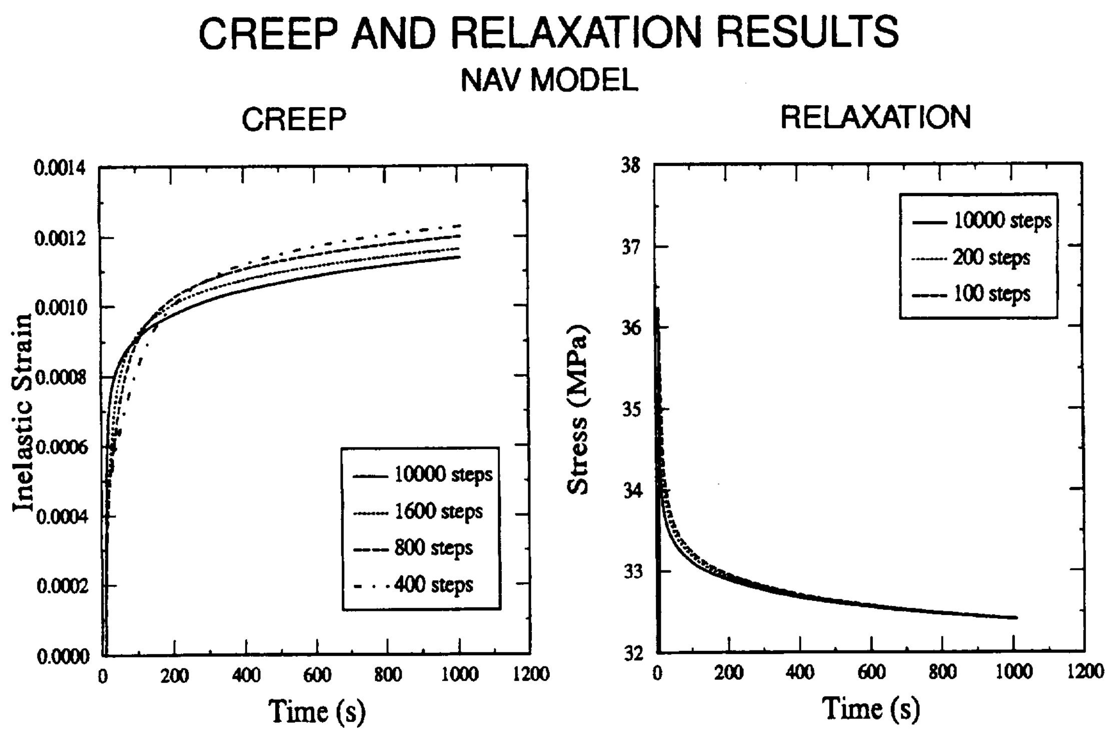

CREEP AND RELAXATION RESULTS
NAV MODEL
CREEP
RELAXATION
Inelastic Strain
Time (s)
10000 steps
1600 steps
800 steps
400 steps
Stress (MPa)
10000 steps
200 steps
100 steps

Fig. 15

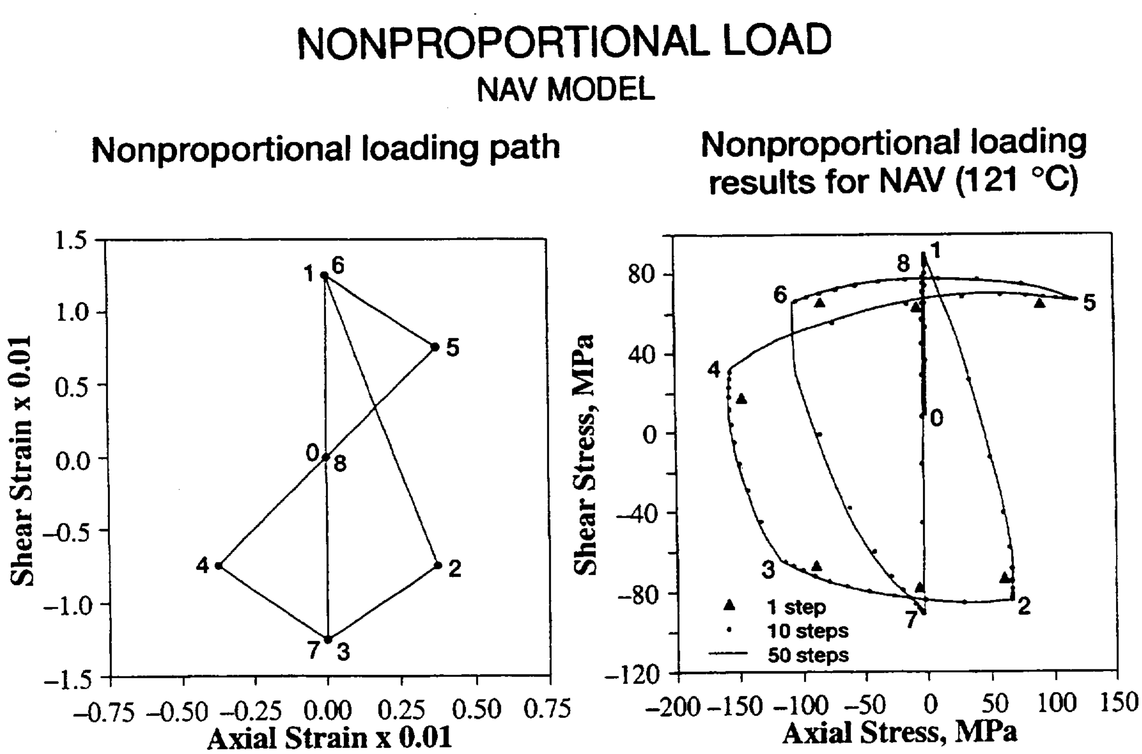

NONPROPORTIONAL LOAD
NAV MODEL
Nonproportional loading path
Nonproportional loading results for NAV (121 °C)
Shear Strain x 0.01
Axial Strain x 0.01
Shear Stress, MPa
Axial Stress, MPa
1 step
10 steps
50 steps

Fig. 16

LINE SEARCH COMPARISONS

CYCLIC TEST (NAV MODEL)

Method	CPU	GIT	SUB	LIT
Subincrementing	14.5	8	15	5
Line search	1.0	3	0	22

CREEP TEST (GVIPS MODEL)

Method	CPU	GIT	SUB	LIT
Subincrementing	1.5	4	4	2
Line search	1.0	4	0	2

RELAXATION TEST (GVIPS MODEL)

Method	CPU	GIT	SUB	LIT
Subincrementing	1.1	3	2	2
Line search	1.0	3	0	2

Fig. 17

SUMMARY/CONCLUSIONS

- Implicit integration algorithm provides unconditional stability
  - for both the creep and relaxation tests the explicit integrator (forward Euler) failed at 100 steps
  - implicit succeed using only 2 steps
  - "large" time increments for an efficient solution (computation time savings)
- Accuracy is consistent with first order formulation
  - creep: 2 steps, 8% error   relaxation: 2 steps, 2% error
- Important for life prediction studies that require many analysis load cycles and an efficient integrator
- Current algorithm used in material parameter estimator
  - analysis is performed repeatedly during optimization: requires efficiency
  - parameters vary during optimization: requires robust integrator

Fig. 18

FUTURE RESEARCH

- Currently, implicit integration algorithm has been successfully implemented into the MARC user subroutine HYPELA
- Next: Implement algorithm into ABAQUS user subroutine UMAT
- Organize computer code to allow easier implementation of new constitutive models
- Develop for combined differential/integral hereditary representations

Fig. 19