Anisotropic eddy viscosity models

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A general discussion on the structure of the eddy viscosity tensor in anisotropic flows is presented. The systematic use of tensor symmetries and flow symmetries is shown to reduce drastically the number of independent parameters needed to describe the rank 4 eddy viscosity tensor. The possibility of using Onsager symmetries for simplifying further the eddy viscosity is discussed explicitly for the axisymmetric geometry.

1. Introduction

Contrary to most of the works presented in this volume, this note does not result from a planned project for the summer program. It developed instead from discussions during the course of the workshop by many participants concerning the representation of anisotropy in the modeling of the subgrid-scale stress in Large Eddy Simulation (LES). This study is thus an attempt to present a systematic discussion of the influence of anisotropy on the structure of the eddy viscosity tensor. Some of the results presented here are not really original since they have been derived in other contexts (viscoelastic media or magnetized plasmas). However, we found several motivations for reproducing the general study of tensor symmetries in the special case of the eddy viscosity tensor.

First, we remark that there is often evidence of anisotropy at the subgrid level. The most obvious case arises when the grid itself is anisotropic. In that case, even if the flow does satisfy the classical local isotropy assumption, the subgrid velocity would be anisotropic by construction. Since most LES's use a non-uniform grid with anisotropic stretching, the effects of anisotropy should be taken into account in a very wide class of problems.

Second, the discussions we had during the workshop showed that few attempts have been made to introduce the anisotropy at the tensor level in the relation between the subgrid scale stress and the resolved strain tensor. On the contrary, most of the studies on the influence of anisotropy have focused on possible modifications to the isotropic eddy viscosity amplitude (Deardorff, 1970, 1971; Scotti *et al.*, 1993).

Finally, the development of the dynamic procedure (Germano, 1992; Ghosal *et al.*, 1995; Lilly, 1992) allows the introduction of multi-parameter models for the subgrid scale stress. Therefore, there is no practical reason for practitioners to limit their models to an isotropic eddy viscosity.

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2. Anisotropic eddy viscosity

In this work we only consider the subgrid scale modeling of an incompressible fluid. If the exact description of the large scale pressure is not required, the trace of the subgrid scale tensor may be added to the pressure, which is then calculated in order to ensure the incompressibility. The only tensor that needs to be modeled is

$$\tau_{ij}^* = \overline{u_i u_j} - \overline{u}_i \overline{u}_j - \frac{1}{3} \left(\overline{u_k u_k} - \overline{u}_k \overline{u}_k \right) \delta_{ij}. \tag{2.1}$$

The usual modeling procedure consists in giving an expression for τ_{ij}^* in terms of the spatial derivatives of the resolved velocity field $\partial_i \overline{u}_j$. These quantities are usually decomposed into a symmetric resolved strain tensor,

$$\overline{S}_{ij} = \frac{1}{2} \left(\partial_i \overline{u}_j + \partial_j \overline{u}_i \right), \tag{2.2}$$

and an antisymmetric resolved rotation tensor,

$$\overline{R}_{ij} = \frac{1}{2} \left(\partial_i \overline{u}_j - \partial_j \overline{u}_i \right) = \frac{1}{2} \epsilon_{ijk} \overline{w}_k, \tag{2.3}$$

where $\overline{w}_k$ is the vorticity and ϵ_{ijk} is the Levi-Civita fully antisymmetric tensor with $\epsilon_{123} = 1$. The most general tensorial relation in an anisotropic system thus reads:

$$\tau_{ij}^* = \nu_{ijkl} \overline{S}_{kl} + \mu_{ijkl} \overline{R}_{kl}. \tag{2.4}$$

For three dimensional turbulence, a naive analysis of this relation would lead to the conclusion that both ν and μ are described by 81 independent parameters. However, very strong simplifications can be derived by using the *tensor symmetry properties* of τ_{ij}^*, $\overline{S}_{ij}$ and $\overline{R}_{ij}$, as well as the *symmetries of the flow*. These simplifications do not require any assumption (as far as the model (2.4) is accepted). A more debatable simplification might apply if the *Onsager reciprocal symmetries* (Onsager, 1931) are assumed to hold for the eddy viscosity tensors. This will be discussed at the end of this section.

2.1 Tensor symmetries

The tensors τ_{ij}^* and $\overline{S}_{ij}$ are symmetric and traceless while the tensor $\overline{R}_{ij}$ is antisymmetric. This implies that the eddy viscosity tensor ν_{ijkl} has the following properties:

$$\begin{aligned} \nu_{ijkl} &= \nu_{jikl}, \\ \nu_{ijkl} &= \nu_{ijlk}, \\ \nu_{iikl} &= 0, \\ \nu_{ijkk} &= 0. \end{aligned} \tag{2.5}$$

Thus, for a given value of $(k, l) = (k^*, l^*)$, the matrix $a_{ij} = \nu_{ijk^* l^*}$ is traceless and symmetric. Consequently, it has 5 independent components. Similarly, for a given

value of $(i,j) = (i^*, j^*)$, the matrix $b_{kl} = \nu_{i^* j^* kl}$ is also traceless and symmetric. The full tensor ν_{ijkl} is thus described by $5 \times 5 = 25$ independent parameters. The same analysis can be performed for the tensor μ_{ijkl}, which has the following symmetries:

$$\mu_{ijkl} = \mu_{jikl},$$
$$\mu_{ijkl} = -\mu_{ijlk}, \tag{2.6}$$
$$\mu_{iikl} = 0.$$

Now, the tensor μ_{ijkl} is symmetric and traceless for its first two indices, while it is antisymmetric for its last two indices. Consequently, the full tensor μ_{ijkl} is described by $5 \times 3 = 15$ independent parameters.

2.2 Flow symmetries

This 25+15 parameter eddy viscosity tensor may be strongly simplified by using the symmetries of the flow. Let us consider some simple cases.

2.2.1 Isotropic turbulence

Any isotropic tensor can only be constructed with the unit tensor δ_{ij}. Thus, the most general isotropic tensor of rank 4 can be written as follows:

$$T_{ijkl} = a_1 \delta_{ij}\delta_{kl} + a_2 \delta_{ik}\delta_{jl} + a_3 \delta_{il}\delta_{jk}. \tag{2.7}$$

If we impose the symmetry relations (2.5), it turns out that the eddy viscosity tensor ν reduces to

$$\nu_{ijkl} = -a\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right), \tag{2.8}$$

while the symmetry relations (2.6) imply that the tensor μ vanishes. Consequently, the subgrid scale stress reads:

$$\tau_{ij}^* = -2a\overline{S}_{ij}, \tag{2.9}$$

where a is the usual isotropic eddy viscosity (Smagorinsky, 1963).

The simplest anisotropic situation arises when only one direction can be distinguished from the other. This axisymmetric geometry is thus characterized by a vector pointing to the anisotropy direction. We will show that the nature of this vector will strongly affect the structure of the eddy viscosity tensor. In particular, anisotropy induced by a pseudovector (like a magnetic field or a rotation) must be treated differently from the anisotropy induced by an axial vector (like a mean flow).

2.2.2 Axisymmetry induced by an axial vector

We first consider the case of an axisymmetry characterized by an axial vector **n**. An axisymmetric tensor of rank 4 can only be a function of this vector **n** and of the unit tensor δ_{ij}. Its most general form, compatible with the symmetry between the first two indices, reads:

$$T'_{ijkl} = b_1 \delta_{ij}\delta_{kl} + b_2 \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right)$$
$$+ b_3 \delta_{ij}n_k n_l + b_4 n_i n_j \delta_{kl} + b_5 \left(\delta_{ik}n_j n_l + \delta_{jk}n_i n_l \right) \qquad (2.10)$$
$$+ b_6 \left(\delta_{il}n_j n_k + \delta_{jl}n_i n_k \right) + b_7 n_i n_j n_k n_l \,.$$

Imposing the constraints (2.5) and defining $b_2 = -c_1$, $b_6 = -c_2$ and $b_7 = -c_3$ lead to the following expressions:

$$b_1 = (6c_1 - 4c_2 n^2 - c_3 n^4)/9 \,,$$
$$b_3 = b_4 = (4c_2 + c_3 n^2)/3 \,, \qquad (2.11)$$
$$b_5 = -c_2 \,.$$

If the constraints (2.6) are imposed on μ_{ijkl}, only two parameters are different from zero and are opposite ($b_5 = -b_6$). Thus, by introducing $b_5 = c_4$ in μ, the subgrid-scale stress reads:

$$\tau^*_{ij} = -2c_1 \overline{S}_{ij} - 2c_2 \left(n_i \overline{s}_j + \overline{s}_i n_j - \frac{2}{3}\delta_{ij}\overline{s}_k n_k \right)$$
$$- c_3 \left(n_i n_j - \frac{1}{3}\delta_{ij}n^2 \right) \overline{s}_k n_k - 2c_4 \left(\overline{r}_i n_j + n_i \overline{r}_j \right) \,, \qquad (2.12)$$

where $\overline{s}_i = \overline{S}_{ik}n_k$ and $\overline{r}_i = \overline{R}_{ik}n_k$. The effect on the resolved energy balance of the first three terms is fully determined by the sign of the parameters c_1, c_2, and c_3. Indeed, these terms correspond to dissipation (resp. creation) of resolved energy if and only if c_1, c_2, and c_3 are positive (resp. negative). On the contrary, the sign of the term proportional to c_4 in the resolved energy balance depends simultaneously on the sign of c_4 and on the flow through the factor $\overline{s}_k \overline{r}_k$:

$$\tau^*_{ij}\overline{S}_{ij} = -c_1|S|^2 - 4c_2\overline{s}^2 - c_3(\overline{s}_k n_k)^2 - 4c_4\overline{s}_k\overline{r}_k \,. \qquad (2.13)$$

If the anisotropy is weak (n is relatively small), only terms up to n^2 must be retained; since $\overline{s}_i, \overline{r}_i = O(n)$, the term proportional to c_3 can be neglected in this case.

2.2.3 Axisymmetry induced by a pseudovector

We now consider that the anisotropy direction is represented by a pseudovector **p**. The most general axisymmetric tensor of rank 4 will be a function of the vector p_i, the unit tensor δ_{ij}, and the Levi-Civita tensor ϵ_{ijk}. The situation is thus more complicated and more parameters need to be introduced. The notations will be simplified by introducing the antisymmetric tensor $V_{ij} = \epsilon_{ijk}p_k$ so that the most general tensor compatible with the symmetry between the first two indices reads:

$$T''_{ijkl} = d_1 \delta_{ij}\delta_{kl} + d_2 \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right) + d_3 \delta_{ij}V_{kl} + d_4 \left(\delta_{ik}V_{jl} + \delta_{jk}V_{il} \right)$$
$$+ d_5 \left(\delta_{il}V_{jk} + \delta_{jl}V_{ik} \right) + d_6 \left(\epsilon_{ikl}p_j + \epsilon_{jkl}p_i \right) + d_7 \delta_{ij}p_k p_l + d_8 p_i p_j \delta_{kl}$$
$$+ d_9 \left(\delta_{ik}p_j p_l + \delta_{jk}p_i p_l \right) + d_{10} \left(\delta_{il}p_j p_k + \delta_{jl}p_i p_k \right) \qquad (2.14a)$$
$$+ d_{11} \left(V_{ik}V_{jl} + V_{il}V_{jk} \right) + d_{12} \left(V_{ik}p_j p_l + V_{jk}p_i p_l \right)$$
$$+ d_{13} \left(V_{il}p_j p_k + V_{jl}p_i p_k \right) + d_{14}p_i p_j V_{kl} + d_{15}p_i p_j p_k p_l \,.$$

We will not discuss the complete tensor T'' with 15 independent parameters. Let us assume that the anisotropy is weak enough to keep only terms proportional to the vector p_i. In this case, T'' reduces to

$$
\begin{aligned}
T''_{ijkl} \approx\; & d_1 \delta_{ij}\delta_{kl} + d_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\\
& + d_3\delta_{ij}V_{kl} + d_4\left(\delta_{ik}V_{jl} + \delta_{jk}V_{il}\right)\\
& + d_5\left(\delta_{il}V_{jk} + \delta_{jl}V_{ik}\right) + d_6\left(\epsilon_{ikl}p_j + \epsilon_{jkl}p_i\right).
\end{aligned}
\tag{2.14b}
$$

Imposing the constraints (2.5) and defining $d_2 = -e_1$, $d_5 = -e_2$ and $d_6 = -e_3$ lead to the following expression:

$$
\begin{aligned}
d_1 &= 2e_1/3\,,\\
d_3 &= 2e_3\,,\\
d_4 &= -2e_3 - e_2\,,
\end{aligned}
\tag{2.15}
$$

while imposing the constraints (2.6) with the new definition $d_5 = -e_4$, and $d_6 = -e_5$ leads to

$$
\begin{aligned}
d_1 &= d_2 = 0\,,\\
d_3 &= (2e_5 - 4e_4)/3\,,\\
d_4 &= e_4\,.
\end{aligned}
\tag{2.16}
$$

The subgrid scale stress thus reads:

$$
\begin{aligned}
\tau_{ij}^* = \;& -2e_1\overline{S}_{ij} - 2(e_2 + e_3)\left(\overline{S}_{ik}V_{jk} + \overline{S}_{jk}V_{ik}\right)\\
& + 2(e_4 + e_5)\left(\overline{R}_{ik}V_{jk} + \overline{R}_{jk}V_{ik} - \frac{2}{3}\delta_{ij}\overline{R}_{kl}V_{kl}\right).
\end{aligned}
\tag{2.17a}
$$

Although the total eddy viscosity contains 5 parameters, only three of them appear independently in the expression for τ_{ij}^*. Let us note that the expression of τ_{ij}^* can be simplified by using the resolved vorticity:

$$
\begin{aligned}
\tau_{ij}^* = \;& -2e_1\overline{S}_{ij} - 2(e_2 + e_3)\left(\overline{S}_{ik}V_{jk} + \overline{S}_{jk}V_{ik}\right)\\
& + 2(e_4 + e_5)\left(\overline{w}_i p_j + \overline{w}_j p_i - \frac{2}{3}\delta_{ij}\overline{w}_k p_k\right).
\end{aligned}
\tag{2.17b}
$$

It is interesting to note that the anisotropic corrections appear at first order in the anisotropy direction p_i. Thus, we conclude that a pseudovector anisotropy (like a rotation or a magnetic field) should affect the eddy viscosity more rapidly than an axial vector anisotropy (like a grid or a flow anisotropy).

2.3 Onsager reciprocal symmetries

Strictly speaking, the Onsager reciprocal symmetries do not apply to turbulence. Indeed, they have been derived for describing the irreversible return to equilibrium in macroscopic system, and they strongly rely on the microscopic reversibility of particle motions as well as on the linearity of the transport laws. However, in an

attempt to simplify the eddy viscosity picture as much as possible, it is tempting to assume the existence of such relations for the tensors ν and μ. We will not try to justify further the use of such relations and present the form of the eddy viscosity tensors fulfilling these relations as a approximate simplification. The Onsager reciprocal relation will imply the following additional relations:

$$
\begin{aligned}
\nu_{ijkl}(\mathbf{n}) &= \nu_{klij}(\mathbf{n}), \\
\mu_{ijkl}(\mathbf{n}) &= \mu_{klij}(\mathbf{n}), \\
\nu_{ijkl}(\mathbf{p}) &= \nu_{klij}(-\mathbf{p}), \\
\mu_{ijkl}(\mathbf{p}) &= \mu_{klij}(-\mathbf{p}).
\end{aligned}
\tag{2.18}
$$

When applied to the previous results, these relations imply $c_4 = 0$ and $e_5 = -e_4$. Thus, they strongly simplify the tensor μ_{ijkl} but they do not affect the tensor ν_{ijkl} in the case presented here.

3. Anisotropic eddy viscosity and dynamic model

It has been mentioned in the introduction that the use of the dynamic procedure gives a direct access to a multiple-parameter eddy viscosity. In this section we present the dynamic derivation of the eddy viscosity tensor in the simplest anisotropic geometry: the weak axisymmetric anisotropy induced by an axial vector. Moreover, the problem is further simplified by assuming the existence of Onsager symmetries for the tensor ν_{ijkl} and μ_{ijkl}. In that case, we have shown in the previous section that the subgrid stress tensor reduces to

$$
\tau_{ij}^* = -2c_1 \overline{S}_{ij} - 4c_2 n^2 \overline{S}_{ij}^{\parallel}, \tag{3.1}
$$

where

$$
\overline{S}_{ij}^{\parallel} = \frac{1}{2n^2}\left(n_i \overline{S}_{jl} n_l + n_j \overline{S}_{il} n_l\right) - \frac{1}{3n^2}\overline{S}_{kl} n_k n_l \delta_{ij}. \tag{3.2}
$$

With the new tensor† $\overline{S}_{ij}^{\perp} = \overline{S}_{ij} - \overline{S}_{ij}^{\parallel}$ and the parameters $\nu_1 = c_1 + 2n^2 c_2$ and $\nu_2 = c_1$, the subgrid stress tensor may be rewritten as

$$
\tau_{ij}^* = -2\nu_1 \overline{S}_{ij}^{\parallel} - 2\nu_2 \overline{S}_{ij}^{\perp}. \tag{3.3}
$$

With this formulation, the resolved energy dissipation reads $\epsilon = -\tau_{ij}^* \overline{S}_{ij} = \nu_1 R_1 + \nu_2 R_2$, where

$$
R_1 = \sum_{ij} \overline{S}_{ij} \widehat{\overline{S}}_{ij}^{\parallel} = \frac{s^2}{n^2} \geq 0, \tag{3.4a}
$$

$$
R_2 = \sum_{ij} \overline{S}_{ij} \widehat{\overline{S}}_{ij}^{\perp} = \sum_{ij} \overline{S}_{ij}\left(\widehat{\overline{S}}_{ij} - \widehat{\overline{S}}_{ij}^{\parallel}\right) = \frac{1}{2}|\overline{S}|^2 - \frac{s^2}{n^2} \geq 0. \tag{3.4b}
$$

† This notation should not lead to the conclusion that $\overline{S}_{ij}^{\perp}$ and $\overline{S}_{ij}^{\parallel}$ are orthogonal. It is easy to show that $\sum_{ij} \overline{S}_{ij}^{\parallel}\overline{S}_{ij}^{\perp} \neq 0$ in general.

The last inequality is a direct consequence of the Cauchy-Schwartz inequality :

$$s^2 = \sum_i (\sum_k \overline{S}_{ik} n_k)^2 \le \sum_i \sum_k \left(\overline{S}_{ik}^2 \right) \left(\sum_l n_l^2 \right) = n^2 |\overline{S}|^2/2 \,. \qquad (3.5)$$

Sufficient conditions for having a positive resolved energy dissipation are thus $\nu_1 \ge 0$ and $\nu_2 \ge 0$. In order to devise the simplest dynamic procedure, we suppose that both ν_1 and ν_2 scale following the Kolmogorov law:

$$\nu_1 = -C_1 \Delta^{4/3} \,, \qquad (3.6a)$$

$$\nu_2 = -C_2 \Delta^{4/3} \,. \qquad (3.6b)$$

The choice for the length scale Δ in C_1 and C_2 (which are *not* dimensionless) is unimportant because the dynamic model will take care of the amplitudes. Only the power $4/3$ is important. With these definitions, the model becomes

$$\tau_{ij}^* = C_1 \rho_{ij} + C_2 \eta_{ij} \,, \qquad (3.7)$$

where

$$\rho_{ij} = -2\Delta^{4/3} \overline{S}_{ij}^{\parallel} \,, \qquad (3.8a)$$

$$\eta_{ij} = -2\Delta^{4/3} \widehat{\overline{S}}_{ij}^{\perp} \,. \qquad (3.8b)$$

Assuming a volume-averaged version of the dynamic model, the error with respect to the Germano identity is given by (Germano, 1992; Ghosal *et al.*, 1995; Lilly, 1992):

$$E_{ij}(C_1, C_2) \equiv L_{ij} + C_1 M_{ij} + C_2 N_{ij} \,, \qquad (3.9)$$

where

$$\begin{aligned} M_{ij} &= -2\,\Delta^{4/3}\,(1 - \alpha^{4/3})\widehat{\overline{S}}_{ij}^{\parallel} \,, \\ N_{ij} &= -2\,\Delta^{4/3}\,(1 - \alpha^{4/3})\widehat{\overline{S}}_{ij}^{\perp} \,, \end{aligned} \qquad (3.10)$$

where α is the ratio between test and grid filters. By minimizing E_{ij}^2, we have the two coupled equations:

$$\begin{aligned} \langle L_{ij} M_{ij} \rangle + C_1 \langle M_{ij} M_{ij} \rangle + C_2 \langle N_{ij} M_{ij} \rangle &= 0 \,, \\ \langle L_{ij} N_{ij} \rangle + C_1 \langle M_{ij} N_{ij} \rangle + C_2 \langle N_{ij} N_{ij} \rangle &= 0 \,. \end{aligned} \qquad (3.11)$$

Since M_{ij} is not aligned with N_{ij}, these two equations are not linearly proportional and they may be used for determining both C_1 and C_2.
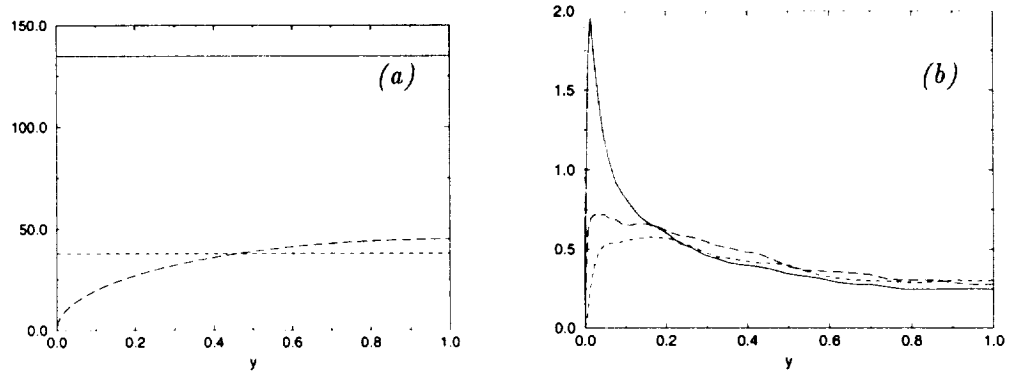
FIGURE 1. Comparison of the grid anisotropy described in terms of the grid spacing in each direction vs. the flow anisotropy described in terms of the square root of the diagonal components of the Leonard tensor. Clearly, the flow anisotropy is only important close to the wall in the streamwise direction. The grid anisotropy is also more important close to the wall, but mainly in the wall-normal direction. a) Dx+:———— ; Dy+:— — — ; Dz+:- - - - . b) Square root of Lxx:———— ; square root of Lyy:- - - - ; square root of Lzz:— — — .

4. Application to the channel flow

The dynamic formulation presented in the previous section has been implemented for channel flow with a friction Reynolds number of 1030 (cf. Cabot, 1994). The weak axisymmetric anisotropy is probably a very rough approximation for the channel geometry, so that the results presented here must be regarded as very preliminary tests. Moreover, it is not clear in channel flow which direction is the dominant anisotropic one (see Fig. 1). Indeed, channel flow is characterized by two anisotropic directions: the streamwise and the wall-normal directions. Both choices for n have been tested.

The rms values of streamwise velocity component (u') are presented in Fig. 2. The rms values of the wall-normal and spanwise velocity components $(v'$ and $w')$ seem to be insensitive to the model and are not shown here. It appears that the results from the isotropic model and the anisotropic model based on the wall-normal direction are almost indistinguishable. The results for the anisotropic model based on the streamwise direction seem to be better close to the wall. This could indicate that the flow anisotropy has more influence than the grid anisotropy. However, no definitive conclusion can be made since the model based on the streamwise direction does not perform well in the core region. Also, preliminary results indicate a long-time lack of stability for this latter model.

Finally, we present the results for the two eddy viscosity coefficients $(\nu_1$ and $\nu_2)$ in Fig. 3. For both models, the condition of positive dissipation $(\nu_1 \geq 0$ and $\nu_2 \geq 0)$ are mostly well satisfied. It is not yet known if the weakly negative values of ν_2 in the model based on the streamwise direction are responsible for its lack of stability.
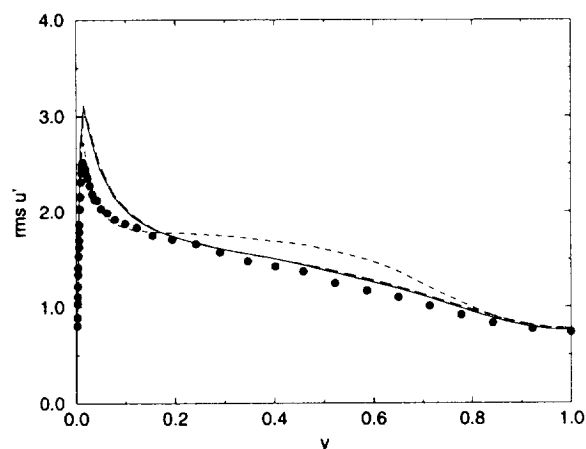
FIGURE 2. Comparison of profiles for the rms of streamwise velocity (u') between experimental data, the isotropic dynamic model and two versions of the anisotropic dynamic model based on the streamwise and wall-normal anisotropy directions. Experimental data (Hussain & Reynolds, 1970):● ; isotropic LES:——— ; anisotropic LES (n=str.dir.):———— ; anisotropic LES (n=wall norm. dir.):——— — .
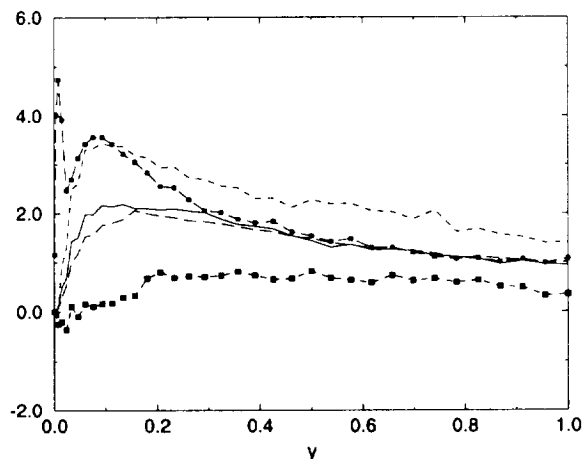

FIGURE 3. Comparison of eddy viscosity coefficients in three different models. Isotropic model: $\nu - t$ ——— ; anisotropic model (streamwise): ν_1 ———— , ν_2 ■; anisotropic model (wall normal): ν_1 ———— , ν_2 ■

5. Discussion

The use of an anisotropic eddy viscosity model has been shown to complicate dramatically the relation between the subgrid stress tensor and the resolved velocity derivatives. In particular, in the fully anisotropic geometry, 40 independent effective transport coefficients must be introduced. However, when some approximations are used, it is possible to simplify the problem drastically. As an example, we have tested the weakly anisotropic axisymmetrical geometry. In that case, the eddy

viscosity tensor reduces to a two parameter quantity. A dynamic procedure has been proposed for this problem and some tests have been made in channel flow.

These numerical tests have clearly shown that the determination of the anisotropy direction remains an important issue in the simplified anisotropic model presented in §3. Indeed, even when the flow is fully anisotropic, the model discussed in §3 may be regarded as the first tensorial invariant correction to the isotropic eddy viscosity. The use of this model could then be seen as the result of a "local axisymmetric assumption" which should be at least as robust as the local isotropic assumption. However, in that case it is probably crucial to chose the vector **n** in an appropriate way. It is also possible that the vector **n** varies with space. An interesting extension to this work would be the derivation of a dynamic procedure giving explicit expressions not only for the eddy viscosity amplitudes but also for the vector **n**.

At this point the simplest test for anisotropic models would be the homogeneous rotating turbulence. In that case, the anisotropy direction is clearly determined and is given in terms of the rotation pseudovector.

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