Characteristics of Three-node Smoothing Element Under Penalty Constraints

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Abstract

The project is based upon research of Tessler et al. (1994) on an improved variational formulation for post-processing stress predictions in Finite Element Analysis. The methodology, called Smoothing Element Analysis (SEA), employs a three-node smoothing finite element. The present effort focused on verifying the basic *constant strain criterion* for the three-node smoothing element subject to a set of internal *penalty constraints*. The convergence characteristics of the element are assessed by first deriving the constrained form of the assumed element stress and stress gradient fields, and then by verifying the validity of the constant strain criterion once the element penalty constraints are explicitly imposed. The analytical investigation is carried out with the use of the symbolic manipulation code *Mathematica*.

Introduction

An improved variational formulation called Smoothing Element Analysis (SEA), developed by Tessler et al., (1994), serves as a foundation for the enhancement of finite-element obtained deformation and stress response. In the case of stress predictions, C^{-1} - continuous stress field from a finite element solution is enforced into a C^1-continuous stress field with continuous stress gradients. These enhanced results are ideally suited for error estimation since the stress gradients can be used to assess equilibrium satisfaction. The approach is employed as a post-processing step in finite element analysis. The variational statement combines the discrete-least squares, and penalty-constraint functionals, thus enabling automated recovery of smooth stresses and stress gradients.

The practical issues whose adequate resolution is essential for a successful application of the approach are:

(1) The SEA mesh and the number of the discrete stresses extracted from the Finite Element Analysis (FEA) mesh should be properly interrelated in order to produce a determined system of SEA equations. To fulfill this requirement, Tessler et al. (1994) proposed specific guidelines. An automated generation of the SEA mesh may also be necessary to make the post-processing transparent to the user.

(2) The FEA stresses need to be extracted at the discrete elemental locations that are best suited for the recovery. Optimal (i.e., superconvergent) Barlow points and Gauss integration points have been successfully used.

(3) The smoothing element should not exhibit *locking* — a pathological stiffening phenomenon commonly exhibited in penalty-constrained elements. In this connection, a judicious choice of the element shape functions is key to avoiding *locking*.

The purpose of this effort is investigate the influence of penalty constraints on the characteristics of the smoothing element used in Tessler et al. (1994). Particularly, the convergence characteristics of Tessler's smoothing element are assessed by way of deriving the constrained form of the assumed element stress and stress gradient fields, and by verifying the validity of the constant strain criterion once the element penalty constraints are explicitly imposed. This analytical investigation is facilitated by the use of the symbolic manipulation code Mathematica.

Error Functional

In this section an error functional proposed by Tessler et al. (1994) for a two-dimensional plane formulation is reviewed. It is assumed that within a two-dimensional region $\Omega = \{\mathbf{x} \in \Re^2\}$, where $\mathbf{x}=\{x_i\}$, i=1,2, represents a position vector in Cartesian coordinates, a finite element-derived stress field $\sigma^h(\mathbf{x})$ has been obtained by means of a discretization of Ω with characteristic element size h. The smoothed stress field, $\sigma^s(\mathbf{x})$, is to be constructed from $\sigma^h(\mathbf{x})$ via a variational formulation. The variational statement involves scalar quantities only, and so each component of $\sigma^h(\mathbf{x})$ is smoothed independently. Hence, in the following reference is made only to components $\sigma^h$and σ^s. The finite element stress field is sampled at $\mathbf{x}_q$, q = 1, 2, ..., N, to obtain the set of stresses $\{\sigma_q^h\}$, i.e., $\sigma_q^h \equiv \sigma^h(\mathbf{x}_q)$. The sampled stresses are those extracted at the Gauss integration points, Barlow points, or other element locations in the finite element analysis. To minimize the error functional, we adopt the finite element methodology and therefore discretize Ω with n_{el} "recovery" or "smoothing" finite elements such that $\Omega = \cup_{e=1}^{n_{el}} \Omega^e$, where Ω^e is the domain of smoothing element e Within our recovery element model, we use C^0-continuous interpolation functions for the stress, σ^s, and the independent quantities θ_i^s, I=1, 2, whose mathematical interpretation will be readily established. The error functional to be minimized can be written as

$$\Phi = \frac{1}{\gamma}\sum_{q=1}^{N} w_q\left[\sigma_q^h - \sigma^s(\mathbf{x}_q)\right]^2 + \lambda\sum_{e=1}^{n_{el}}\int_{\Omega^e}\rho(\mathbf{x})\left[(\sigma_{,x}^s - \theta_x^s)^2 + (\sigma_{,y}^s - \theta_y^s)^2\right]d\Omega \tag{1}$$

where w_q and $\rho(\mathbf{x})$ are the appropriate weight functions; γ is a normalization factor; λ is a dimensionless parameter; and a comma denotes partial differentiation. Because the highest partial derivative in equation (1) is of order one, the field variables need only be approximated with C^0-continuous shape functions.

The first term in equation (1) represents a discrete least-squares functional in which the squared 'error' between the smoothed stress field and the sample data is computed for all sampled stresses. The term can be normalized in several different ways; presently, the normalization factor equals the total number of the sampled stresses, i.e., $\gamma = N$. The discrete weights w_q are introduced so that sample data known to be of higher accuracy can be assigned more weight than less accurate data.

The second term in equation (1) represents a penalty functional which, for λ sufficiently large, enforces the derivatives of the smoothed stress field $\sigma_{,i}^s$ to approach the corresponding θ_i^s variable pointwise, i.e.,

$$\sigma_i^s \rightarrow \theta_i^s \quad (i=\text{x,y}) \text{ in } \Omega^e \tag{2}$$

Theoretically, the greater the value of λ, the closer the correlation between $\sigma^s_{,i}$ and θ^s_i, where C^1 continuity of σ^s is achieved as $\lambda \rightarrow \infty$. In practice, λ needs to be sufficiently large in order to enforce conditions (2), yet it should not be excessively large to cause ill-conditioning of the *smooth* solution. Because θ^s_i are interpolated with continuous functions, the smoothed stress field, for all practical purposes, is C^1 continuous. The weight function $\rho(\mathbf{x})$ is introduced in the functional to allow the enforcement of C^1 continuity to be somewhat relaxed in certain regions of Ω and more strictly enforced in others, if so desired. Also note that by specifying the weights w_q and $\rho(\mathbf{x})$ to vanish in regions outside a given domain of interest, a *local* or *patch* analysis is admitted. Presently, we only consider the special case where $w_q = 1$ and $\rho(\mathbf{x}) = 1$, that is all stress data are treated equally and the C^1 continuity is enforced throughout the Ω domain.

Assumed Element Fields

Although the functional (1) admits C^0-continuous shape functions for the field variables σ^s, θ^s_x, and θ^s_y, the constraints (2) impose certain restrictions on the suitable choice of shape functions. In a similar plate theory formulation, constraints of this type are known to cause *locking* (i.e., severe stiffening) when conventional isoparametric interpolations are used. (In the present context, *locking* would manifest itself in a smoothed stress field, σ^s, that grossly underestimates the 'true' stress distribution.) When σ^s is interpolated with a polynomial one degree higher than those for the θ^s_x and θ^s_y variables, using *anisoparametric* interpolations, the *locking* effect is alleviated or completely eliminated (Tessler, 1985).

Another important consideration is the nodal configuration that is best suited for the smoothing element. It turns out that a three-node triangle is well-suited for this purpose because (a) from a modeling standpoint, it represents the most versatile element topology, and (b) it permits a one-to-one linear mapping between the global and element local (area-parametric) coordinates, thus allowing a straightforward identification of the sampled stress data within the smoothing element.

The anisoparametric interpolations for a three-node element involve quadratic approximation of σ^s and linear approximations of θ^s_x and θ^s_y which can be expressed in matrix form as

$$\sigma^s = \mathbf{z}\sigma^e + \mathbf{m}\theta^e_x + \mathbf{l}\theta^e_y, \quad \theta^s_i = \mathbf{z}\theta^e_i \;\; (i=1,2) \tag{3}$$

where σ^e, θ^e_i are 3x1 vectors of nodal degrees-of-freedom (dof), $\mathbf{z}$ is selected as a row-vector of a linear shape function, and $\mathbf{m}$ and $\mathbf{l}$ are selected as row-vectors of quadratic shape functions. Their explicit forms given in terms of area-parametric coordinates are

$$\mathbf{z} = \{z_1, z_2, z_3\}, \quad \mathbf{m} = \{m_1, m_2, m_3\}, \quad \mathbf{l} = \{l_1, l_2, l_3\} \tag{3.1}$$

where

$$z_i = \frac{1}{2A}(c_i + b_i x + a_i y), \quad m_i = \frac{1}{2}(a_k z_i z_j - a_j z_i z_k), \quad l_i = \frac{1}{2}(b_j z_i z_k - b_k z_i z_j)$$
$$a_i = x_k - x_j, \quad b_i = y_j - y_k, \quad c_i = x_j y_k - x_k y_j \quad (i = 1,2,3, \quad j = 2,3,1, \quad k = 3,1,2)$$

and A denotes the area of a triangular element.

Note that these interpolations are consistent with a three-node element which has only three dof per node, even though σ^s is quadratic and θ_i^s are linear functions. Moreover, equations (3) ensure that the gradient of the smoothed stress, $\sigma^s_{,i}$, is the same degree polynomial as that representing θ_i^s, i.e., they are both linearly distributed across Ω^e. This naturally leads to a reasonable expectation that penalty constraints (2) can be adequately fulfilled without over constraining (*locking*) the element.

Edge Penalty Constraints

A straightforward manipulation of the two constraint equations in (2), in which equations (3) are introduced, produces three edge-wise constraints per element. For the element edge defined by nodes i and j, the edge constraint equation has a simple form in terms of the nodal dof corresponding to the edge (Pomeranz, 1995; Tessler, 1985):

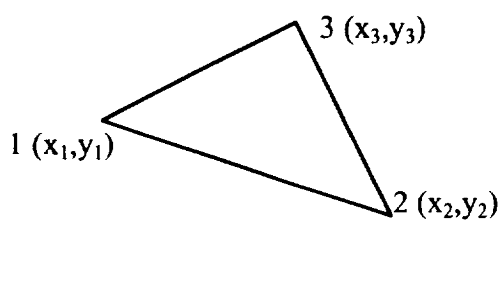
$$\sigma_i^e - \sigma_j^e \rightarrow \tfrac{1}{2}(x_i - x_j)(\theta_{xi}^e + \theta_{xj}^e) + \tfrac{1}{2}(y_i - y_j)(\theta_{yi}^e + \theta_{yj}^e) \tag{4}$$

where x_k and y_k $(k = i, j)$ are the nodal coordinates.

The three edge constraint equations ensure that there are only six independent dof per element, thus properly describing the complete parabolic field of σ^s. They also facilitate a simple calculation of the total number of independent dof in the mesh. The key aspect of these constraints is that they control the mechanisms of *locking*. Their assessment in the context of assembly of elements can provide proper insight into preferable discretization patterns for such elements. For example, a fully non-locking behavior is achieved by producing SEA meshes made of quadrilateral *macro-elements* that are formed with four triangles in a cross-diagonal pattern.

Constant Strain Criterion

Let us consider an arbitrary triangular element as shown in the diagram below.



The satisfaction of the *constant strain criterion* in the finite element method ensures convergence of the method as the mesh is refined. In this case, it is expected that each individual finite element accommodates constant strains. Mathematically, the criterion is verified by summing up on all shape functions for each field that is approximated, and the resulting sum should add up to unity. This can be readily verified for the unconstrained element fields in (3).

The constraint equations for the edges of this element can be written as follows.

Edge 1-2

$$\sigma_1 - \sigma_2 = \tfrac{1}{2}\big((x_1 - x_2)(\theta_{x1} + \theta_{x2}) + (y_1 - y_2)(\theta_{y1} + \theta_{y2})\big)$$

Edge 2-3

$$\sigma_2 - \sigma_3 = \tfrac{1}{2}\big((x_2 - x_3)(\theta_{x2} + \theta_{x3}) + (y_2 - y_3)(\theta_{y2} + \theta_{y3})\big)$$

Edge 3-1

$$\sigma_3 - \sigma_1 = \tfrac{1}{2}\big((x_3 - x_1)(\theta_{x3} + \theta_{x1}) + (y_3 - y_1)(\theta_{y3} + \theta_{y1})\big) \qquad (5)$$

Using Mathematica, the three constraint equations are solved for σ_1, σ_2, and θ_{x3}. When these solutions are substituted into the original definitions for σ^s, θ_x^s, and θ_y^s, the following expressions of the three element fields are derived

$$\begin{aligned}
\sigma^s &= g_1\sigma_3 + g_2(x)\theta_{x1} + g_3(x)\theta_{x2} + g_4(x,y)\theta_{y1} + g_5(x,y)\theta_{y2} + g_6(x,y)\theta_{y3} \\
\theta_x^s &= d_1(x)\theta_{x1} + d_2(x)\theta_{x2} + d_3(x,y)\theta_{y1} + d_4(x,y)\theta_{y2} + d_5(x,y)\theta_{y3} \\
\theta_y^s &= e_1(x,y)\theta_{y1} + e_1(x,y)\theta_{y2} + e_3(x,y)\theta_{y3}
\end{aligned} \qquad (6)$$

where g, d, and e are shape functions whose expressions are summarized in the Appendix.

To verify the constant strain criterion for the resulting element fields, the summation of the g, d, and e shape functions is carried out with the use of Mathematica. The resulting equations are as follows

$$g \equiv \sum_{i=1}^{6} g_i = 1 + x - x_3 + y - y_3, \quad d \equiv \sum_{i=1}^{5} d_i = 1, \quad e \equiv \sum_{i=1}^{3} e_i = 1 \qquad (7)$$

Note that both θ_x^s and θ_y^s fulfill the constant strain criterion for a finite size element since d $=1$ and e=1. On the other hand, g only approaches unity in the limit as the element size diminishes to zero, i.e.,

$$x \to x_3, \quad y \to y_3 \qquad (8)$$

giving rise to $g \to 1$. Thus, the element convergence is ensured as the smoothing mesh is refined.

Conclusions

The project has been a continuation of the research of Tessler et al. (1994) on an improved variational formulation for post-processing stress predictions in Finite Element Analysis. The effort focused on verifying the basic *constant strain criterion* for the three-node smoothing element subject to a set of internal *penalty constraints*. The convergence characteristics of the element were assessed by first deriving the constrained form of the assumed element stress and stress gradient fields through a process of simplifications using *Mathematica*. The element penalty constraints were explicitly imposed using the formulas set out by Tessler et al. (1994). Then the validity of the constant strain criterion for the element was verified. As the smoothing mesh is refined, the constant strain criterion is satisfied.

References

Tessler, A., Riggs, H. R., Macy, S. C., 1994, "A variational Method for Finite Element Stress Recovery and Error Estimation," *Computer Methods in Applied Mechanics and Engineering*, Vol. 111 pp. 369-382.

Tessler, A., 1985, "A Priori Identification of Shear Locking and Stiffening in Triangular Mindlin Elements, *Computer Methods in Applied Mechanics and Engineering*, Vol. 53, pp.183-200.

Pomeranz, S., 1995, Personal Communication.

Wolfram, S., 1988, *Mathematica™ A System for Doing Mathematics by Computer*, Addison-Wesley Publishing Company, Inc., Redwood City, CA.

Appendix

The following are the shape functions in (6) as solved by Mathematica™.

g_1= 1

g_2= ((-x + x3)*(x - 2*x2 + x3))/(2*(-x1 + x2))

g_3= ((x - x3)*(-x + 2*x1 - x3))/(2*(x1 - x2))

g_4= (x1*x2*y^2 - x2^2*y^2 - x1*x3*y^2 + x2*x3*y^2 - 2*x*x1*y*y2 + 2*x*x2*y*y2 + 2*x1*x3*y*y2 - 2*x2*x3*y*y2 + x^2*y1*y2 - 2*x*x2*y1*y2 + 2*x2*x3*y1*y2 - x3^2*y1*y2 - x^2*y2^2 + 2*x*x1*y2^2 - 2*x1*x3*y2^2 + x3^2*y2^2 + 2*x*x1*y*y3 - 2*x*x2*y*y3 - 2*x1*x2*y*y3 + 2*x2^2*y*y3 - x^2*y1*y3 + 2*x*x2*y1*y3 - 2*x2*x3*y1*y3 + x3^2*y1*y3 + x^2*y2*y3 - 2*x*x1*y2*y3 + 2*x1*x3*y2*y3 - x3^2*y2*y3 + x1*x2*y3^2 - x2^2*y3^2 - x1*x3*y3^2 + x2*x3*y3^2)/(2*(-x1 + x2)*(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3))

g_5=(-(x1^2*y^2) + x1*x2*y^2 + x1*x3*y^2 - x2*x3*y^2 + 2*x*x1*y*y1 - 2*x*x2*y*y1 - 2*x1*x3*y*y1 + 2*x2*x3*y*y1 - x^2*y1^2 + 2*x*x2*y1^2 - 2*x2*x3*y1^2 + x3^2*y1^2 + x^2*y1*y2 - 2*x*x1*y1*y2 + 2*x1*x3*y1*y2 - x3^2*y1*y2 - 2*x*x1*y*y3 + 2*x1^2*y*y3 + 2*x*x2*y*y3 - 2*x1*x2*y*y3 + x^2*y1*y3 - 2*x*x2*y1*y3 + 2*x2*x3*y1*y3 - x3^2*y1*y3 - x^2*y2*y3 + 2*x*x1*y2*y3 - 2*x1*x3*y2*y3 + x3^2*y2*y3 - x1^2*y3^2 + x1*x2*y3^2 + x1*x3*y3^2 - x2*x3*y3^2)/(2*(-x1 + x2)*(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3))

g_6= (-(x1*y) + x3*y + x*y1 - x3*y1 - x*y3 + x1*y3)/(2*(-x1 + x2)) + (x2*y - x3*y - x*y2 + x3*y2 + x*y3 - x2*y3)/(2*(-x1 + x2)) + ((-x + x3)*(y1 - y2)*(-(x1*y) + x2*y + x*y1 - x2*y1 - x*y2 + x1*y2))/(2*(-x1 + x2)*(x2*y1 - x3*y1 - x1*y2 + x3*y2 + x1*y3 - x2*y3)) + ((x1*y - x2*y - x*y1 + x2*y1 + x*y2 - x1*y2)*(y - y3))/(2*(x2*y1 - x3*y1 - x1*y2 + x3*y2 + x1*y3 - x2*y3))

d_1= (x - x2)/(x1 - x2)

d_2= (x - x1)/(-x1 + x2)

d_3= ((-(x1*y) + x2*y + x*y1 - x2*y1 - x*y2 + x1*y2)*(y2 - y3))/((-x1 + x2)*(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3))

d_4= ((x1*y - x2*y - x*y1 + x2*y1 + x*y2 - x1*y2)*(-y1 + y3))/((-x1 + x2)*(x2*y1 - x3*y1 - x1*y2 + x3*y2 + x1*y3 - x2*y3))

d_5= ((y1 - y2)*(-(x1*y) + x2*y + x*y1 - x2*y1 - x*y2 + x1*y2))/((-x1 + x2)*(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3))

e_1= (x2*y - x3*y - x*y2 + x3*y2 + x*y3 - x2*y3)/(x2*y1 - x3*y1 - x1*y2 + x3*y2 + x1*y3 - x2*y3)

e_2= (x1*y - x3*y - x*y1 + x3*y1 + x*y3 - x1*y3)/(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3)

e_3= (-(x1*y) + x2*y + x*y1 - x2*y1 - x*y2 + x1*y2)/(-(x2*y1) + x3*y1 + x1*y2 - x3*y2 - x1*y3 + x2*y3)

Notation

In the above expressions, x1=x_1, y1=y_1, x1^2=x_1^2, and the asterisk (*) denotes multiplication.