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THE PROPAGATION OF A LIQUID BOLUS THROUGH AN ELASTIC TUBE AND AIRWAY REOPENING

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ABSTRACT

We use lubrication theory and matched asymptotic expansions to model the quasi-steady propagation of a liquid bridge through an elastic tube. In the limit of small capillary number, asymptotic expressions are found for the pressure drop across the bridge and the thickness of the liquid film left behind, as functions of the capillary number, the thickness of the liquid lining ahead of the bridge and the elastic characteristics of the tube wall. For a given precursor thickness, we find a critical propagation speed, and hence a critical imposed pressure drop, above which the bridge will eventually burst, and hence the tube will reopen.

INTRODUCTION

An airway in a lung may crudely be described as an elastic tube with a viscous liquid lining. It is well-known that this lining is susceptible to an instability driven by capillarity at its free surface. If the initial lining thickness is sufficiently large, the instability culminates in the formation of a liquid bridge (known in the medical context as a bolus) which blocks the tube. This phenomenon, known as airway closure, can lead to respiratory difficulty. It is known to be exacerbated by increased liquid content in the lung and by increased airway wall flexibility (ref. 1). In normal gravity, airway closure is generally confined to the lower regions of the lung. However, in microgravity airway closure still occurs and appears to be more homogeneously distributed (ref. 2).

Once such a liquid bolus has been formed, it propagates along the airway under the pressure drops imposed across it during respiration. As it does so, it leaves behind a thin liquid film. If the thickness of the trailing film exceeds that of the liquid lining ahead of the bolus, then the bolus must decrease in volume as it propagates, leading ultimately to reopening of the airway. In this paper we model the propagation of a liquid bolus along an elastic tube and hence deduce minimum criteria for airway reopening. In particular, we find that a more compliant airway is easier to reopen.

SYMBOLS

See figure 1 for a definition sketch.

a	tube radius
U	bolus propagation speed
σ	liquid surface tension
μ	liquid viscosity
h_1, h_2	trailing film thickness, precursor film thickness
P_1, P_2, P_b	pressure behind bolus, ahead of bolus and in bolus respectively.
w	tube wall displacement
Ed	tube wall stiffness (Youngs modulus times thickness)

Dimensionless parameters

Capillary number $Ca = \mu U/\sigma$

Dimensionless pressure drop $\Delta P = a\,(P_1 - P_2)\,/\,2\sigma$

Dimensionless wall flexibility $G = \dfrac{\sigma}{Ed}\left(\dfrac{a}{h_1}\right)^2 (3\,Ca)^{2/3}$

ASYMPTOTIC ANALYSIS

Throughout this paper we assume that the flow is axisymmetric and that inertia may be neglected (*i.e.* that the Reynolds number is small); our governing equations are therefore the axisymmetric Stokes equations. At the free surface of the liquid, we have the classical balance between liquid stress and interfacial tension. We perform an asymptotic analysis of the problem, assuming that the capillary number is a small parameter. This approach has been adopted by many previous authors, and applied to such problems as the flow of bubbles in tubes (ref. 3, ref. 4), the dynamics of free surfaces in Hele-Shaw cells (ref. 5) and plate withdrawal (ref. 6). Our analysis represents a generalisation of these papers to allow for flexibility of the tube wall. In this paper we employ a very simple (essentially linear) relation between the inwards wall displacement w and the pressure drop across it, namely

$$P_1 - P_b - \frac{\sigma}{a} = \frac{Ed}{(a-w)}\,\frac{w}{a}. \tag{1}$$

The key observation is that at small capillary number, in most of the flow domain viscous dissipation is dominated by surface tension. Indeed in the leading-order problem, obtained by setting the capillary number to zero, there is no flow and the problem is simply one of capillary statics. Formally, nondimensionalising the Stokes equations (using U, a and $2\sigma/a$ as scales for velocity, length and pressure respectively) reveals that

$$P_b \sim \text{const.} + O(Ca), \tag{2}$$

and so the two free surfaces are, to leading order, surfaces of constant mean curvature, that curvature being the ratio between the constant pressure difference across each surface and the interfacial tension σ. Also imposing that the surfaces be axisymmetric and analytic, the only admissible constant-mean-curvature surfaces are hemispheres, with leading-order mean curvature $2/a$.

We can anticipate that the scalings employed above will break down where the liquid free surfaces become close to the tube wall, since the hemispheres described above cannot be joined analytically to the uniform films ahead of and behind the bolus. Instead, there are transition regions (see figure 1) between the uniform films and the hemispheres, in which the pressure ceases to be constant as viscous dissipation becomes important. However, asymptotic simplification can still be achieved since in the transition regions the liquid layer is thin, with a slowly-varying free surface, so that we can employ lubrication theory.

Consider first the rear meniscus. If the film thickness h and wall displacement w are nondimensionalised with h_1 and axial distance z with $h_1\,(3Ca)^{-1/3}$, then in a frame moving with the bolus h

is found at leading order to satisfy a variant of the Landau-Levich equation:

$$w'''(z) + h'''(z) = \frac{h-1}{h^3}. \tag{3}$$

Coupled to this is the leading-order dimensionless wall law from **(1)**, namely

$$w = G(w'' + h''), \tag{4}$$

where G is the dimensionless flexibility: as $G \to 0$, the tube becomes rigid and the classical Landau-Levich equation is recovered.

Equations **(3)** and **(4)** combine to a single ordinary differential equation for $\eta := h + w$ (*i.e.* the free-surface profile):

$$\eta''' = \frac{\eta - G\eta'' - 1}{(\eta - G\eta'')^3}. \tag{5}$$

An initial condition for **(5)** is that the film become uniform far away from the bolus, and by linearising about $\eta = 1$ it is apparent that this one initial condition specifies the problem completely. Moreover the unique solution satisfying this condition has the asymptotic behaviour

$$\eta \sim 1 + e^{lz} \quad \text{as} \quad z \to -\infty, \tag{6}$$

where l is the unique real, positive root of the cubic

$$\lambda^3 + G\lambda^2 - 1 = 0; \tag{7}$$

equation **(6)** is used as an initial condition in integrating **(5)** numerically.

The solution of **(5)** with initial condition **(6)** is found to behave quadratically for large, positive z:

$$\eta \sim \frac{1}{2}Az^2 + Bz + C \quad \text{as} \quad z \to \infty, \tag{8}$$

where A, B and C are numerically-determined constants. Notice that these constants are not all uniquely determined because the origin for z may be chosen arbitrarily; however A and the combination $F \equiv AC - B^2/2$ are independent of the origin chosen for z and so are determined uniquely for any fixed G.

The final step is to match the transition region with the outer, bolus solution. Formally, we apply Van Dyke's matching rule (ref. 7) to the three-term outer solution (in which the free surface is hemispherical up to $O(Ca)$) and the one-term inner solution found above. Matching the curvature gives the well-known asymptotic relation for the trailing film thickness:

$$\frac{h_1}{a} \sim A(G)\,(3\,Ca)^{2/3}. \tag{9}$$

Moreover, the higher-order matching gives asymptotic expressions for the apparent contact radius and contact angle of the meniscus. From these, the first perturbation to the meniscus curvature can be found, and hence the pressure difference across the rear meniscus is given asymptotically by

$$\frac{a}{2\sigma}(P_1 - P_b) \sim 1 + F(G)(3Ca)^{2/3}. \tag{10}$$

Now we apply the same asymptotic arguments to the front meniscus. The same differential equation **(5)** is found for η (here nondimensionalised with the precursor film thickness h_2), with G replaced by $\tilde{G} := G h_1^2/h_2^2$ and the initial condition of uniform η now imposed for large positive z. In contrast to the rear meniscus, this problem admits a one-parameter family of solutions, with asymptotic behaviour

$$\eta \sim 1 + \alpha e^{-z/2\tilde{l}^2} \cos(\sqrt{(4\tilde{l}^2 - 1)}z/2\tilde{l}^2) \quad \text{as} \quad z \to \infty, \tag{11}$$

($\tilde{l}$ is the real positive solution of **(7)** with G replaced by $\tilde{G}$). Once again the transition film behaves quadratically as it approaches the meniscus:

$$\eta \sim \frac{1}{2}\tilde{A}z^2 + \tilde{B}z + \tilde{C} \quad \text{as} \quad z \to -\infty. \tag{12}$$

Now, however, the translation-invariant constants $\tilde{A}$ and $\tilde{F} \equiv \tilde{A}\tilde{C} - \tilde{B}^2/2$ are both functions of the initial condition α. Matching the parabola **(12)** with the front meniscus gives

$$\tilde{A} = \frac{h_2}{a(3Ca)^{2/3}}; \tag{13}$$

the shooting parameter α must be adjusted until $\tilde{A}$ satisfies this (recall that h_2 is a physically specified quantity, while h_1 is a quantity to be found). The pressure difference across the front meniscus is given by

$$\frac{a}{2\sigma}(P_2 - P_b) \sim 1 + \tilde{F}(3Ca)^{2/3}. \tag{14}$$

Here, $\tilde{F}$ is in effect a function of $\tilde{A}$: for each fixed $\tilde{G}$ they may be plotted against one another using α as a parameter. Hence the total pressure drop driving the bolus is related to its propagation speed by

$$\frac{a}{2\sigma}(P_1 - P_2) \sim \left[F(G) - \tilde{F}\left(\tilde{G}; \frac{h_2}{a(3Ca)^{2/3}}\right)\right](3Ca)^{2/3}. \tag{15}$$

For a rigid tube, when $G \equiv \tilde{G} \equiv 0$, this means that, given the precursor layer thickness h_2 and the imposed pressure drop $P_1 - P_2$, the propagation speed of the bolus is given by **(15)**; then the trailing film thickness h_1 can be found from **(9)**. In general the situation is more complicated; **(15)** and **(9)** are coupled since h_1 appears in G.

REOPENING CRITERION

The condition for the bolus eventually to rupture is that h_1 be greater than h_2. From **(9)** we can deduce a critical capillary number above which this condition is satisfied:

$$\textbf{reopening} \;\rightarrow\; Ca > Ca_c = \frac{1}{3}\left(\frac{h_2}{a}\right)^{3/2} f\left(\frac{a}{h_2}\frac{\sigma}{Ed}\right), \qquad \textbf{(16)}$$

where f is a numerically-determined function. Hence from **(15)** the minimum pressure difference ΔP_c required to reopen the tube takes the form

$$\Delta P_c = \frac{2\sigma h_2}{a^2} g\left(\frac{a}{h_2}\frac{\sigma}{Ed}\right), \qquad \textbf{(17)}$$

where g is a second numerically-determined function.

Typical critical pressure drops are plotted against the flexibility parameter σ/Ed in figure 2. It can be seen that increased wall flexibility decreases the critical pressure drop, *i.e.* makes the tube easier to reopen. The effect appears to be very slight compared to the increase in critical pressure drop caused by increasing precursor layer thickness; this is not surprising given that the present analysis is only valid for relatively stiff walls.

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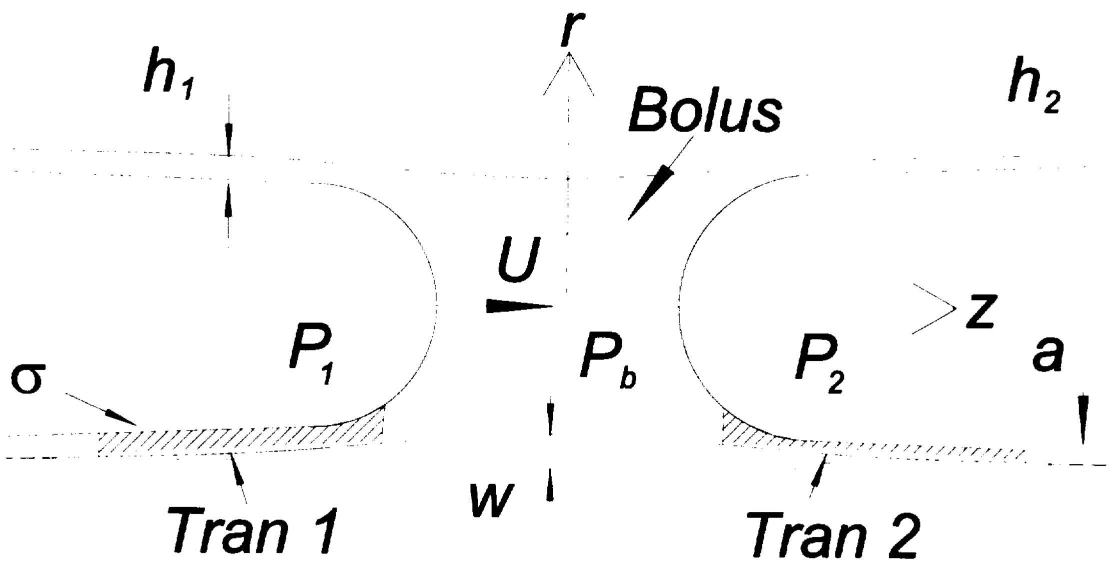

Figure 1: Definition sketch of a bolus in a flexible tube.

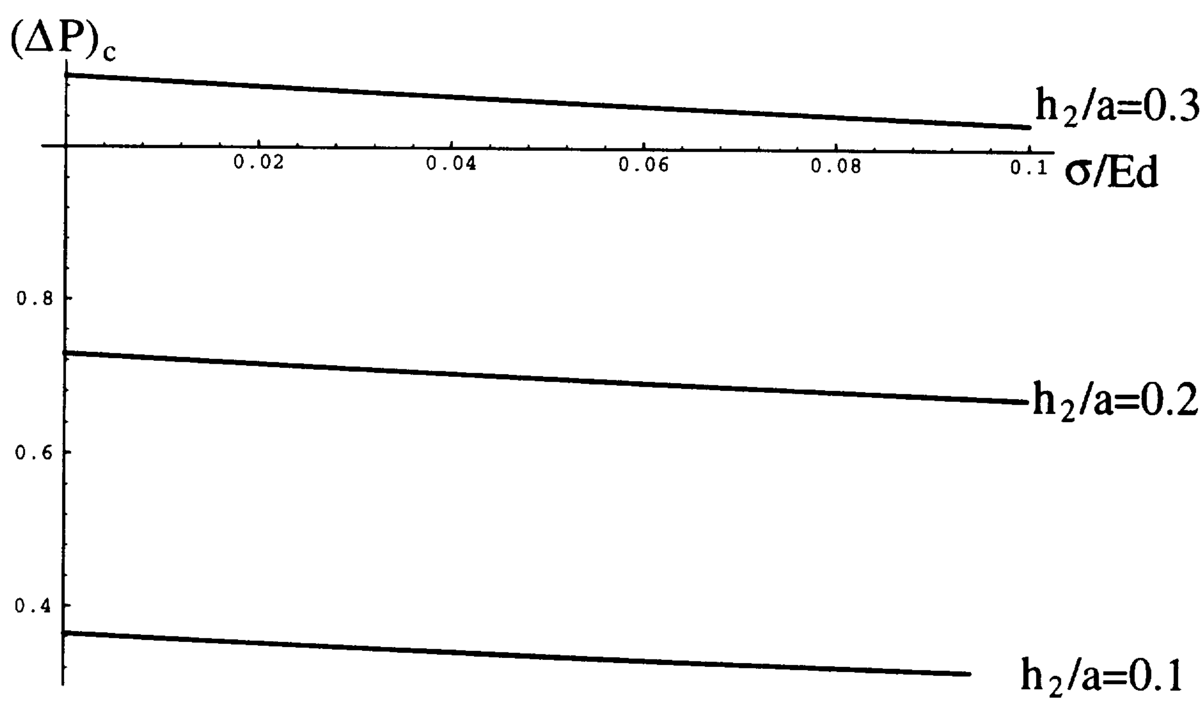

Figure 2: Critical dimensionless pressure drop $(\Delta P)_c = a(P_1 - P_2)/2\sigma$ versus flexibility parameter σ/Ed for various values of the dimensionless precursor thickness.

Convective Flows