EQUILIBRIUM FLUID INTERFACE BEHAVIOR UNDER LOW- AND ZERO-GRAVITY CONDITIONS. II

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ABSTRACT

The mathematical basis for the forthcoming Angular Liquid Bridge investigation on board *Mir* is described. The anticipated liquid behavior used in the apparatus design is illustrated.

INTRODUCTION

We describe here recent mathematical results that form the basis of our forthcoming space experiment, developed jointly with Mark Weislogel of NASA Lewis Research Center, which is scheduled for the Glovebox on the *Mir* 23/NASA 4 Mission in December, 1996. Our mathematical work is based on the classical Young-Laplace-Gauss formulation for an equilibrium free surface of liquid partly filling a container or otherwise in contact with solid support surfaces. In this formulation, when gravity is absent or can be neglected, which is the situation we consider here, the mechanical energy E of the system is given by

$$E = \sigma(S - S^* \cos\gamma). \tag{1}$$

The interfacial liquid-vapor surface tension parameter σ and the relative adhesion coefficient $\cos\gamma$ of the liquid with the container walls are assumed to depend only on the material properties, which are taken here to be homogeneous (the same value of $\cos\gamma$ on all parts of the container, as will be the case for the experiment). S and S^* are, respectively, the areas of the liquid-vapor free surface and of the solid-liquid interface.

Equilibrium configurations are those providing stationary values of the energy functional E subject to the condition of fixed liquid volume [1]. The equilibrium liquid-vapor free surfaces so determined are surfaces of constant mean curvature meeting the bounding walls with contact angle γ. We consider here values of the contact angle $0 < \gamma < \pi$. Of particular interest in our mathematical studies are situations in which small changes in contact angle or geometry can result in large changes, possibly discontinuous, of the equilibrium fluid configuration. Impetus for the present experiment arises largely from recent doctoral dissertations of two students associated with our study, John McCuan [2] and Lianmin Zhou [3], from whose contrasting results striking inferences can be drawn.

ANGULAR LIQUID BRIDGE

In his work, McCuan found conditions under which an equilibrium tubular bridge in a wedge domain (Fig. 1) would be possible in zero gravity, and he gave the shape such a bridge might take. This work is a completely rigorous mathematical study, based on the classical formulation.

Consider a wedge domain with opening angle 2α, $0 < 2\alpha < \pi$. The results McCuan proved contain the following (if the contact angles on the two sides of the wedge are different, the following results hold if γ on the left of the inequalities is their average):

If $\gamma > \pi/2 + \alpha$, a bridge in the shape of a portion of a sphere making contact angle γ with the walls exists.

If $\gamma \leq \pi/2 + \alpha$, no physically realizable bridge is possible.

It has not yet been proved whether or not other shape bridges may be possible when $\gamma > \pi/2 + \alpha$, or whether the spherical bridges are stable (provide a local minimum for the energy). However, our numerical results and those of H. Mittelmann (private communication), obtained using the Surface Evolver software package [4], indicate that the spherical bridges are stable, at least for the representative cases we considered. Also, no bridge shapes other than the sphere have been found numerically. Note that McCuan's results imply that a bridge is possible only for $\gamma > \pi/2$. A spherical liquid bridge is shown in Fig. 4 for the case $\alpha = 25°$, $\gamma = 130°$.

BRIDGE BETWEEN PARALLEL PLATES – DISCONTINUOUS BEHAVIOR

The above results for liquid bridges in a wedge compare in a remarkable way with those for bridges between parallel plates (Fig. 2). This latter problem was studied initially from a rigorous mathematical point of view by Athanassenas [5] and by Vogel [6], and later using a more physical approach by Langbein [7]. (Note that in these papers, as is the case in [3] and here, the boundary conditions at the plates are prescribed contact angle, which arises from the variational condition for (1). For fixed end conditions, as considered in much of the materials science literature, the behavior of solutions is different.) In her doctoral dissertation, Zhou obtained definitive mathematical results that imply the following:

For any value of the contact angle γ and for any liquid volume V greater than or equal to a critical value $V_0(\gamma)$, a unique stable liquid bridge exists between two parallel plates of given separation.

It is known that any equilibrium bridge must be rotationally symmetric [6], [8] and that its free surface is a Delaunay surface [3], [9], [10]. For $\gamma > \pi/2$ and for a specific liquid volume $V_s(h)$ depending on the plate spacing h, the free surface is simply a portion of the surface of a sphere. For other values of the volume the Delaunay surface is different from a sphere.

These results, when combined with the results for the wedge, imply that a bridge between parallel plates may change its configuration and position markedly when one of the plates is tilted, even by a small amount, or it even may cease to exist as a bridge altogether; *a liquid bridge between parallel plates can behave discontinuously with respect to tilting of the plates.* In stability studies such as [3], [6], and [7], limited to the parallel plate geometry, this liquid bridge instability with respect to plate tilt is not observed.

As a specific example to illustrate the possibilities, consider the case $\gamma > \pi/2$ and a bridge with volume V_s between parallel plates of spacing h, so that the bridge is spherical. Suppose the top plate is tilted clockwise by an angle $2\alpha < 2\gamma - \pi$ about a pivot line in the plate that is a distance $\frac{1}{2}h \tan\alpha$ from the symmetry axis of the bridge. Then this particular bridge remains an equilibrium one for the new tilted plate configuration, without any change in the radius of the sphere or in the bridge's position on the lower plate. However, a bridge with any volume V different from V_s (and with the same contact angle) would change both position and shape discontinuously in altering to a spherical bridge in conjunction with the tilt, shifting to the right for $V < V_s$ or to the left for $V > V_s$.

For $\gamma \leq \pi/2$ an initial bridge would always behave discontinuously with respect to the tilt, regardless of volume, as it cannot persist as a bridge. It has to be expected that the liquid will jump to the edge of the plates in this case. If the tilted plates touch forming a wedge, then configurations described in the following section may form. The above phenomena are ones we wish to study in our forthcoming experiment.

OTHER CONFIGURATIONS

When the conditions for a bridge in a wedge are not satisfied, liquid may assume a position as a blob in the shape of a portion of a sphere in contact with the edge, see Fig. 3. The condition for such a configuration to be possible is that $|\gamma - \pi/2| \leq \alpha$. (Recall we consider here only the case $0 < 2\alpha < \pi$.) Although the edge blobs have not been studied with the same mathematical completeness as have the bridges, they have been noted in [11] and [12] and for some examples studied numerically. Our numerical computations indicate that, as for the angular bridges, the spherical edge blobs are stable, and as yet we have found no other edge blob shapes numerically.

In our earlier work, which considers fluid behavior in the neighborhood of the vertex of a wedge, we have shown that if $\alpha + \gamma < \pi/2$, then fluid cannot remain as a blob in the edge but must spread arbitrarily far along the edge [1], [10]. See also [12] and the references there for a discussion of stability of liquid columns in a wedge.

ANTICIPATED EXPERIMENT BEHAVIOR

The liquid behavior one might expect in a physical experiment in space, based on the Laplace-Young-Gauss formulation, is summarized in Fig 4. This figure illustrates the information discussed above, based in part on mathematically rigorous results and, where these are not available, on computational evidence for particular cases. The numerical solutions depicted in Fig. 4 were obtained using the Surface Evolver software package. The computations were carried out with initial approximations and transitions between configurations similar to those in which the experiment is designed to proceed, thereby enhancing appropriateness of the numerically based predictions on uniqueness and stability.

The upper two rows of Fig. 4 depict the nonwetting case $\gamma > \pi/2$: A liquid bridge between parallel plates is convex (part of a sphere for a specific fluid volume). Spherical tubular bridges and edge blobs exist for tilted plates, for the range of values indicated. Edge spread is not possible. For fixed $\gamma > \pi/2$, transition from tubular bridges to edge blobs occurs as α increases through the value $\gamma - \pi/2$.

For the wetting case $\gamma < \pi/2$, a liquid bridge between parallel plates is concave. A tubular bridge between tilted plates is not possible, but the (spherical) edge blob and edge spread are. For fixed $\gamma < \pi/2$, the transition from edge blob to unbounded edge spread occurs as α decreases through the value $\pi/2 - \gamma$. Computed edge blobs are shown (from different viewing perspectives) for the case $\alpha = 25°$, $\gamma = 100°$ in the second row and for $\alpha = 20°$, $\gamma = 75°$ in the bottom row.

The planned experiment will explore the transition between the configurations for a nonwetting and for a wetting fluid. As discussed above, when initially parallel plates are tilted, the fluid is predicted to behave discontinuously in general, the exception being the special case of a spherical bridge and a particular pivot line. The other transitions, horizontally across the second and fourth rows of Fig. 4 as α changes value, are gradual, as can be demonstrated by the explicit spherical solutions.

CONCLUDING REMARKS

We have described fluid behavior predicted mathematically and computationally for the forthcoming Angular Liquid Bridge investigation on board the *Mir* 23/NASA 4 Mission. The predictions, which include discontinuous behavior, are based on the classical Young-Laplace-Gauss formulation. In the experiment there will be an opportunity to check the predictions against physical behavior and to observe the effects of hysteresis and other phenomena not included in the classical formulation.

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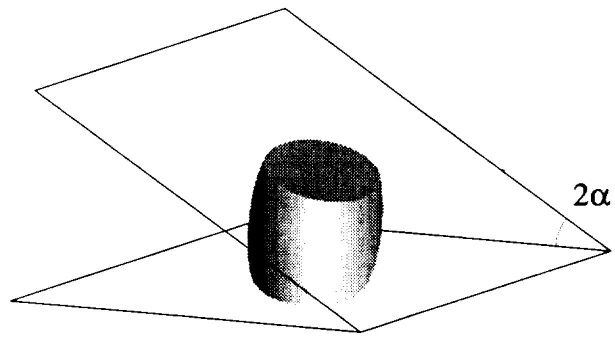

Figure 1. Tubular bridge in a wedge.

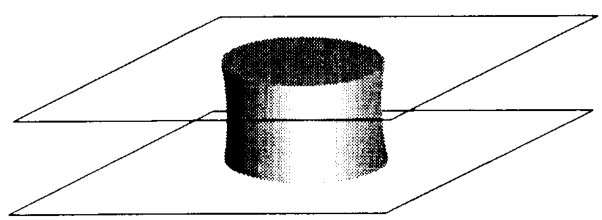

Figure 2. Bridge between parallel plates.

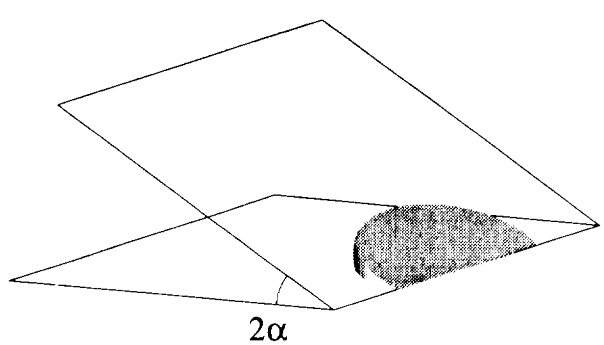

Figure 3. Edge blob.

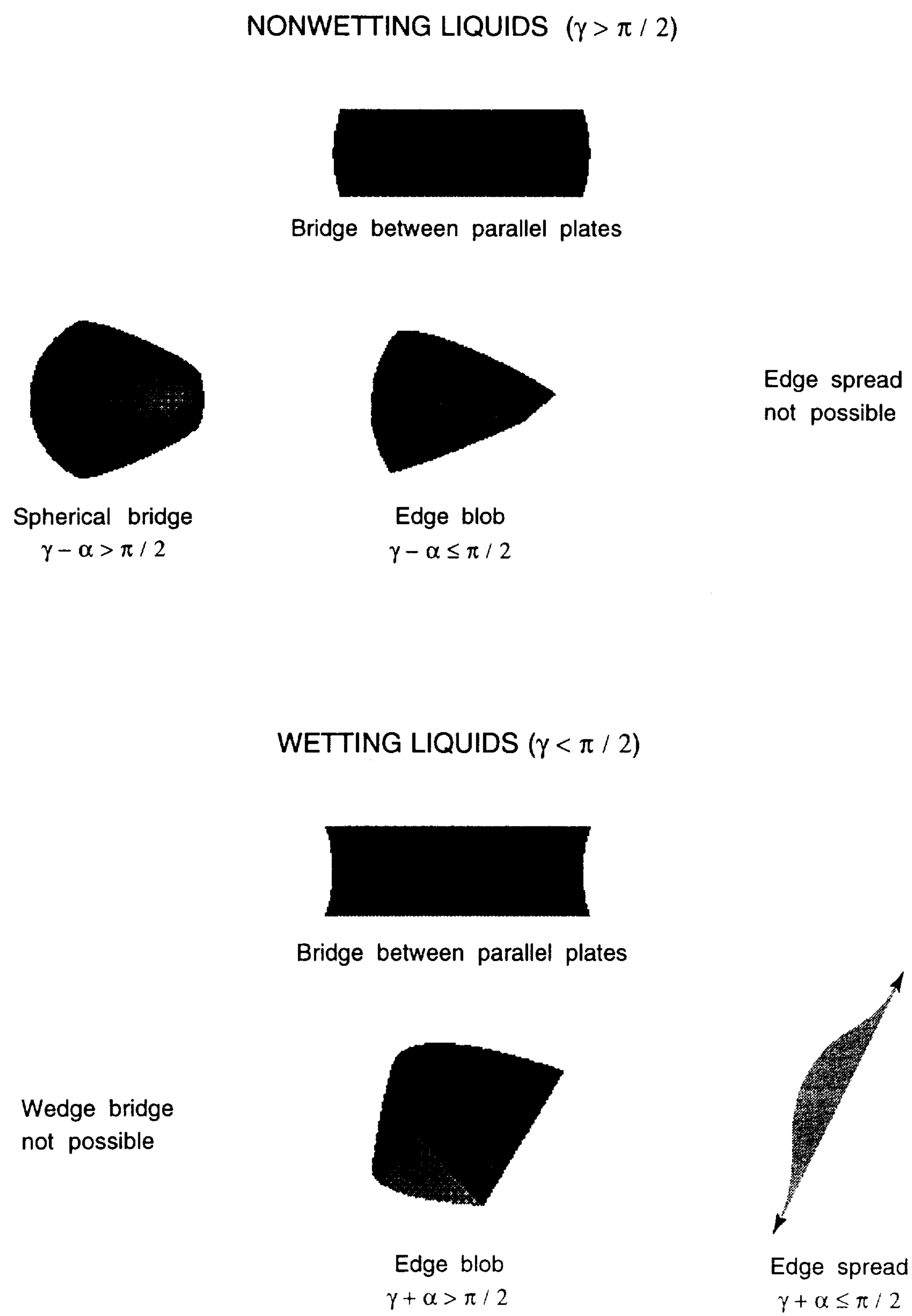

Figure 4. Fluid configurations. Upper two rows: nonwetting liquids; lower two rows: wetting liquids.