Refractive Index Effects on Radiation in an Absorbing, Emitting, and Scattering Laminated Layer

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A simple set of equations is derived for predicting temperature distributions and radiative energy flow in a two-region semitransparent laminated layer in the limit of zero heat conduction. The composite is heated on its two sides by unequal amounts of incident radiation. The two layers of the composite have different refractive indices, and each material absorbs, emits, and isotropically scatters radiation. The interfaces are diffuse, and all interface reflections are included. To illustrate the thermal behavior that is readily calculated from the equations, typical results are given for various optical thicknesses and refractive indices of the layers. Internal reflections have a substantial effect on the temperature distribution and radiative heat flow.

Introduction

The use of ceramic materials for parts or coatings is of interest for high-temperature applications. Some of these materials have reinforcing fibers or are laminated so it is necessary to consider heat transfer in composite regions. The surrounding temperatures are high, which provides substantial radiative heating. Since some of the materials are semitransparent, their temperature distributions depend on the internal radiative heat flow. The refractive index of a semitransparent material can have a considerable effect on its temperature distribution. The refractive index governs the amount of transmission into the interior of the material and affects the internal reflections that occur. Of more significance, the emission within a material depends on the *square* of its refractive index; hence internal emission can be many times that for a blackbody radiating into a vacuum. Since radiation exiting through an interface into a vacuum cannot exceed that of a blackbody, the amount of internal reflection can be substantial. It redistributes energy within the layer and tends to make the temperature distribution more uniform than for a material with refractive index close to one.

To obtain the general solution for the temperature distribution in a composite semitransparent layer requires solving the integral equations of energy transfer in each region, including heat conduction, and matching temperatures and heat flows at the internal interface. A spectral calculation can be carried out in each significant wavelength band and the total energy flow found by summing over the bands. A numerical solution by computer is required. A simple limiting solution is obtained here that is helpful and informative. The purpose of this paper is to show that in the limit with no heat conduction and for gray layers with refractive indices greater than one, the solution for a composite region can be obtained from information that is already known and results calculated very easily. The effects of surface reflections and isotropic scattering are included. The solution of integral equations is not required. The limiting result can be helpful in initiating iterative computer solutions for a more general analysis and for checking the validity of numerical solutions.

The development here builds on the analysis of Siegel and Spuckler (1992) where it was shown that for radiative equilibrium in a gray layer with diffuse interfaces, the temperature distribution and radiative heat flux for any index of refraction is obtained very simply from available results for an index of refraction of one. The present paper applies these ideas to a two-layer laminate subjected to external heating on its outer surfaces. Each of the layers emits, absorbs, and isotropically scatters radiation. For simplicity the medium surrounding the laminate has a refractive index of one.

The outer surfaces of the laminate, and the interface between the two layers, are each assumed diffuse. This is probably a reasonable approximation for unpolished materials that are bonded together. Transmitted radiation, or radiation emitted from the interior, is assumed to be diffuse when it reaches the inner surface of an interface. If the index of refraction of the material is greater than that of the surrounding medium, some of the internal radiation is in angular directions for which there is total internal reflection. This retains energy within the layer and tends to equalize its temperature distribution.

Some of the important early work on calculating heat transfer behavior in semitransparent layers was done by Gardon (1958) who developed an analysis for temperature distributions during heat treating and cooling of glass plates including index of refraction effects. The interfaces were optically smooth, and the specular reflections at the interfaces were computed from the Fresnel reflection relations. A similar application (Fowle et al., 1969), predicted heating in a window of a re-entry vehicle. Recent papers by Rokhsaz and Dougherty (1989), Ping and Lallemand (1989), and Crosbie and Shieh (1990) further examined the effects of Fresnel boundary reflections and having a refractive index larger than one. Many analyses of both steady and transient heat transfer in single or multiple plane layers such as Amlin and Korpela (1979), and Tarshis et al. (1969), have used diffuse conditions at the interfaces as in the present study. Thomas (1992) set up a solution procedure to include a ceramic interface that is partially specular and diffuse. Ho and Ozisik (1987) carried out a numerical analysis of radiation and conduction in a two-region laminate without including refractive index effects and hence without internal interface reflections.

Analysis

A laminated plane layer is made of two different materials that have thicknesses D_1 and D_2 as shown in Fig. 1. Both sublayers absorb, emit, and isotropically scatter radiation. The
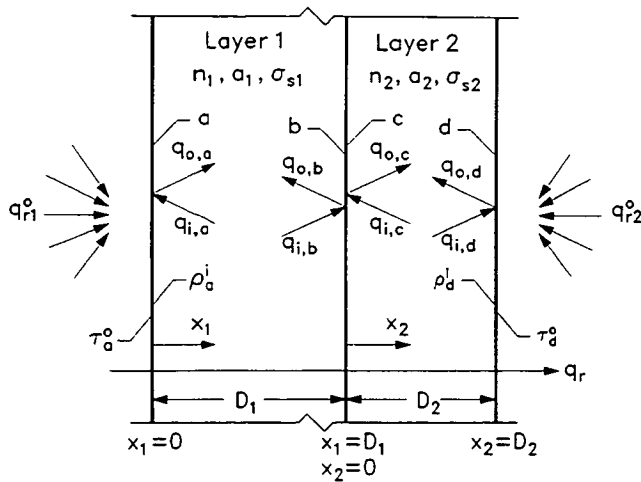
Fig. 1 Laminated two-layer geometry, coordinate system, and nomenclature for heat fluxes at interfaces

limiting case is considered here where the temperature distribution within the composite is dominated by radiation so heat conduction is neglected. Each material has a constant refractive index; it is the effect of the two different n values that is investigated here along with having different optical thicknesses for the layers. The materials provide significant scattering, so the interfaces between the two layers and between each layer and the surrounding air or vacuum (with $n = 1$) are assumed diffuse. The laminated layer is subjected to diffuse radiation from the surroundings of q_{r1}^o and q_{r2}^o on the two outer boundaries $x_1 = 0$ and $x_2 = D_2$; for convenience here $q_{r1}^o > q_{r2}^o$. Inside each layer there are outgoing and incoming fluxes, q_o and q_i, at each interior interface. Since scattering is included, the local optical depth in each layer is related to its x coordinate by $\kappa = (a + \sigma_s)x$. Using an individual coordinate in each of the two layers facilitates the formulation and solution that follows.

The temperature distribution in each layer is governed by an integral equation given by Siegel and Howell (1981), modified with index of refraction factors as,

$$n_1^2 \sigma T_1^4(\kappa_1) = \frac{1}{2}\left[q_{o,a}E_2(\kappa_1) + q_{o,b}E_2(\kappa_{D1} - \kappa_1)\right]$$

$$+ \frac{n_1^2}{2}\int_0^{\kappa_{D1}} \sigma T_1^4(\kappa_1^*)E_1(|\kappa_1 - \kappa_1^*|)d\kappa_1^* \quad (1a)$$

$$n_2^2 \sigma T_2^4(\kappa_2) = \frac{1}{2}\left[q_{o,c}E_2(\kappa_2) + q_{o,d}E_2(\kappa_{D2} - \kappa_2)\right]$$

$$+ \frac{n_2^2}{2}\int_0^{\kappa_{D2}} \sigma T_2^4(\kappa_2^*)E_1(|\kappa_2 - \kappa_2^*|)d\kappa_2^* \quad (1b)$$

The radiative flux is a *constant* through the entire laminated layer for the present conditions of radiative equilibrium. The relation of flux to temperature distribution in each of the two layers is obtained by evaluating the equation for q_r for a plane layer from Siegel and Howell (1981) at $x_1 = 0$ and $x_2 = 0$,

$$q_r = q_{o,a} - 2q_{o,b}E_3(\kappa_{D1}) - 2n_1^2 \int_0^{\kappa_{D1}} \sigma T_1^4(\kappa_1)E_2(\kappa_1)d\kappa_1 \quad (2a)$$

$$q_r = q_{o,c} - 2q_{o,d}E_3(\kappa_{D2}) - 2n_2^2 \int_0^{\kappa_{D2}} \sigma T_2^4(\kappa_2)E_2(\kappa_2)d\kappa_2 \quad (2b)$$

The following dimensionless groups are now defined:

$$\Phi_1(\kappa_1) = \frac{n_1^2\sigma T_1^4(\kappa_1) - q_{o,b}}{q_{o,a} - q_{o,b}}; \quad \Phi_2(\kappa_2) = \frac{n_2^2\sigma T_2^4(\kappa_2) - q_{o,d}}{q_{o,c} - q_{o,d}} \quad (3a, b)$$

$$\Psi_1 = \frac{q_r}{q_{o,a} - q_{o,b}}; \quad \Psi_2 = \frac{q_r}{q_{o,c} - q_{o,d}} \quad (4a, b)$$

Equations (1) and (2) then transform to (note that within each layer $X_j = x_j/D_j = \kappa_j/\kappa_{Dj}$ where $j = 1$ or 2)

$$\Phi_j(X_j, \kappa_{Dj}) = \frac{1}{2}E_2(\kappa_{Dj}X_j)$$

$$+ \frac{1}{2}\kappa_{Dj}\int_0^1 \Phi_j(X_j^*, \kappa_{Dj})E_1(\kappa_{Dj}|X_j - X_j^*|)dX_j^* \quad (j = 1, 2) \quad (5)$$

$$\Psi_j(\kappa_{Dj}) = 1 - 2\kappa_{Dj}\int_0^1 \Phi_j(X_j, \kappa_{Dj})E_2(\kappa_{Dj}X_j)dX_j \quad (j = 1, 2) \quad (6)$$

It is noted that Eqs. (5) and (6) are the same for each layer,

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Nomenclature

a = absorption coefficient of layer, m^{-1}

a, b, c, d = the four internal sides of the interfaces in the two-layer laminate

D = thickness of plane layer, m

E_1, E_2, E_3 = exponential integral functions,
$$E_m(x) = \int_0^1 \mu^{m-2}\exp(-x/\mu)d\mu$$

$F(n)$ = function of refractive index defined in Eq. (15a)

n = index of refraction

q = heat flux, W/m^2

q_r^o = externally incident radiation flux, W/m^2

T = absolute temperature, K

T_{s1}, T_{s2} = temperatures of surrounding radiating regions, K

x_1, x_2 = coordinates normal to

boundaries of individual plane layers, m; $X_1 = x_1/D_1$, $X_2 = x_2/D_2$

X = coordinate through two-layer laminate, $X = X_1$ in first layer, $X = 1 + X_2$ in second layer

κ = optical depth = $(a + \sigma_s)x$; κ_D = optical thickness = $(a + \sigma_s)D$

κ_e = extinction coefficient in the complex index of refraction; $\bar{n} = n - i\kappa_e$

σ = Stefan-Boltzmann constant, $W/m^2 \cdot K^4$

σ_s = scattering coefficient of layer, m^{-1}

ρ^i = reflectivity of interface for internally incident radiation

τ^o = transmissivity of surface for externally incident radiation

Φ = dimensionless temperature distribution, Eq. (3)

Ψ = dimensionless radiative heat flux, Eq. (4)

Subscripts

i, o = incoming and outgoing radiation

j = index 1 or 2 for each of the two sublayers in composite

r = radiative quantity

$1, 2$ = layers adjacent to hotter and cooler surroundings; hotter and cooler surroundings

Superscripts

i = at inside surface of an interface

o = at outside surface of an interface

$*$ = dummy variable of integration

and that $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ are not functions of n. Hence to obtain Φ and Ψ for all $n \geq 1$ it is necessary to solve Eq. (5) only once for each κ_D and use each result to determine Ψ from Eq. (6). Tabulated results for some $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ are available from references such as Heaslet and Warming (1965) and Siegel and Spuckler (1992).

Although solutions for Φ and Ψ are available, the present analysis has not yet provided $T(\kappa_1)$, $T(\kappa_2)$, and q_r since Eqs. (3) and (4) contain the outgoing boundary fluxes $q_{o,a}$, $q_{o,b}$, $q_{o,c}$, and $q_{o,d}$ that are unknown. These fluxes must be expressed in terms of known quantities; this is accomplished by looking at the boundary and interface conditions in detail.

At the two outer boundaries the internal fluxes are related to the transmission of external flux and the reflection of internal flux by

$$q_{o,a} = q_{r1}^o \tau_a^o + q_{i,a}\rho_a^i; \quad q_{o,d} = q_{r2}^o \tau_d^o + q_{i,d}\rho_d^i \qquad (7a, b)$$

At the inside surfaces of the outer boundaries there are the following relations between the radiative flux and the outgoing and incoming fluxes:

$$q_r = q_{o,a} - q_{i,a}; \quad q_r = -q_{o,d} + q_{i,d} \qquad (8a, b)$$

Equations (7a) and (8a) are combined to eliminate $q_{i,a}$ and similarly Eqs. (7b) and (8b) to eliminate $q_{i,d}$. This yields

$$q_{o,a} = \frac{1}{1 - \rho_a^i}(q_{r1}^o \tau_a^o - q_r \rho_a^i) \qquad (9a)$$

$$q_{o,d} = \frac{1}{1 - \rho_d^i}(q_{r2}^o \tau_d^o + q_r \rho_d^i) \qquad (9b)$$

By using similar relations at interfaces b and c on either side of the internal interface, the following relations are found between the outgoing fluxes and the radiant flux q_r being transferred:

$$q_{o,b} = \frac{1}{1 - \rho_b^i}[q_{o,c}\tau_b^o + q_r(\rho_b^i - \tau_b^o)] \qquad (10a)$$

$$q_{o,c} = \frac{1}{1 - \rho_c^i}[q_{o,b}\tau_c^o - q_r(\rho_c^i - \tau_c^o)] \qquad (10b)$$

The radiant flux q_r can now be obtained. The $q_{o,a}$ from Eq. (9a) and $q_{o,b}$ from Eq. (10a) are substituted into Eq. (4a). The $q_{o,c}$ is then eliminated by using Eq. (4b), and the $q_{o,d}$ is eliminated by using Eq. (9b). The resulting expression is solved for q_r to yield

$$q_r = \frac{\dfrac{\tau_a^o}{1 - \rho_a^i}q_{r1}^o - \dfrac{\tau_b^o}{1 - \rho_b^i}\dfrac{\tau_d^o}{1 - \rho_d^i}q_{r2}^o}{\dfrac{1}{\Psi_1} + \dfrac{\rho_a^i}{1 - \rho_a^i} + \dfrac{\rho_b^i}{1 - \rho_b^i} + \dfrac{\tau_b^o}{1 - \rho_b^i}\left(\dfrac{\rho_d^i}{1 - \rho_d^i} - 1 + \dfrac{1}{\Psi_2}\right)} \qquad (11)$$

As a result of total internal reflections at interfaces when radiation passes into a medium with a lower refractive index, there are the following relations at the four surfaces of the interfaces:

$$\tau_a^o = (1 - \rho_a^i)n_1^2; \quad \tau_b^o = (1 - \rho_b^i)\left(\frac{n_1}{n_2}\right)^2$$

$$\tau_c^o = (1 - \rho_c^i)\left(\frac{n_2}{n_1}\right)^2; \quad \tau_d^o = (1 - \rho_d^i)n_2^2 \qquad (12a\text{-}d)$$

Equation (12) is used to eliminate the τ's from Eq. (11) and the final result for q_r becomes:

$$\frac{q_r}{q_{r1}^o - q_{r2}^o} = \frac{n_1^2}{\dfrac{1}{\Psi_1} + \dfrac{\rho_a^i}{1 - \rho_a^i} + \dfrac{\rho_b^i}{1 - \rho_b^i} + \left(\dfrac{n_1}{n_2}\right)^2\left(\dfrac{\rho_d^i}{1 - \rho_d^i} - 1 + \dfrac{1}{\Psi_2}\right)} \qquad (13a)$$

By starting the previous algebra with Eq. (4b) rather than Eq. (4a) an alternative expression for q_r is found as:

$$\frac{q_r}{q_{r1}^o - q_{r2}^o} = \frac{n_2^2}{\dfrac{1}{\Psi_2} + \dfrac{\rho_c^i}{1 - \rho_c^i} + \dfrac{\rho_d^i}{1 - \rho_d^i} + \left(\dfrac{n_2}{n_1}\right)^2\left(\dfrac{\rho_a^i}{1 - \rho_a^i} - 1 + \dfrac{1}{\Psi_1}\right)} \qquad (13b)$$

The q_r is the same from expressions (13a) and (13b); evaluating both relations is a check on the solution.

The temperature distributions in the two layers are now obtained. Starting with Eq. (3b), $T_2(\kappa_2)$ can be obtained from $n_{20}^2\sigma T_2^4(\kappa_2) = q_{o,d} + (q_{o,c} - q_{o,d})\Phi_2(\kappa_2)$. The $q_{o,c} - q_{o,d}$ is eliminated by using Eq. (4b) to give $n_{20}^2\sigma T_2^4(\kappa_2) = q_{o,d} + (q_r/\Psi_2)\Phi_2(\kappa_2)$. The $q_{o,d}$ is now eliminated by using Eq. (9b), and τ_d^o is eliminated by using Eq. (12d). The final result for $T_2(\kappa_2)$ is placed into the dimensionless form

$$\frac{\sigma T_2^4(\kappa_2) - q_{r2}^o}{q_{r1}^o - q_{r2}^o} = \frac{q_r}{q_{r1}^o - q_{r2}^o}\left[\frac{\rho_d^i}{1 - \rho_d^i} + \frac{\Phi_2(\kappa_2)}{\Psi_2}\right]\frac{1}{n_2^2} \qquad (14a)$$

In a similar fashion the dimensionless $T_1(\kappa_1)$ distribution is obtained. Starting with Eq. (3a) and eliminating $q_{o,b}$ using Eq. (4a) gives $n_{10}^2\sigma T_1^4(\kappa_1) = q_{o,a} + (q_r/\Psi_1)[\Phi_1(\kappa_1) - 1]$. The $q_{o,a}$ is eliminated using Eq. (9a), and τ_a^o is eliminated using Eq. (12a). Then $-n_1^2 q_{r2}^o$ is added to both sides of the equation. The final dimensionless result for $T_1(\kappa_1)$ is,

$$\frac{\sigma T_1^4(\kappa_1) - q_{r2}^o}{q_{r1}^o - q_{r2}^o} = 1 - \frac{q_r}{q_{r1}^o - q_{r2}^o}\left[\frac{\rho_a^i}{1 - \rho_a^i} - \frac{\Phi_1(\kappa_1) - 1}{\Psi_1}\right]\frac{1}{n_1^2} \qquad (14b)$$

To use these relations, values of ρ^i are needed for various refractive indices. The externally incident radiation is diffuse. Although the interfaces are not optically smooth, it is assumed that each bit of roughness acts as a smooth facet so that the reflectivity can be obtained from the Fresnel interface relations for a nonabsorbing dielectric medium. By integrating the reflected energy over all incident directions the relation for $\rho^i(n)$ is (Siegel and Howell, 1981),

$$\rho^i(n) \equiv F(n) = \frac{1}{2} + \frac{(3n+1)(n-1)}{6(n+1)^2} + \frac{n^2(n^2-1)^2}{(n^2+1)^3}\ln\left(\frac{n-1}{n+1}\right)$$
$$- \frac{2n^3(n^2+2n-1)}{(n^2+1)(n^4-1)} + \frac{8n^4(n^4+1)}{(n^2+1)(n^4-1)^2}\ln(n) \qquad (15a)$$

This is for radiation passing into a material of higher refractive index where $n = n_h/n_s$ (n_h and n_s are the "higher" and "smaller" n values). When going in the reverse direction from a higher to a smaller n value, the ρ^i is given by (Richmond, 1963)

$$\rho^i(n) = 1 - \frac{1}{n^2}[1 - F(n)] \quad (n = n_h/n_s) \qquad (15b)$$

The relations in Eq. (15) are for perfect dielectrics, that is, for materials that do not attenuate radiation internally. As discussed by Cox (1965), in the spectral regions where ceramic materials are semitransparent to radiation the extinction coefficient, κ_e, in the complex index of refraction is usually not large enough to significantly affect the surface reflectivity. The absorption coefficient, a, in a material is related to κ_e by $a = 4\pi\kappa_e/\lambda_0$. Since wavelengths λ_0 for thermal radiation are in the micrometer range, only a small value of κ_e is required for this relation to yield a large value for a. If the extinction coefficient κ_e is large enough to influence the interface reflectivity relations, the absorption coefficient will be so large that the radiating layer is essentially opaque unless its thickness is much smaller than the ceramic layers considered here.

The required relations have now been provided to evaluate Eqs. (13) and (14) for the heat flux and temperature distribution in a laminated two-layer region. The equations contain the functions $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ that are available in the literature as tabulated results for a single layer with $n = 1$. For a laminated layer, these functions are inserted corresponding to the optical thickness of each sublayer of the composite. It is not

necessary to solve any integral equations to evaluate the present solution.

A convenient alternative to using the exact numerical solutions for the functions Φ and Ψ in Eqs. (13) and (14) is to use Φ and Ψ available from the diffusion approximation. The accuracy of this approximation is discussed in the next section. The diffusion relations for a plane layer are (Siegel and Howell, 1981)

$$\Psi(\kappa_D) = \frac{1}{\frac{3}{4}\kappa_D + 1} \quad (16a)$$

$$\Phi(X, \kappa_D) = \Psi(\kappa_D)\left[\frac{3}{4}\kappa_D(1-X) + \frac{1}{2}\right] \quad (16b)$$

When used with Eqs. (13) and (14), Eqs. (16) provide rapid predictions of heat flux and temperature distributions for any optical thicknesses and refractive indices of the sublayers in the laminate.

Results and Discussion

For a plane layer in radiative equilibrium with absorption and isotropic scattering and for $n = 1$, the $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ are known functions or can be calculated from Eqs. (5) and (6). These functions are the building blocks required to evaluate the present solution for a two-layer region including refractive index effects and interface reflections. The exact solutions for $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ obtained by solving Eqs. (5) and (6) were examined in detail by Heaslet and Warming (1965). The numerical results used here for $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ (Siegel and Spuckler, 1992) are in excellent agreement with previous work and were carried out to obtain values for a large range of optical thicknesses, $0.1 \leq \kappa_D \leq 100$.

An alternative set of basic functions for $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$, which was found to be accurate for the present conditions and is very simple to implement, is to use the diffusion results in Eqs. (16a, b). A comparison of exact and diffusion functions is given in Table 1 for various optical thicknesses. The diffusion results are reasonably accurate, but there are some differences when the optical thickness is small. Results for two-layer laminated regions were calculated from the present analysis using both exact and diffusion functions for the required $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$. There was little difference in the results for the laminate, so for convenience the diffusion functions were used for evaluating the illustrative results presented here. For simplicity and sufficient accuracy the diffusion functions are recommended for temperature distribution and heat flux calculations using the analytical Eqs. (13) and (14).

The diffusion functions also have an advantage in the theory used for developing the present equations for two layers where the interfaces are assumed diffuse. Since in the diffusion approximation the radiation is everywhere diffuse, there is no discontinuity in radiative behavior when energy crosses a diffuse interface. In the exact solution for a single plane layer the diffuse energy leaving an internal boundary is no longer diffuse after it travels across the layer. The radiative energy passing across from one layer into another thus undergoes a change in its angular distribution when crossing a diffuse interface between the two regions. This can have a small effect on the temperatures adjacent to the internal interface. This effect is not present when using the diffusion functions for Φ and Ψ.

For the graphic results shown here, $\Phi(X, \kappa_D)$ and $\Psi(\kappa_D)$ were obtained from Eqs. (16), but exact results such as those in Table 1 could have been used. The Ψ_1 and Ψ_2 were substituted into Eq. (13a) to obtain the dimensionless radiative flux through the two layers; the heat flux was checked by evaluating the alternative Eq. (13b). The dimensionless flux was substituted into Eqs. (14a, b) to obtain dimensionless temperature dis-

Table 1 Comparison of results from exact (E) and diffusion (D) solutions

(a) Dimensionless temperature distribution, Φ

x	Optical Thickness							
	$\kappa_D = 0.1$		$\kappa_D = 1$		$\kappa_D = 10$		$\kappa_D = 100$	
	E	D	E	D	E	D	E	D
0	.5710	.5349	.7582	.7143	.9495	.9412	.9948	.9934
0.1	.5541	.5279	.6946	.6714	.8511	.8529	.8974	.8947
0.2	.5397	.5209	.6429	.6286	.7627	.7647	.7994	.7961
0.3	.5262	.5140	.5942	.5857	.6750	.6765	.6998	.6974
0.4	.5130	.5070	.5468	.5429	.5874	.5882	.6000	.5987
0.5	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
0.6	.4870	.4930	.4532	.4571	.4125	.4118	.4000	.4013
0.7	.4738	.4861	.4058	.4143	.3249	.3235	.3002	.3026
0.8	.4603	.4791	.3571	.3714	.2372	.2353	.2007	.2040
0.9	.4459	.4721	.3054	.3286	.1488	.1471	.1026	.1053
1.0	.4290	.4651	.2419	.2857	.0505	.0588	.0052	.0066

(b) Dimensionless heat flux, Ψ

Optical Thickness κ_D	Ψ	
	E	D
0.1	.9157	.9302
1	.5534	.5714
10	.1168	.1177
100	.0122	.0132

Table 2 Interface reflectivities

n_1	n_2	ρ_a^i	ρ_b^i	ρ_c^i	ρ_d^i
1	1	0	0	0	0
1	1.5	0	.0918	.5964	.5964
1	2	0	.1606	.7902	.7902
1	3	0	.2762	.9196	.9196
1.5	3	.5964	.1606	.7902	.9196
2	1	.7902	.7902	.1606	0
2	4	.7902	.1606	.7902	.9604
3	1.5	.9196	.7902	.1606	.5964
4	2	.9604	.7902	.1606	.7902

tributions in each layer. The values of the interface reflectivities were evaluated from Eqs. (15a) and (15b) depending on whether radiation is passing into a region of higher or lower refractive index. The reflectivity values at the four surfaces in Fig. 1 are given in Table 2 for the combinations of n_1 and n_2 values used in Figs. 2–5. In the results that follow, the temperature distributions are plotted against a continuous X coordinate where $X = X_1$ in layer 1 and $X = 1 + X_2$ in layer 2; thus $0 \leq X \leq 2$.

Reciprocal Behavior of Dimensionless Temperatures. Figure 2 illustrates a reciprocal behavior for the temperature results. Compared to Fig. 2(a), both the n values and the optical thicknesses of the two layers have been interchanged in Fig. 2(b). It is noted that rotating Fig. 2(a) 180 deg gives dimensionless temperature distributions that are the same as in Fig. 2(b). It was also found that the radiative heat flux through the laminate does not depend on the order of placement of the two layers relative to the side with larger incident radiation.

Effect of Optical Thickness. Figure 2(a) also illustrates the effect of the optical thicknesses of the two layers. The solid and dashed profiles are respectively for optical thicknesses of the first layer of $\kappa_{D1} = 1$ and 10. The families of curves correspond to optical thicknesses in the second layer of $\kappa_{D2} = 0.1, 1, 3, 10, 30,$ and 100. As optical thickness increases, the
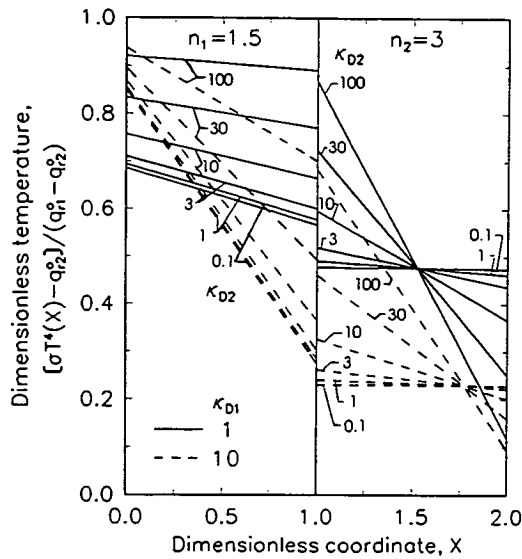
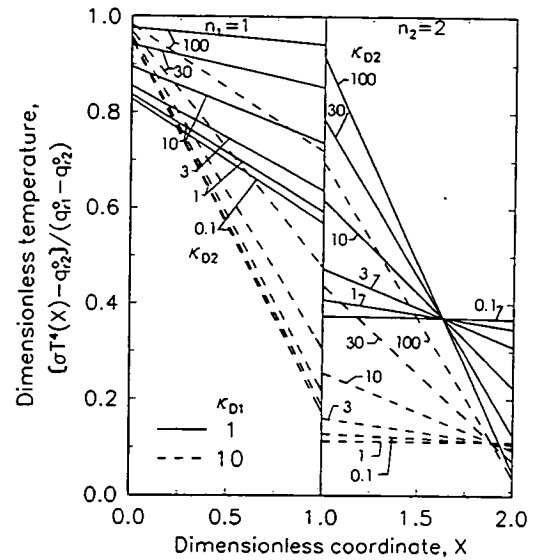Fig. 2(a) Layer with lower index of refraction adjacent to higher source of radiant energy

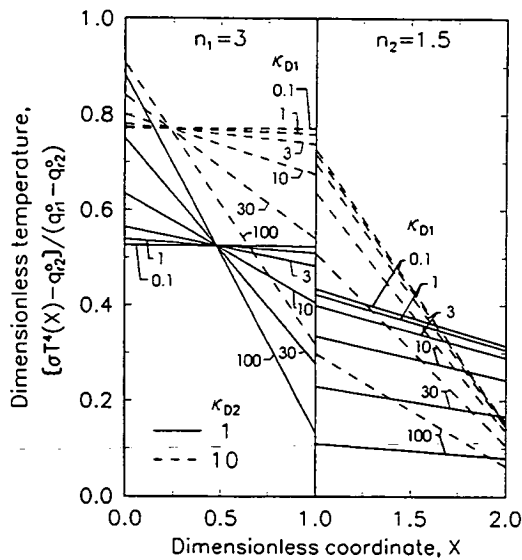Fig. 3(a) Indices of refraction: $n_1 = 1$ and $n_2 = 2$

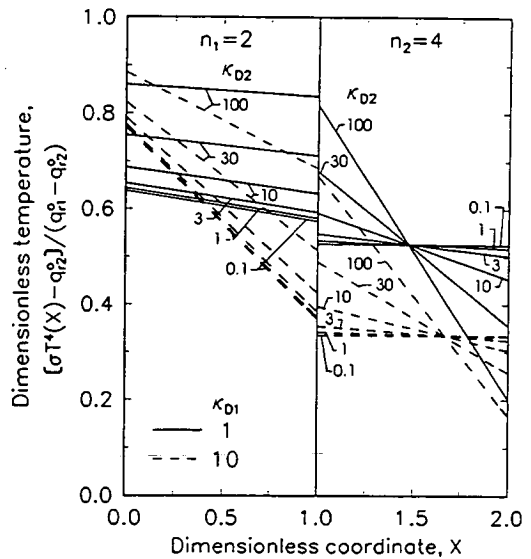Fig. 2(b) Demonstration of reciprocity by interchanging n and κ_D values

Fig. 2 Dimensionless temperature distributions for indices of refraction $n_1 = 1.5$ and $n_2 = 3$, two optical thicknesses in layer 1, and six optical thicknesses in layer 2 (values reversed in Fig. 2(b))

Fig. 3(b) Indices of refraction: $n_1 = 2$ and $n_2 = 4$

Fig. 3 Effect of magnitude of index of refraction on dimensionless temperature distributions in layers as a function of optical thicknesses; ratio $n_2/n_1 = 2$

resistance to radiative heat flow increases so the temperature gradients in the second layer increase. For $\kappa_{D2} = 0.1$ the second layer is optically thin and its temperature distribution is essentially uniform. For $\kappa_{D2} = 100$, however, there is a steep gradient in the second layer. When $\kappa_{D1} = 10$ the large temperature changes in the first layer cause the temperatures in the second layer to be lower than those for $\kappa_{D1} = 1$. The dimensionless radiative heat fluxes are given in Table 3 and they illustrate how the heat flux decreases with increasing optical thickness.

Effect of Magnitude of Refractive Index. The effect of the size of the refractive indices n_1 and n_2 is illustrated in Figs. 2(a), 3(a, b), which form a set of three arrays of temperature distributions. The optical thicknesses are as described in the previous section. The ratio of n_2/n_1 is kept equal to 2 for these three sets of profiles, but the refractive indices increase by factors of 1:1.5:2 in the order of Figs. 3(a), 2(a), and 3(b). Larger refractive indices increase the amount of internal re-

flection at the two outer interfaces as a result of total reflection when radiation is leaving a material with a higher refractive index. This tends to equalize the temperature distributions within the layers as the n values increase. The reflections at the interface between the layers remain the same because n_2/n_1 is constant.

For $\kappa_{D1} = 1$, increasing the refractive index provides decreased temperatures in the first layer. An increased amount of the incident radiant energy is reflected away at the first interface and does not heat the laminate. Internal reflections increase energy absorption within the laminated region, thereby increasing the average temperature level in the second layer.

Effect of Order of Refractive Indices in Heat Flow Direction. In Fig. 4 the κ_D values are the same as in Fig. 3. The n values, however, are interchanged so the largest refractive index is now adjacent to the environment that provides the largest incident radiation. This changes the amount of totally reflected energy within the layers as compared with Fig. 3. Since total internal reflection is increased in the first layer and
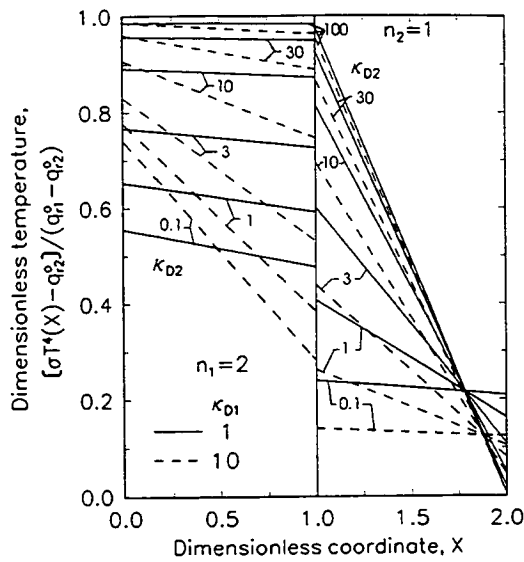
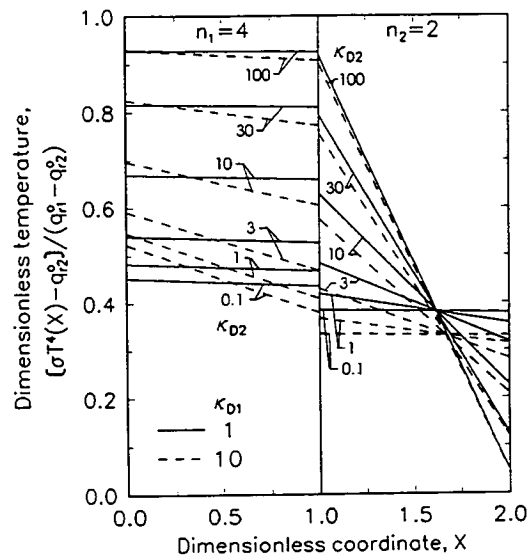Fig. 4(a) Indices of refraction: $n_1 = 2$ and $n_2 = 1$

Fig. 4(b) Indices of refraction: $n_1 = 4$ and $n_2 = 2$

Fig. 4 Effect of inverting the indices of refraction as compared with Fig. 3; ratio $n_2/n_1 = 1/2$
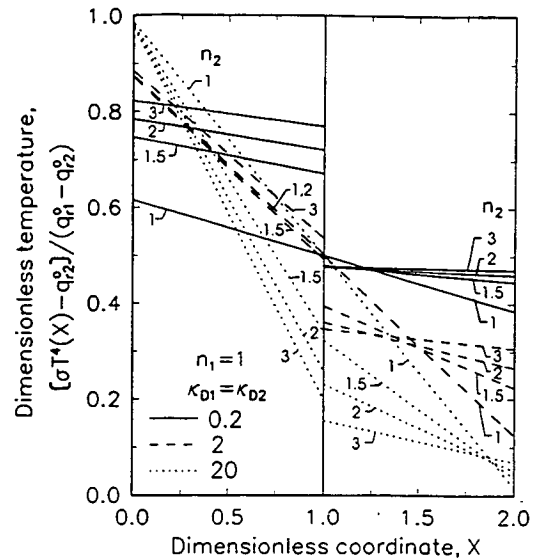

Fig. 5 Effect of index of refraction ratio n_2/n_1 on dimensionless temperature distributions for optical thickness varying from thin to thick

Table 3 Dimensionless radiative heat flux, $q_r/(q_{r1}^o - q_{r2}^o)$

κ_{D2}	$n_1 = 1$ $n_2 = 2$	$n_1 = 1.5$ $n_2 = 3$	$n_1 = 2$ $n_2 = 4$	$n_1 = 2$ $n_2 = 1$	$n_1 = 4$ $n_2 = 2$
			$\kappa_{D1} = 1$		
0.1	0.3447	0.3574	0.3393	0.4175	0.3545
1	0.3257	0.3480	0.3345	0.3257	0.3345
3	0.2903	0.3290	0.3243	0.2188	0.2972
10	0.2102	0.2760	0.2931	0.1018	0.2138
30	0.1175	0.1890	0.2299	0.0403	0.1187
100	0.0462	0.0899	0.1311	0.0129	0.0464

κ_{D2}	$n_1 = 1$ $n_2 = 2$	$n_1 = 1.5$ $n_2 = 3$	$n_1 = 2$ $n_2 = 4$	$n_1 = 2$ $n_2 = 1$	$n_1 = 4$ $n_2 = 2$
			$\kappa_{D1} = 10$		
0.1	0.1036	0.1725	0.2158	0.2449	0.3084
1	0.1018	0.1703	0.2138	0.2102	0.2931
3	0.0981	0.1656	0.2096	0.1598	0.2641
10	0.0869	0.1510	0.1961	0.0869	0.1961
30	0.0655	0.1206	0.1657	0.0377	0.1130
100	0.0352	0.0708	0.1073	0.0127	0.0455

n_2	$\kappa_{D1} = \kappa_{D2} = 0.2$	$\kappa_{D1} = \kappa_{D2} = 2$	$\kappa_{D1} = \kappa_{D2} = 20$
		$n_1 = 1$	
1	0.7692	0.2500	0.0323
1.5	0.5065	0.2548	0.0427
2	0.4310	0.2495	0.0479
3	0.3548	0.2316	0.0518

decreased in the second, the temperatures become more uniform in the first layer and have steeper gradients in the second. The temperature profiles in the first layer are not as close together as on Fig. 3.

Effect of Changing Refractive Index in One Layer. For all the distributions in Fig. 5 the refractive index in the first layer is retained at $n_1 = 1$. In the second layer the values are: $n_2 = 1$, 1.5, 2, and 3. The optical thicknesses in the two layers are equal. Three families of curves are shown for $\kappa_{D1} = \kappa_{D2} = 0.2$, 2, 20. Hence the solid curves for each layer are optically thin, the dashed curves are for an intermediate optical thickness, and the short dashed curves are for a rather thick region.

Consider the solid profiles for $\kappa_{D1} = \kappa_{D2} = 0.2$. When $n = 1$ there are no interface reflections and a continuous temperature profile is obtained through the laminated region. For $n_2 > 1$ the profiles become discontinuous at the internal interface, and as n_2 becomes larger the discontinuity increases. For $\kappa_{D1} = \kappa_{D2} = 0.2$, an increased n_2 raises the temperatures in the first layer and makes the temperatures more uniform in the second layer. For the short dashed curves with $\kappa_{D1} = \kappa_{D2} = 20$, there is a different effect. As n_2 increases, the temperatures

in the first layer are reduced, while the temperatures in the second layer become more uniform as before. For an intermediate optical thickness, $\kappa_{D1} = \kappa_{D2} = 2$, there is a small decrease in temperatures in the first layer as n_2 is increased, and then with a further increase in n_2 the temperatures increase. The radiative heat fluxes in Table 3 show a similar reversal in trend when going from an optically thin to an optically thick layer.

Conclusions

A convenient solution was obtained for temperature distributions and radiative heat fluxes in a laminated two-layer region with radiative energy incident on each outer boundary. The interfaces are assumed diffuse with the intent of modeling interface reflections of a ceramic material. The limiting case

considered here is for a two-layer laminate in radiative equilibrium (no heat conduction) that emits, absorbs, and isotropically scatters radiation. The solution yields results for two laminated layers by using known dimensionless temperature and heat flux functions for a single semitransparent radiating layer with $n = 1$. These functions are combined through the coupling at the interfaces to yield the solution for a two-layer laminated region with $n > 1$. The coupling relations include transmission through the interfaces and all internal reflections within the layers. The result is a set of simple algebraic expressions for the temperature distribution in each of the layers, and for the radiative heat flux through the laminated region. Results were evaluated using the basic functions for a single layer obtained by both exact and diffusion methods. Unless the optical thickness is much smaller than one, the diffusion results are within engineering accuracy, and they provide an especially simple way to compute results for two layers.

Illustrative results are given for a variety of refractive indices and optical thicknesses of the layers. A reciprocity relation was found for the temperature distributions. If the order of the refractive indices of the layers is reversed as well as their optical thicknesses, the temperature distributions become inverted mirror images. It was found that increasing the refractive index in each layer, while keeping their ratio constant, generally makes the temperature profiles more uniform as the result of increased internal reflections. When the refractive index of the second layer is increased, and the first is kept unchanged, the temperatures in the layers can either increase or decrease depending on the layer optical thicknesses.

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