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Carleson Measure and Balayage

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The balayage of a Carleson measure lies of course in bounded mean oscillation (BMO). We show that the converse statement is false. We also make a two-sided estimate of the Carleson norm of a positive measure in terms of *certain* balayages.

1 Introduction and Notation

In this note, we consider a question that naturally appeared in the recent work of Frazier–Nazarov–Verbitsky [3]. The question is:

How does the Carleson norm of a positive measure in the disk relate to the *bounded mean oscillation* (BMO) norm of its balayage on the circle?

A related question is:

How can one describe measures on the disk (say, positive measures) whose balayage is a BMO function?

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We show that the seemingly answer: “These are exactly the Carleson measures” is false. The Carleson property is indeed of course sufficient, but not at all necessary.

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However, we can characterize the Carleson property in terms of the *BMO* norms of the balayages of restrictions of the measure.

Throughout the paper, we will use the notation $\lesssim$, $\gtrsim$ for one-sided estimates up to an absolute constant, and the notation $\approx$ for two-sided estimates up to an absolute constant.

We will use the setting of the upper half plane $\mathbb{R}^2_+$ rather than the unit disk. Given a positive regular Borel measure μ on the upper half plane $\mathbb{R}^2_+ = \{(t, y) \in \mathbb{R}^2 : y > 0\}$, its *balayage* is defined as the function

$$S_\mu(t) = \int_{\mathbb{R}^2_+} p_{x,y}(t) \mathrm{d}\mu(t, y),$$

where $p_{x,y}(t) = \frac{1}{\pi} \frac{x}{y^2+(t-x)^2}$ is the Poisson kernel for $\mathbb{R}^2_+$. We say that μ is a *Carleson measure* if there exists a constant $C > 0$ such that for each interval $I \subset \mathbb{R}$, the inequality

$$\mu(Q_I) \leq C|I| \tag{1}$$

holds. Here, Q_I denotes the *Carleson square* $\{(x, y) : x \in I, 0 < y \leq |I|\}$ over I. It is easy to see that it is sufficient to consider dyadic intervals in this definition. We denote the infimum of all constants $C > 0$ such that (1) holds for all dyadic intervals by $\mathrm{Carl}(\mu)$.

Recall that the space of functions of BMO $(\mathbb{R})$ is defined as

$$\left\{ b \in L^2(\mathbb{R}) : \sup_{I \subset \mathbb{R} \text{ interval}} \frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I| \mathrm{d}t < \infty \right\},$$

with $\|b\|_{\mathrm{BMO}} = \sup_{I \subset \mathbb{R} \text{ interval}} \frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I| \mathrm{d}t$. By the John–Nirenberg inequality, the L^1 norm in the definition of BMO can be replaced by any $\| \cdot \|_p$ norm, $1 \leq p < \infty$. We thus obtain a family of equivalent norms on BMO$(\mathbb{R})$, with equivalent constants depending on p.

The connection between the properties of a measure μ and its balayage S_μ have long been studied. In particular, it is well known that the BMO norm of S_μ is controlled by the Carleson constant of μ,

$$\|S_\mu\|_{\mathrm{BMO}} \lesssim \mathrm{Carl}(\mu). \tag{2}$$

For this and other basic facts on BMO functions, we refer the reader to [4].

A partial reverse of (2) was found in [2], [7], and in the dyadic case, [5]. Namely, it was shown that for each $b \in \mathrm{BMO}$, there exists an $L^\infty(\mathbb{R})$ function ϕ and a Carleson measure μ such that $b = \phi + S_\mu$, $\|\phi\|_\infty + \mathrm{Carl}(\mu) \lesssim \|b\|_{\mathrm{BMO}}$. If we allow μ to be a complex measure, one even has the representation $b = S_\mu$ with $\mathrm{Carl}(\mu) \lesssim \|b\|_{\mathrm{BMO}}$ [6].

The purpose of this note is to show that reverse inequality to (2) in the strict sense does not hold, and to give a characterization of the Carleson property of a measure μ in terms of the BMO norm of the balayage of *restrictions* of μ.

2 The Dyadic Balayage

We start by examining the dyadic case. We will use the standard Whitney-type decomposition of the upper half plane, indexed by the set $\mathcal{D}$ of left-half open dyadic intervals in $\mathbb{R}$,

$$T_I = \left\{(x, y) : x \in I, \frac{|I|}{2} < y \leq |I|\right\} \text{ for } I \in \mathcal{D}.$$

That means, T_I is the "top half" of the Carleson square Q_I defined above.

For a positive regular Borel measure μ on $\mathbb{R}^2_+$, we define the *dyadic balayage* by

$$S^{\mathrm{d}}_\mu(t) = \sum_{I \in \mathcal{D}} \frac{\chi_I(t)}{|I|} \mu(T_I) \quad (t \in \mathbb{R}),$$

which is well defined as a function taking values in $[0, \infty]$. By comparing box kernel and Poisson kernel, one easily verifies the pointwise estimate $S^{\mathrm{d}}_\mu \lesssim S_\mu$.

We recall the definition of *dyadic BMO*, $\mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$, as the class of $L^2(\mathbb{R})$ functions for which

$$\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}} = \sup_{I \in \mathcal{D}} \frac{1}{|I|} \int_I |b(t) - \langle b \rangle_I|^2 \mathrm{d}t = \sup_{I \in \mathcal{D}} \frac{1}{|I|} \|P_I b\|^2 = \sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \in \mathcal{D}, J \subseteq I} |b_I|^2$$

is finite. Here, h_J denotes the L^2-normalized Haar function, $b_J := (b, h_J)$ denotes the corresponding Haar coefficient of function b, and P_I denotes the orthogonal projection on to $\overline{\mathrm{span}\{h_J : J \subseteq I\}}$. Again, by the John–Nirenberg inequality the L^2 norm in the definition can be replaced by any L^p norm, $1 \leq p < \infty$, yielding an equivalent norm.

We say that a sequence of nonnegative numbers $(\alpha_I)_{I\in\mathcal{D}}$ is a *Carleson sequence*, if there exists a constant $C > 0$ such that

$$\frac{1}{|I|}\sum_{J\in\mathcal{D}, J\subseteq I} a_I \leq C \text{ for each } I\in\mathcal{D}.$$

Again, we denote the infimum of such constants by $\mathrm{Carl}((a_I))$. With this notation, one verifies immediately the following well-known lemma.

Lemma 2.1. Let $b \in L^2(\mathbb{R})$. Then the following are equivalent:

1. μ is a Carleson measure
2. $(\mu(T_I))_{I\in\mathcal{D}}$ is a Carleson sequence
3. $b_\mu = \sum_{I\in\mathcal{D}} h_I \mu(T_I)^{1/2} \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$.

In this case, $\mathrm{Carl}(\mu) = \mathrm{Carl}((\mu(T_I))) = \|b_\mu\|^2_{\mathrm{BMO}^{\mathrm{d}}}$. □

Notice that with the above definition of b_μ,

$$S^{\mathrm{d}}_\mu = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|}\mu(T_I) = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|}|(b\mu)_I|^2 = \mathcal{S}[b_\mu],$$

where $\mathcal{S}$ denotes the square of the dyadic square function, $\mathcal{S}[f] = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|}|f_I|^2$ for $f \in L^2(\mathbb{R})$. In this sense, we have identified the dyadic balayage of a positive regular Borel measure μ with the square of a dyadic square function of b_μ. Conversely, for any $f \in L^2(\mathbb{R})$, $\mathcal{S}[f]$ can be written as a dyadic balayage of a measure μ_f, for example by letting $\mu_f = \sum_{I\in\mathcal{D}} |f_I|^2 \delta_{z(I)}$, $z(I)$ denoting the center of T_I.

The well-known dyadic analog of (2) is therefore equivalent to the inequality

$$\|\mathcal{S}[b]\|_{\mathrm{BMO}^{\mathrm{d}}} \lesssim \|b\|^2_{\mathrm{BMO}^{\mathrm{d}}}, \tag{3}$$

which can be now be proved as a simple application of the John–Nirenberg inequality. Notice that for any dyadic inverval $I \in \mathcal{D}$, all summands in $\mathcal{S}[b] = \sum_{J\in\mathcal{D}} \frac{\chi_J}{|J|}|b_J|^2$ except those corresponding to dyadic intervals $J \subset I$ are constant on I. Thus

$$\begin{aligned}\frac{1}{|I|}\int_I |\mathcal{S}[b](t) - \langle \mathcal{S}[b]\rangle_I|\mathrm{d}t &= \frac{1}{|I|}\int_I |\mathcal{S}[P_I b](t) - \langle \mathcal{S}[P_I b]\rangle_I|\mathrm{d}t \\ &\leq \frac{1}{|I|}\int_I \mathcal{S}[P_I b](t)\mathrm{d}t + \langle \mathcal{S}[P_I b]\rangle_I = 2\frac{1}{|I|}\int_I \sum_{J\subseteq I}\frac{\chi_J(t)}{|J|}|b_J|^2\mathrm{d}t = 2\|P_I b\|_2^2 \leq 2\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}},\end{aligned}$$

which proves (3).

Here are the main results of this section, which concern the reverse inequality to (3). The first says that the BMO norm of the dyadic balayage can be very much smaller than the Carleson constant of a measure, even if one increases the BMO norm by the L^2 norm.

Theorem 2.2. Let $\varepsilon > 0$. Then there exists a Carleson measure μ on $\mathbb{R}^2_+$ with $\mathrm{Carl}(\mu) = 1$, $\|S^{\mathrm{d}}_\mu\|_{\mathrm{BMO}} + \|S^{\mathrm{d}}_\mu\|_2 < \varepsilon$. □

Proof. By Lemma 2.1 and the argument following it, we want to find a $\mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$ function b of norm 1 such that both the $\mathrm{BMO}^{\mathrm{d}}$ norm and the L^2 norm of $\mathcal{S}[b]$ are small. To this end, let $I_0 = (0, 1]$, $I_{-1} = (-2, 0]$, $I_k = (2^k - 1, 2^{k+1} - 1]$ for $k > 0$ and $I_k = (-2^{-k}, -2^{-k-1}]$ for $k < 0$. In particular, $|I_k| = 2^{|k|}$ for all $k \in \mathbb{N}$. Let r_1 denote the first Rademacher function on $\mathbb{R}$, $r_1 = \sum_{j\in\mathbb{Z}}(-\chi_{(j,j+\frac{1}{2}]} + \chi_{(j+\frac{1}{2},j+1]})$, and let $r_n = r_1(2^{n-1}\cdot)$ be the nth Rademacher function on $\mathbb{R}$. Let $N \in \mathbb{N}$, N to be determined later, and let

$$b = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{N-|k|} \chi_{I_k}(t) r_n(t).$$

One verifies without difficulty that $\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}} = N$. Clearly,

$$\mathcal{S}[b] = \sum_{k=-\infty}^{\infty} \sum_{n=1}^{N-|k|} \chi_{I_k} = \sum_{k=0}^{N} (N-k)\chi_{I_k \cup I_{-k}}.$$

This is a "dyadic log", and it is not difficult to show that

$$\|\mathcal{S}[b]\|_{\mathrm{BMO}} \le C,$$

where C is an absolute constant independent of N. Notice that we have an estimate here not only for the dyadic BMO norm, but for the full BMO norm.

Now choose N so large that $\frac{C}{N} < \frac{\varepsilon}{2}$ and replace b by $\frac{1}{N^{1/2}} b$. This already guarantees that $\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}} = 1$, $\|\mathcal{S}[b]\|_{\mathrm{BMO}} < \frac{\varepsilon}{2}$. To deal with the desired L^2 estimate, observe that the estimates achieved so far do not change at all if b is dilated with an integer power of 2. By choosing a suitable power 2^K of 2, $K \in \mathbb{N}$, and replacing b by $b(2^K\cdot)$, we obtain the desired estimate

$$\|b\|^2_{\mathrm{BMO}^{\mathrm{d}}} = 1, \quad \|\mathcal{S}[b]\|_{\mathrm{BMO}} + \|\mathcal{S}[b]\|_2 < \varepsilon.$$ ■

The next theorem says that we can retrieve the Carleson constant of a measure up to an absolute constant from its dyadic balayage, if we restrict the measure to certain sets.

Theorem 2.3. Let μ be Carleson measure μ on $\mathbb{R}^2_+$. Then

$$\mathrm{Carl}(\mu) \approx \sup_{E \subseteq \mathbb{R}^2_+, E \text{ Borel set}} \| S^{\mathrm{d}}_{\mu_E} \|_{\mathrm{BMO}^{\mathrm{d}}} \approx \sup_{I \in \mathcal{D}} \| S^{\mathrm{d}}_{\mu_{Q_I}} \|_{\mathrm{BMO}^{\mathrm{d}}}.$$

Here, μ_E stands for the restriction of μ to E, given by $\mu_E(A) = \mu(E \cap A)$. □

Proof. Clearly, $\mathrm{Carl}(\mu_E) \le \mathrm{Carl}(\mu)$ for each Borel set $E \subseteq \mathbb{R}^2_+$, so

$$\sup_{I \in \mathcal{D}} \| S^{\mathrm{d}}_{\mu_{Q_I}} \|_{\mathrm{BMO}^{\mathrm{d}}} \le \sup_{E \subseteq \mathbb{R}^2_+, E \text{ Borel set}} \| S^{\mathrm{d}}_{\mu_E} \|_{\mathrm{BMO}^{\mathrm{d}}} \lesssim \sup_{E \subseteq \mathbb{R}^2_+, E \text{ Borel set}} \mathrm{Carl}(\mu_E) \le \mathrm{Carl}(\mu).$$

To prove the reverse inequality, let $I \in \mathcal{D}$. Observe that $S^{\mathrm{d}}_{\mu_{Q_I}}$ is supported on the closure of I. Therefore, with I' denoting the dyadic sibling of I, we have

$$\| S^{\mathrm{d}}_{\mu_{Q_I}} \|_{\mathrm{BMO}^{\mathrm{d}}} \ge |\langle S^{\mathrm{d}}_{\mu_{Q_I}} \rangle_I - \langle S^{\mathrm{d}}_{\mu_{Q_I}} \rangle_{I'}| = \langle S^{\mathrm{d}}_{\mu_{Q_I}} \rangle_I = \frac{1}{|I|} \int_I \sum_{J \in \mathcal{D}, J \subseteq I} \frac{\chi_J(t)}{|J|} \mu(T_J) \mathrm{d}t = \frac{1}{|I|} \mu(Q_I).$$

Thus, $\mathrm{Carl}(\mu) \lesssim \sup_{I \in \mathcal{D}} \| S^{\mathrm{d}}_{\mu_{Q_I}} \|_{\mathrm{BMO}^{\mathrm{d}}}$. ■

3 The Algebra of Paraproducts

This section contains a short operator-theoretic motivation for the choice of the counterexample, in particular the appearance of Rademacher functions, in the previous section, in terms of *paraproducts*. Recall that for $b \in L^2(\mathbb{R})$, the standard dyadic paraproduct π_b is defined by

$$\pi_b f = \sum_{I \in \mathcal{D}} h_I b_I \langle f \rangle_I \text{ for } f \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}).$$

It is well known, and indeed a reformulation of the classical Carleson Embedding Theorem, that π_b extends to a bounded linear operator on $L^2(\mathbb{R})$, if and only if $b \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$. In this case, $\|\pi_b\| \approx \|b\|_{\mathrm{BMO}^{\mathrm{d}}}$.

Such dyadic paraproducts have the nice property that $\pi_b^*\pi_b$ is essentially a dyadic paraproduct again, with symbol $\mathcal{S}[b]$ (see [1]):

$$\pi_b^*\pi_b = \pi_{\mathcal{S}[b]} + (\pi_{\mathcal{S}[b]})^* + \mathrm{Diag}(b), \tag{4}$$

where $\mathrm{Diag}(b)$ denotes the diagonal of $\pi_b^*\pi_b$ with respect to the Haar basis, $\mathrm{Diag}(b)h_I = \|\pi_b h_I\|^2 h_I$ for $I \in \mathcal{D}$. Moreover,

$$\|\pi_{\mathcal{S}[b]}\| \approx \|\pi_{\mathcal{S}[b]} + (\pi_{\mathcal{S}[b]})^*\| \approx \|\mathcal{S}[b]\|_{\mathrm{BMO}^{\mathrm{d}}}. \tag{5}$$

As pointed out in the previous section, the problem of finding a Carleson measure with Carleson constant 1 and small $\mathrm{BMO}^{\mathrm{d}}$ norm of the dyadic balayage is equivalent to finding $b \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$ of norm 1 such that $\mathcal{S}[b]$ has small $\mathrm{BMO}^{\mathrm{d}}$ norm.

In light of (4) and (5), this means finding $b \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$ such that $\pi_b^*\pi_b$ is "almost diagonal", in the sense that

$$\|\mathcal{S}[b]\|_{\mathrm{BMO}^{\mathrm{d}}} \approx \|\pi_{\mathcal{S}[b]} + (\pi_{\mathcal{S}[b]})^*\| = \|\pi_b^*\pi_b - \mathrm{Diag}_b\| \ll \|\pi_b^*\pi_b\| = \|\pi_b\|^2 \approx \|b\|^2_{\mathrm{BMO}^{\mathrm{d}}}.$$

Note the elementary identity

$$\pi_b^*\pi_b h_I = \frac{1}{|I|^{1/2}}\left(\sum_{J\subseteq I^+}\frac{\chi_J}{|J|}|b_J|^2 - \sum_{J\subseteq I^-}\frac{\chi_J}{|J|}|b_J|^2\right). \tag{6}$$

The function $\sum_{J\subseteq I^+}\frac{\chi_J}{|J|}|b_J|^2 + \sum_{J\subseteq I^-}\frac{\chi_J}{|J|}|b_J|^2$ is constant on its support I for each I, if b is a sum of Rademacher functions. In this case, the right-hand side $\sum_{J\subseteq I^+}\frac{\chi_J}{|J|}|b_J|^2 - \sum_{J\subseteq I^-}\frac{\chi_J}{|J|}|b_J|^2$ of (6) is always a multiple of h_I, and $\pi_b^*\pi_b$ is diagonal in the Haar basis. In our counterexample, we have to introduce cutoffs on the Rademacher functions in order to control the L^2 norm. This introduces nondiagonal terms, but these can then be controlled by the logarithmic staggering of the cutoffs.

4 The Poisson Balayage

We are now going to construct a compactly supported positive measure μ on the upper half plane such that its Carleson constant $\mathrm{Carl}(\mu)$ is very large (say m), but $\|S_\mu\|_{\mathrm{BMO}} + \|S_\mu\|_{L^1}$ is bounded by absolute constant. From here, one can easily construct finite positive measure μ which is not Carleson, but whose balayage is a nice BMO function.

Fix $m \in \mathbb{N}$. For $0 \le j \le m$, let I_j denote the interval $[-2^j, 2^j]$ and $\tilde{I}_j = I_j \setminus I_{j-1}$. Furthermore, let $\tilde{I}_0 = I_0$ and let $\tilde{I}_{m+1} = \mathbb{R} \setminus I_m$.

Let μ_j denote one-dimensional Lebesgue measure on the segment $I_j \times \{2^{-j}\}$, and let $\mu = \sum_{j=0}^{m} m_j$. Clearly, $\mathrm{Carl}(\mu) = m + 1$.

Here is the elementary technical lemma which will show the desired properties of μ.

Lemma 4.1. There exists an absolute constant $c > 0$ (independent of m) such that

$$|S_{\mu_j}(t) - \chi_{I_j}(t)| \le c 2^{-2j} \text{ for } |t| \le 2^{j-1} \text{ or } |t| \ge 2^{j+1}, \ \ j \in \{0, \dots, m\}. \qquad \square$$

Proof. Observe that

$$S_{\mu_j}(t) = \frac{1}{\pi} \int_{-2^j}^{2^j} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x \le S_{\mu_j}(0) \le 1 \text{ for all } t \in \mathbb{R}, \ \ j \in \{0, \dots, m\}.$$

Now let $|t| \le 2^{j-1}$. Then

$$\begin{aligned} S_{\mu_j}(t) - 1 &= \frac{1}{\pi} \int_{-2^j}^{2^j} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x \\ &= \frac{1}{\pi} \int_{-\infty}^{-2^j} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x + \frac{1}{\pi} \int_{2^j}^{\infty} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x \\ &\le \frac{2}{\pi} \int_0^{\infty} \frac{2^{-j}}{(x + 2^{j-1})^2 + 2^{-2j}} \mathrm{d}x \\ &= \frac{2}{\pi} \int_{2^{2j-1}}^{\infty} \frac{1}{x^2+1} \mathrm{d}x \le \sum_{l=j}^{\infty} \frac{2}{\pi} \int_{2^{2l-1}}^{2^{2l+1}} \frac{1}{x^2+1} \mathrm{d}x \\ &\le \frac{6}{\pi} \sum_{l=j}^{\infty} 2^{2l-1} \frac{1}{(2^{2l-1})^2} = \frac{8}{\pi} 2^{-2j+1}. \end{aligned}$$

If $|t| \ge 2^{j+1}$, then

$$\begin{aligned} S_{\mu_j}(t) &= \frac{1}{\pi} \int_{-2^j}^{2^j} \frac{2^{-j}}{(x-t)^2 + 2^{-2j}} \mathrm{d}x \\ &\le \frac{1}{\pi} \int_{-2^j}^{2^j} \frac{2^{-j}}{2^{2j} + 2^{-2j}} \mathrm{d}x \\ &\le \frac{1}{\pi} 2^{-2j+1}. \end{aligned}$$

■

Writing $S_\mu = \sum_{j=0}^m S_{\mu_j} = \sum_{j=0}^m \chi_{I_j} + \sum_{j=0}^m (S_{\mu_j} - \chi_{I_j})$, we see that the first term is a dyadic log function, and therefore in BMO($\mathbb{R}$) with some absolute norm bound independent of m. To estimate the second term, let $t \in \tilde{I}_k$. By the previous lemma, $|S_{\mu_j}(t) - \chi_{I_j}(t)| \le c2^{-j}$ for $j \notin \{k-1, k, k+1\}$, therefore

$$\sum_{j=0}^m |S_{\mu_j}(t) - \chi_{I_j}(t)| \le \sum_{j=0}^m c2^{-j} + 6 = 2c + 6.$$

Thus, the second term is in $L^\infty(\mathbb{R})$, with L^∞ norm bounded by $2c+6$. Altogether, we find that there is an absolute constant $\tilde{c}$, independent of m, such that $\|S_\mu\|_{\mathrm{BMO}} \le \tilde{c}$. However, an elementary calculation shows that

$$\|S_\mu\|_1 = \sum_{j=0}^m \|S_{\mu_j}\|_1 = \sum_{j=0}^m 2^{j+1} = 2^{m+2} - 2,$$

and we would like to control the L^1 norm of S_μ as well. But by scaling our construction with a small $h > 0$, that is, replacing each μ_j by $\tilde{\mu}_j$, the one-dimensional Lebesgue measure on $[-h2^j, h2^j] \times \{h2^{-j}\}$ and letting $\tilde{\mu} = \sum_{j=0}^m \tilde{\mu}_j$, we obtain a measure $\tilde{\mu}$ with $\mathrm{Carl}(\tilde{\mu}) = \mathrm{Carl}(\mu) = m+1$, $S_{\tilde{\mu}}(t) = S_\mu(\frac{t}{h})$. Thus, we have $\|S_\mu\|_1 = h(2^{m+2} - 2)$ and $\|S_{\tilde{\mu}}\|_{\mathrm{BMO}} = \|S_\mu\|_{\mathrm{BMO}} \le \tilde{c}$.

After choosing an appropriate $h > 0$ and dividing by an appropriate multiple of m, we obtain

Theorem 4.2. Let $\varepsilon > 0$. Then there exists a Carleson measure μ on $\mathbb{R}^2_+$ with $\mathrm{Carl}(\mu) = 1$, $\|S_\mu\|_{\mathrm{BMO}} + \|S_\mu\|_1 < \varepsilon$. □

We will now show a continuous analog to Theorem 2.3.

Theorem 4.3. Let μ be Carleson measure μ on $\mathbb{R}^2_+$. Then

$$\mathrm{Carl}(\mu) \approx \sup_{E \subseteq \mathbb{R}^2_+, E \text{ Borel set}} \|S^{\mathrm{d}}_{\mu_E}\|_{\mathrm{BMO}^{\mathrm{d}}} \approx \sup_{I \subset \mathbb{R} \text{ interval}} \|S_{\mu_{Q_I}}\|_{\mathrm{BMO}}.$$ □

Proof. We only have to prove that $\sup_{I \subset \mathbb{R} \text{ interval}} \|S\mu_{Q_I}\|_{\mathrm{BMO}} \gtrsim \mathrm{Carl}(\mu)$. After translation and dilation of μ, we can assume without loss of generality that $\mu(Q_J) \ge \frac{1}{4}\mathrm{Carl}(\mu)$ for

$J = [1/4, 3/4]$. Let $I = [0, 1]$ and let I' denote the translated interval $[2, 3]$. Then

$$\begin{aligned}
\|S_{\mu_{Q_I}}\|_{\mathrm{BMO}} &\gtrsim |\langle S_{\mu_{Q_I}}\rangle_I - \langle S_{\mu_{Q_I}}\rangle_{I'}| \\
&= \int_0^1 \frac{1}{\pi}\int_{Q_I} \frac{y}{(t-x)^2+y^2} - \frac{y}{(t+2-x)^2+y^2}\,\mathrm{d}\mu(x,y)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{Q_I}\int_{-x}^{1-x} \frac{y(4+4t)}{(t^2+y^2)((t+2)^2+y^2)}\mathrm{d}t\mathrm{d}\mu(x,y) \\
&\geq \frac{1}{\pi}\int_{[1/4,3/4]\times[0,1]}\int_{-x}^{1-x} \frac{y(4+4t)}{(t^2+y^2)((t+2)^2+y^2)}\mathrm{d}t\mathrm{d}\mu(x,y) \\
&\geq \frac{1}{\pi}\int_{[1/4,3/4]\times[0,1]}\int_{-1/4}^{1/4} \frac{y(4+4t)}{(t^2+y^2)((t+2)^2+y^2)}\mathrm{d}t\mathrm{d}\mu(x,y) \\
&\gtrsim \frac{1}{\pi}\int_{[1/4,3/4]\times[0,1]}\int_{-1/4}^{1/4} \frac{y}{t^2+y^2}\mathrm{d}t\,\mathrm{d}\mu(x,y) \\
&\geq \frac{1}{\pi}\int_{[1/4,3/4]\times[0,1]}\int_{-1/4}^{1/4} \frac{1}{t^2+1}\mathrm{d}t\,\mathrm{d}\mu(x,y) \gtrsim \mu(Q_J) \gtrsim \mathrm{Carl}(\mu). \qquad \blacksquare
\end{aligned}$$

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