Chapter V

A Variational Formalism for the Radiative Transfer Equation and a Geostrophic, Hydrostatic Atmosphere: Prelude to MODEL III

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1. Introduction

The approach to the development of MODEL III has been to divide the problem into three steps of increasing complexity. In Chapter IV we successfully developed a variational algorithm for the classical temperature retrieval problem that includes just the radiative transfer equation as a constraint. The radiances for each of the four TOVS MSU microwave channels were dependent variables.

Chapter V summarizes the second step which combines the four radiative transfer equations of the first step with the equations for a geostrophic and hydrostatic atmosphere. This step is intended to bring radiance into a three-dimensional balance with wind, height, and temperature. The use of the geostrophic approximation in place of the full set of primitive equations allows for an easier evaluation of how the inclusion of the radiative transfer equation increases the complexity of the variational equations.

It should be noted that the variational method is a powerful mathematical tool and a powerful method for diagnosing the physical role of the observations in the adjustment. We developed seven different variational formulations for the geostrophic, hydrostatic and radiative transfer equations. The first derivation was too complex to yield solutions that were physically meaningful. For the remaining six derivations, the variational method gave the same physical interpretation - the observed brightness temperatures could provide no meaningful input into a geostrophic, hydrostatic balance - at least through the problem-solving methodology employed in these studies. It would be axiomatic therefore, that the brightness temperatures could provide no meaningful input into a variational assimilation with the primitive equations.

During the writing of this chapter, the equations were reviewed and a conceptual error regarding one of the Lagrange multipliers was discovered.

In the following section, the variational methodology is presented and the Euler-Lagrange equations rederived for the geostrophic, hydrostatic and radiative transfer equations. Then the equations are reduced in number through elimination of variables to produce a single equation for the geopotential height. It is shown that the single equation is too difficult to solve but that a three equation set can be solved iteratively. It is also shown that space-based thermodynamic data can be assimilated into the meteorological data mainstream and that none of the difficulties associated with traditional temperature retrievals will be encountered.

2. A Variational Assimilation Theory for the Geostrophic, Hydrostatic and Radiative Transfer Equations

The variational formalism will be derived for the four radiative transfer equations in integral form. Let the dynamical constraints be,

$$m_2 - \phi_\sigma + \gamma T + \beta = 0 \qquad (2)$$

$$\sum_{j=1}^{J} m_{1j}=\sum_{j=1}^{J} (B_j-\int_0^{\infty} w_j T d\sigma=0) \tag{1}$$

$$m_3=v-\phi_x-F_5=0 \tag{3}$$

$$m_4=u+\phi_y+F_6=0 \tag{4}$$

For additional simplification, set the terrain correction term ß=0. The forcing functions F_5 and F_6 (see Chapter II) are simplified through setting $R_o = 0$.

The integrand of the functional to be minimized is,

$$I=\pi_1(u-u^o)^2+\pi_1(v-v^o)^2+\pi_2(T-T^o)^2+\pi_3(\phi-\phi^o)^2 +\sum_{j=1}^{J}\pi_{4j}(B_j-B_j^o)^2+2\sum_{j=1}^{J}\lambda_{1j}m_{1j}+2\lambda_2 m_2+2\lambda_3 m_3+2\lambda_4 m_4 \tag{5}$$

where the π_i are the relative weights accorded to the observations.

Performing the variations for the eight dependent variables, u, v, Φ, T, and B_j (j=1,4), yields the following Euler-Lagrange equations,

$$\delta u: \quad \pi_1(u-u^o)+\lambda_4=0 \tag{6}$$

$$\delta V: \quad \pi_1(V-V^o)+\lambda_3=0 \tag{7}$$

$$\delta\phi: \quad \pi_3(\phi-\phi^o)-\lambda_{2\sigma}+\lambda_{3x}-\lambda_{4y}=0 \tag{8}$$

$$\delta T: \quad \pi_2(T-T^o)+\gamma\lambda_2-\sum_{j=1}^{J}\lambda_{1j}\int_0^\infty w_j d\sigma=0 \tag{9}$$

$$\delta B: \quad \sum_{j=1}^{J}[\pi_{4j}(B_j-B_j^o)+\lambda_{1j}=0] \tag{10}$$

These eight equations plus the seven original constraints constitute a set of 15 algebraic and linear partial differential equations to be solved. The number of equations may be reduced through the elimination of variables. There results a single diagnostic equation with geopotential height as the dependent variable. We develop a diagnostic equation for the geostrophic, hydrostatic adjustment first and then include the contribution from the radiative transfer equations. Two Lagrange multipliers are eliminated by combining (6), (7), and (8). Then, forming the vorticity from (3) and (4) and combining with (8) gives,

$$\pi_1\nabla^2\phi-\pi_3\phi+\lambda_{2\sigma}-\pi_1(V_x^o-U_y^o)+\pi_3\phi^o+\pi_1(F_{5x}+F_{6y})=0 \tag{11}$$

Reducing the thermodynamic variables is done as follows. Divide (9) by γ and operate by σ. Eliminate brightness temperature between (1) and (10). There results two equations,

$$\frac{\partial}{\partial\sigma}\left[\frac{\pi_2}{\gamma}(T-T^{o})\right]+\lambda_{2\sigma}-\sum_{j=1}^{J}\lambda_{1j}\epsilon_j=0 \tag{12}$$

where,

$$\epsilon_j=\frac{\partial}{\partial\sigma}\left[\frac{1}{\gamma}\int_0^{\infty}w_j d\sigma\right] \tag{13}$$

and,

$$\sum_{j=1}^{J}\left[\pi_{4j}\left(\int_0^{\infty}w_j T d\sigma-B_j^{o}\right)+\lambda_{1j}=0\right] \tag{14}$$

Combining (12) with (14) and substituting (2) gives,

$$\lambda_{2\sigma}=\frac{\pi_2}{\gamma^2}\phi_{\sigma\sigma}+\left(\frac{\pi_2\phi_\sigma}{\gamma^2}\right)_\sigma+\sum_{j=1}^{J}\epsilon_j\pi_{4j}\int_0^{\infty}\frac{w_j\phi_\sigma}{\gamma}d\sigma$$
$$+\left(\frac{\pi_2}{\gamma}T^{o}\right)_\sigma+\sum_{j=1}^{J}\epsilon_j\pi_{4j}B_j^{o} \tag{15}$$

Eliminating the Lagrangian multiplier between (11) and (15) yields a diagnostic equation in the geopotential height,

$$\pi_1\nabla^2\phi+\frac{\pi_2}{\gamma^2}\phi_{\sigma\sigma}+\left(\frac{\pi_2\phi_\sigma}{\gamma^2}\right)_\sigma-\pi_3\phi+\sum_{j=1}^{J}\epsilon_j\pi_{4j}\int_0^\infty\frac{w_j\phi_\sigma}{\gamma}d\sigma+F=0 \tag{16}$$

where,

$$F=\pi_1(v_x^o-u_y^o)+\pi_3\phi^o+\left(\frac{\pi_2}{\gamma}T^o\right)_\sigma+\sum_{j=1}^{J}\epsilon_j\pi_{4j}B_j^o+\pi_1(F_{5x}+F_{6y}) \tag{17}$$

Much effort was spent programming for (16). The resulting solution was not considered satisfactory. Given the complex coefficient structures and the delicate convergence criteria, much additional effort was expended through six subsequent derivations to express the variational formalism in forms easier to understand and easier to solve. These efforts eventually led away from a direct inclusion of the radiative transfer equation in a geostrophic, hydrostatic atmosphere.

In retrospect, it seems that the solution could have been more easily obtained if the 15 equation set was reduced to the following 3 equation set:

$$\pi_1\nabla^2\phi-\pi_3\phi=-\lambda_{2\sigma}+\pi_1(v_x^o-u_y^o)-\pi_3\phi^o-\pi_1(F_{5x}+F_{6y}) \tag{18}$$

$$T=-\frac{1}{\gamma}(\phi_\sigma+\beta) \tag{19}$$

$$\lambda_2 = -\frac{\pi_2}{\gamma}(T-T^{o}) - \sum_{j=1}^{J}\left[\frac{\pi_{4j}}{\gamma}\int_0^{\infty} w_j d\sigma \left(\int_0^{\infty} w_j T d\sigma - B_j^{o}\right)\right] \qquad (20)$$

These equations can be solved iteratively by first setting $\lambda_2 = 0$ and solving (18) for Φ. Then (19) is solved for T and the temperature substituted into (20) to derive an updated λ_2. Then the updated values are entered into (18) and the cycle repeated until a satisfactory level of convergence is attained.

3. Results

There are three important points to consider regarding (18) - (20).

a) NO RETRIEVAL OF TEMPERATURE IS REQUIRED TO BLEND SATELLITE OBSERVED BRIGHTNESS TEMPERATURES INTO THE METEOROLOGICAL DATA MAINSTREAM. This means that none of the problems associated with temperature retrievals will be encountered. The integral term appears on the right hand side of (20) not as a term to be solved. This is analogous to solving the radiative transfer equation for the brightness temperature - a very easy exercise. This single finding may make it worth while to pursue the formal variational approach to assimilation of microwave channel data especially if higher resolution radiance data becomes available in the future.

b) There must be observations of geopotential height or winds or both of equivalent accuracy with the satellite measurements in order for the (18) - (20) to work. Accurate observations of temperature apart from space-based measurements are not necessary. The caveat is that geopotential height must be known at the boundaries of the domain in order to obtain a solution for (18). Lateral boundaries would vanish for the equations written on the sphere and the top boundary conditions can be removed to the top of the atmosphere or to some level where model predictions or climatology give satisfactory estimates.

c) It is highly probable that (18) - (20) converge to a solution. The same equation set with the absence of the second term of (20) (the radiative transfer equation contribution) is known to converge. The second term of (20) is an integral term which should further stabilize the solution.

The main goal of the variational assimilation project was to blend satellite-derived thermodynamic data into the meteorological data mainstream in a dynamically consistent way. The classical variational calculus method used to achieve that goal typically yields sets of complicated equations that require innovative methods for solution and also involve immense programming efforts. Therefore the effort was broken down into several simpler models that could be solved.

The attempt to reduce the equation set from 15 equations to one diagnostic equation in geopotential height resulted in equation (16). After an extensive programming effort, a satisfactory solution was not obtained. I was unable to devise a scheme that could determine whether the problems were mathematical or programmatical. During the six other efforts to derive a more tractable diagnostic equation a conceptual error was made, namely, λ_{1j} was treated as a variable that could be differentiated with respect to σ. The observed brightness temperature dropped out of the equations. This led to the conclusion that the satellite data could not be successfully included in a classical variational assimilation.

With the discovery of this error during the writing of Chapter 5 of this final report, that conclusion is no longer valid. It appears, instead, that the satellite data can be successfully incorporated into a variational assimilation and that the blending can be done without any of the problems typically encountered with temperature retrievals.