# STOCHASTIC ROBUSTNESS: TOWARDS A COMPREHENSIVE ROBUSTNESS TOOL

Laurie Ryan Ray

Department of Mechanical and Aerospace Engineering
Princeton University
Princeton, New Jersey

Stochastic robustness is a simple technique to determine the robustness of linear, time-invariant systems by Monte Carlo methods. *Stochastic stability robustness* has been described previously. These results are extended here to provide insight into control system design for performance. Together, stochastic stability and performance robustness concepts constitute a comprehensive tool that can be used to analyze control system robustness properties. As well, they offer control system design insight that can set the stage for stochastic robustness synthesis. The concept of stochastic stability robustness is reviewed, stochastic performance robustness is introduced, and stochastic robustness synthesis is described qualitatively. Confidence intervals necessary for comparing control laws statistically are presented.

*STOCHASTIC ROBUSTNESS ANALYSIS*
*Stability robustness*
*Performance robustness*

*STOCHASTIC ROBUSTNESS SYNTHESIS*
*Design insight uncovered by stochastic robustness*
*Confidence intervals for comparing control laws*

*SUMMARY*

# STOCHASTIC ROBUSTNESS ANALYSIS

*Stochastic robustness* is a robustness measure based on the probability of satisfactory stability/performance in the face of uncertainty. Stability robustness is described by a single metric: the probability of instability. Because it is a statistical measure of robustness, and because it directly uses knowledge of the statistics of the physical parameter variations, stochastic robustness is inherently intuitive and precise. The physical meaning behind the probability of instability is apparent, and overconservative or insufficiently robust designs can be avoided. Concepts behind stochastic stability robustness can be easily extended to provide insight into control system design for performance. Design specifications such as rise time, overshoot, settling time, dead time, and steady-state error are normally used as indicators of adequate performance and lend themselves to the same kind of analysis as described above. Concepts of stochastic stability robustness analysis can be applied to these criteria giving probabilistic bounds on individual scalar performance criteria. Stochastic robustness concepts can be applied to specific aircraft handling qualities criteria as well. Binomial confidence intervals for the scalar probability of instability have been presented, and these apply to performance robustness criteria as well.

*STOCHASTIC ROBUSTNESS*
A robustness *measure* based on the
*probability* of satisfactory stability/performance,
given the statistics of a plant's parameter variations

*STABILITY ROBUSTNESS:*
*PROBABILITY OF INSTABILITY*

*PERFORMANCE ROBUSTNESS: MANY MEASURES*
Degree of stability
Time response envelopes
Handling qualities criteria

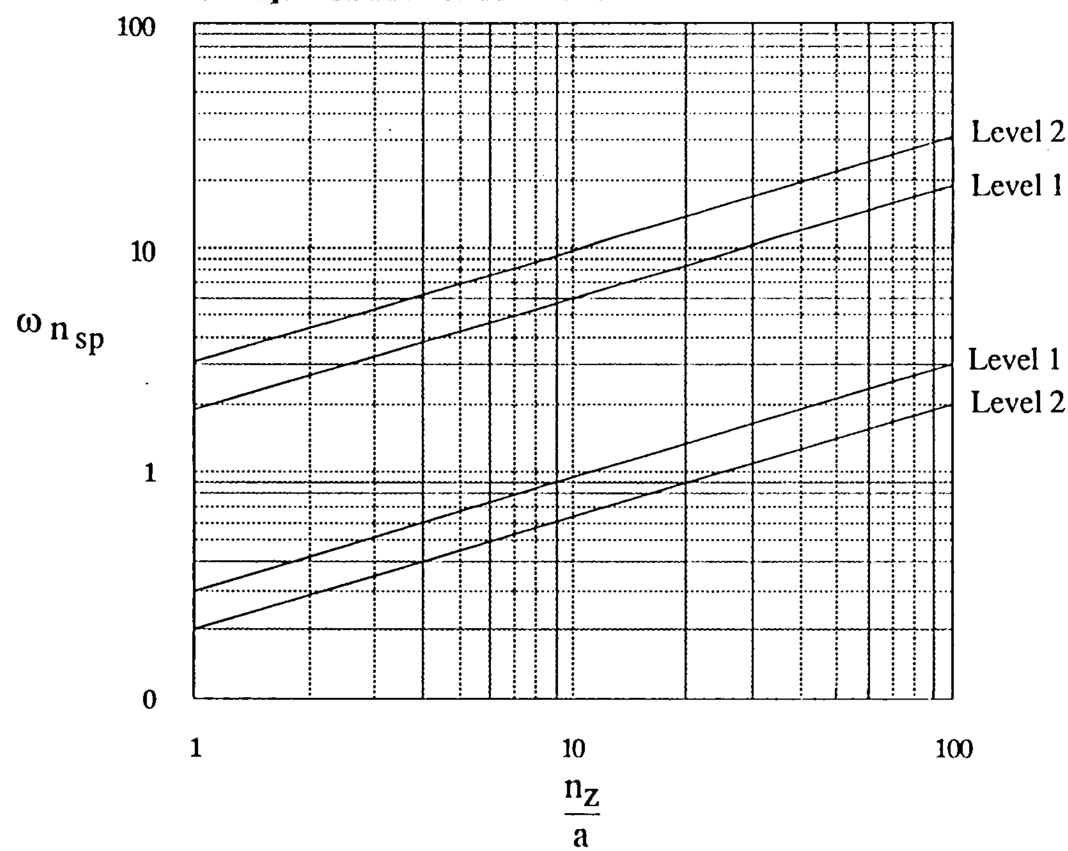
*BINOMIAL CONFIDENCE INTERVALS*
Statistical bounds for the
estimated stability or performance measure

# HANDLING QUALITIES ROBUSTNESS

Principles behind stochastic stability robustness can be directly applied to aircraft handling qualities. Here, the short-period mode is evaluated using the MIL-F-8785C specification that relates short-period handling quality levels to the normal acceleration sensitivity to angle-of-attack vs. short-period undamped natural frequency. Each Monte Carlo evaluation would result in a single point on the graph. The probability of remaining within level 1 or 2 specifications is the performance robustness metric. The abscissa and ordinate quantities can be computed using very little computation beyond eigenvalue evaluation. Hence, performance can be characterized as easily as stability using this metric.

*Principles behind stochastic stability robustness can be directly applied to aircraft handling qualities*

Example: Short-Period Frequency Requirements from MIL-F-8785C

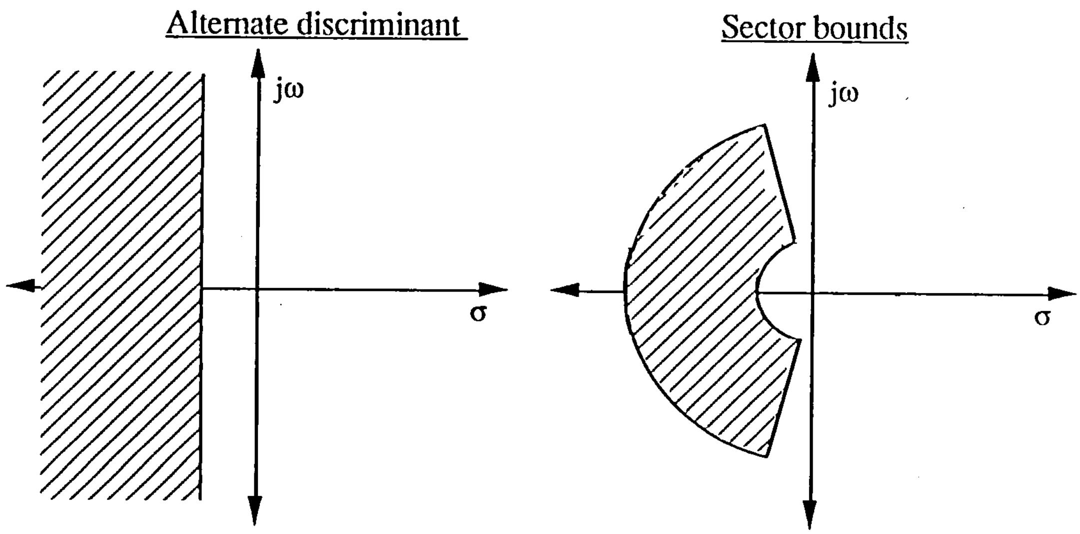



The probability of remaining within Level 1 or Level 2 regions is the scalar performance robustness metric

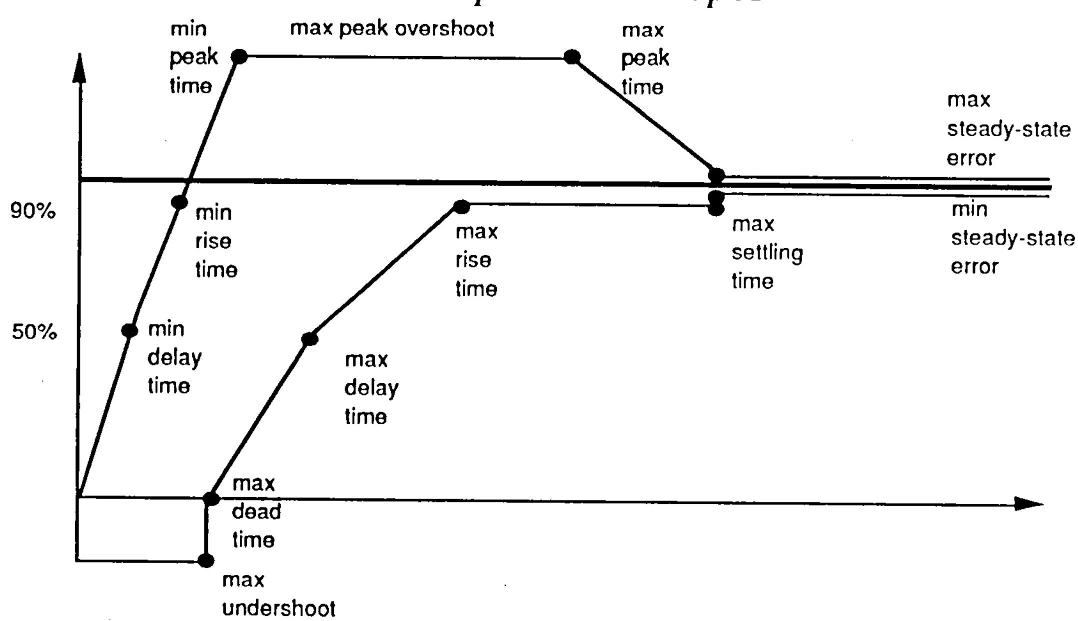
# PERFORMANCE ROBUSTNESS

Making the transition from strict stability to degree of stability is simple. Alternate regions can be described that relate to classical measures of response speed. Alternate discriminants relate to time-to-half or time-to-double. Sector bounds relate directly to damping ratios and natural frequencies. The probability of closed-loop eigenvalues remaining within the alternate region is the scalar performance robustness metric. Time response envelopes can be defined as well, and stochastic robustness analysis gives the probability of a response remaining within the desired envelope. Using any of these metrics, binomial confidence intervals apply.

*Transition from strict stability to degree of stability*




*Time response envelopes*

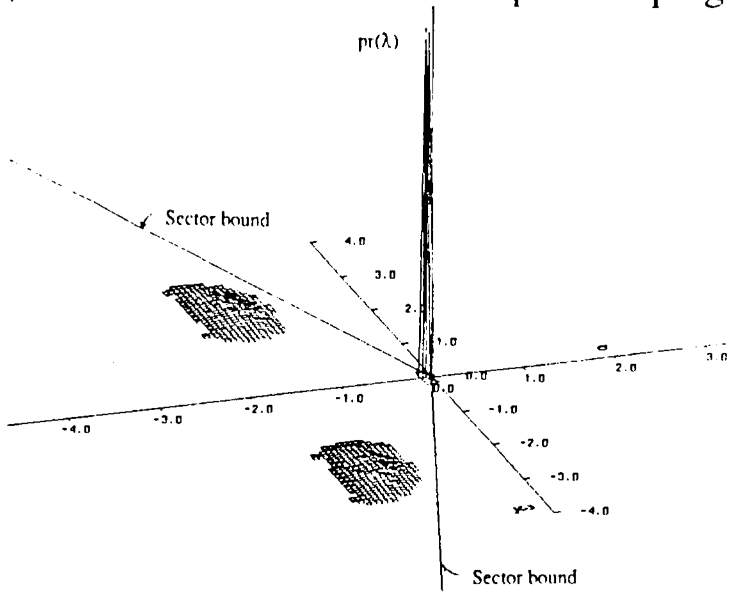


# TWIN-JET TRANSPORT AIRCRAFT: SHORT-PERIOD MODE EVALUATION

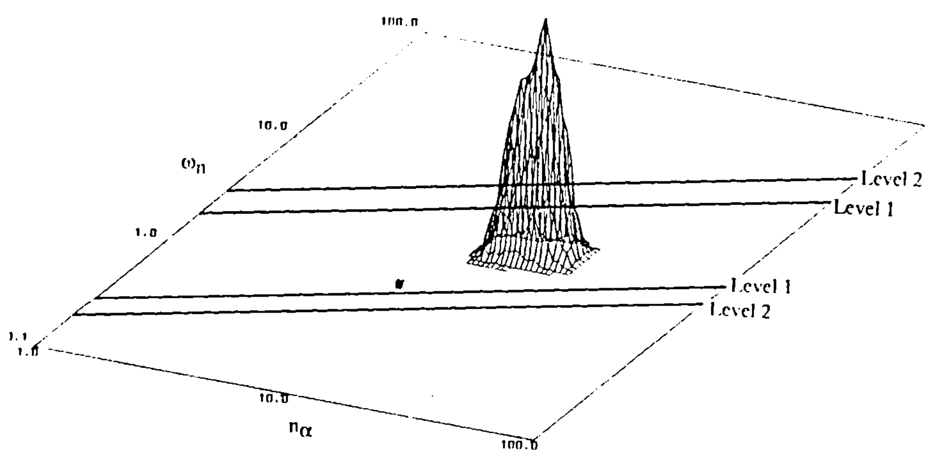
As an example of the above concepts, stochastic performance robustness analysis is applied to a nonlinear longitudinal model of a twin-jet transport aircraft. Each Monte Carlo evaluation consists of linearizing the system around nominal trim conditions and computing the eigenvalues and $\frac{n_z}{\alpha}$. The probability of violating sector bounds and the probability of violating level 1 Mil-spec requirements are shown here.

*Nonlinear, longitudinal rigid-body model, 22 parameters*

..using sector bounds based on Mil-spec damping requirement



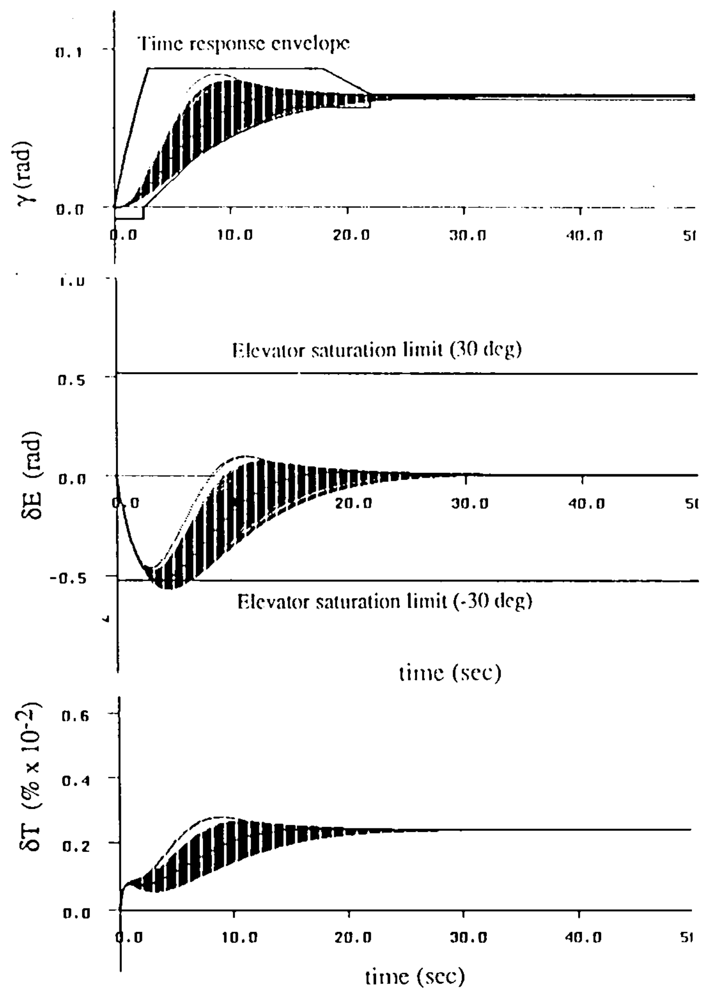

..using Mil-spec short-period frequency requirement



# TWIN-JET TRANSPORT AIRCRAFT: CLOSED-LOOP COMMAND RESPONSE

Closed-loop flight-path angle command response is given here, where the controller was developed using a Proportional-Filter-Implicit-Model-Following LQR control law. Here, the probability of violating the defined time response envelope and the probability of control saturation are the performance robustness metrics.

Flight-path-angle command response distribution




$\hat{P}_{\gamma} = 0.034\ (0.0199, 0.0539)$

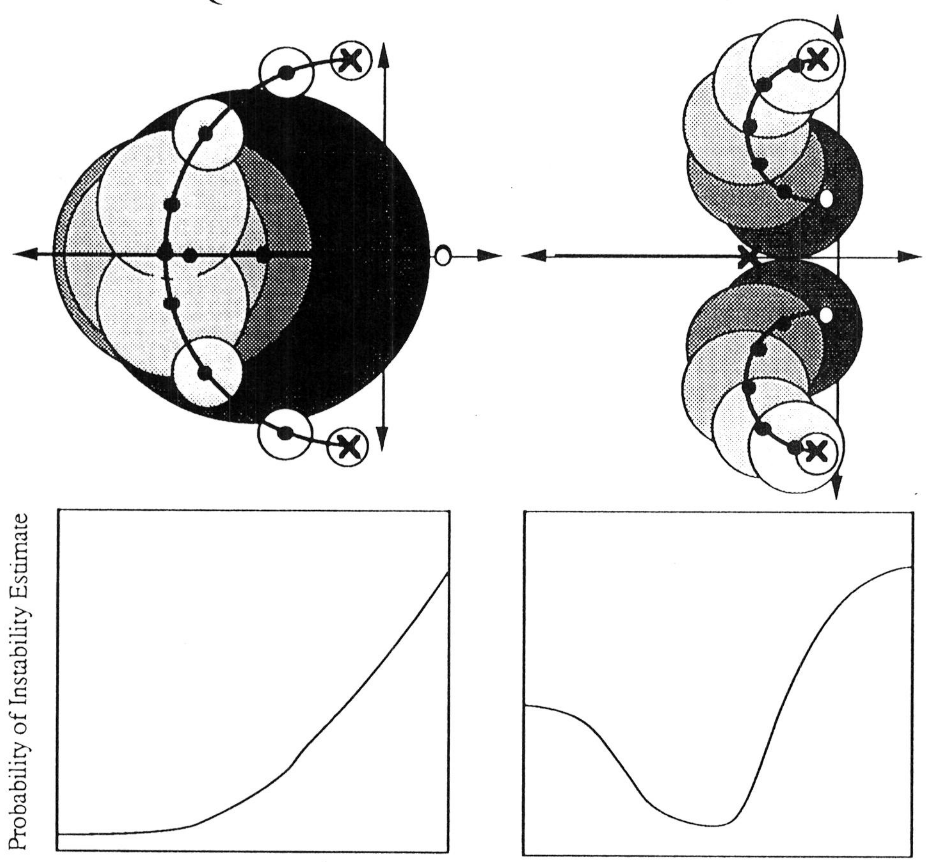
$\hat{P}_{\delta E} = 0.502\ (0.457, 0.547).$

# STOCHASTIC ROBUSTNESS SYNTHESIS

Metrics resulting from stability and performance robustness can be related to controller parameters, providing a foundation for design tradeoffs and optimization. This is illustrated qualitatively here. As gain increases along a root locus, uncertainty is magnified, and the "uncertainty circles" associated with the closed-loop root locations enlarge. The first root locus demonstrates one where the probability of instability increases monotonically with increased gain. The shape of the second root locus makes it possible for the uncertainty circles to initially cross into the right-half plane, remain in the left-half plane as gain increases, and finally cross into the right-half plane again for very large gain. The probability of instability decreases, then increases, and it has a minimum for some value of gain.

*Tradeoffs exist and can provide a foundation for optimization*

QUALITATIVE ILLUSTRATION




Closed-loop roots are enclosed by "uncertainty circles"
a) Stability robustness decreases as gain increases
b) Stability robustness increases, then decreases

# STATISTICAL COMPARISON OF CONTROL LAWS

The above analysis qualitatively demonstrates tradeoffs that can provide a foundation for optimization. The ability to statistically compare control laws and say with certainty that one is better than another is another tool necessary for optimization. The Bonferroni inequality can be used to describe confidence intervals for the difference based on individual confidence intervals. As illustrated, the one control law is statistically better than another when their individual confidence intervals no longer overlap.

*Given binomial confidence intervals for two estimates*

$$\Pr(L_1 \leq \mathbb{P}_1 \leq U_1) = 1 - \alpha_1$$
$$\Pr(L_2 \leq \mathbb{P}_2 \leq U_2) = 1 - \alpha_2$$

*the confidence interval for* $\Delta\mathbb{P} = \mathbb{P}_1 - \mathbb{P}_2$

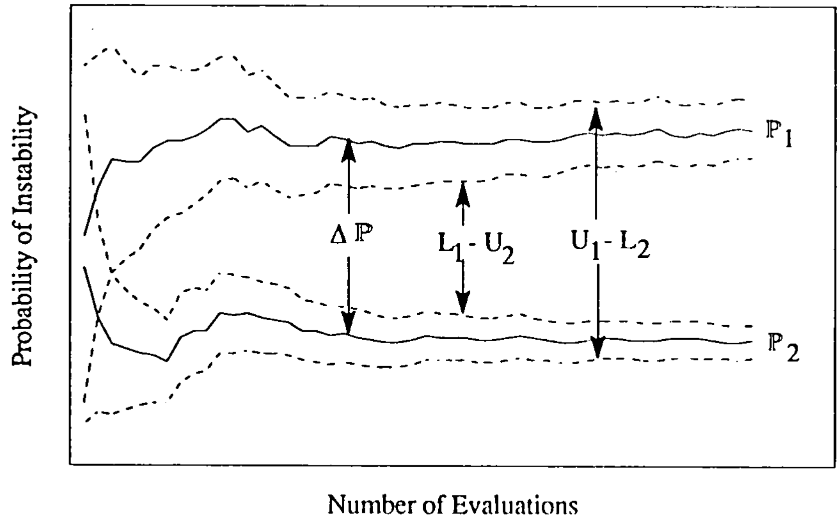
$$\Pr[(L_1 - U_2) \leq \Delta\mathbb{P} \leq (U_1 - L_2)] \geq 1 - \alpha$$

*is derived from the Bonferroni inequality*

$\alpha = \alpha_1 - \alpha_2$ for dependent intervals

$\alpha = \alpha_1 - \alpha_2 + \alpha_1 \alpha_2$ for independent intervals

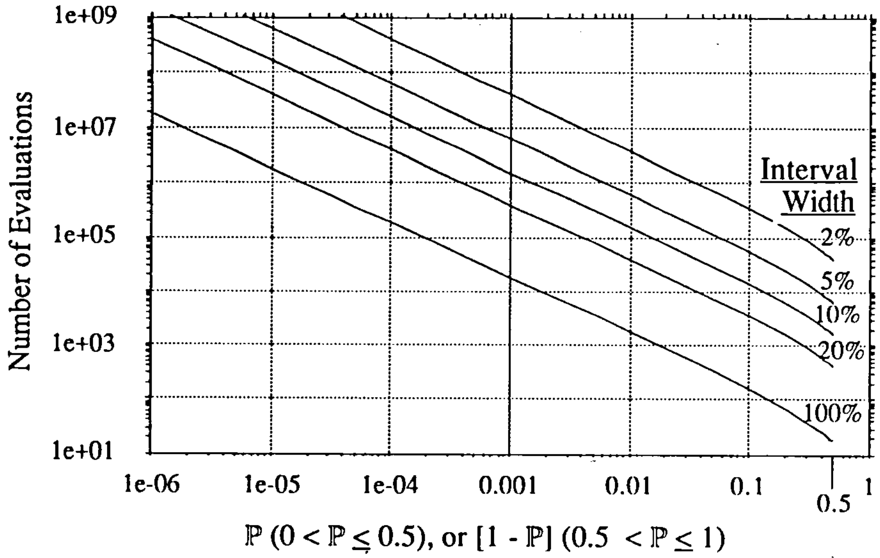
ILLUSTRATION



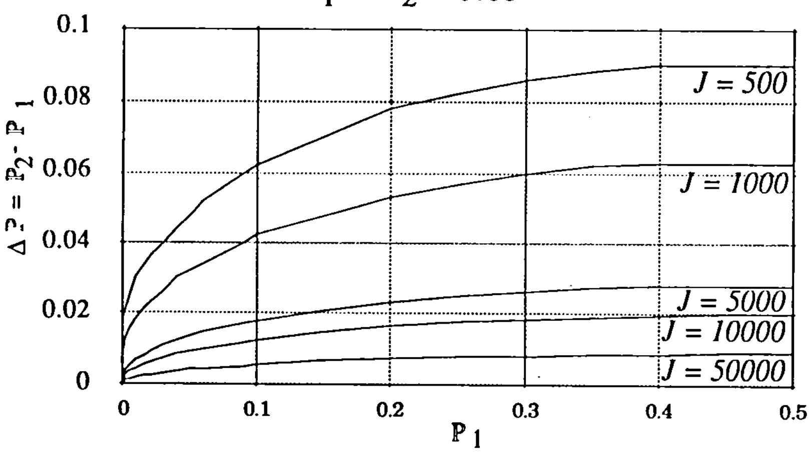
# HOW MANY EVALUATIONS ?

The number of evaluations necessary to distinguish between two control laws is an important factor when considering stochastic robustness synthesis using optimization. Here, the number of evaluations necessary for a required interval width is related to the number of evaluations necessary to distinguish a significant difference.

Number of Evaluations
for a Required Interval Width and $\alpha = 0.05$




Number of Evaluations Required
to Identify a Significant Difference
$\alpha_1 = \alpha_2 = 0.05$



# SUMMARY

Stochastic robustness is a good overall robustness analysis tool. It is intuitive and simple, and makes good use of engineering knowledg. Both stability and performance metrics can be identified, and confidence intervals offer statistical significance to the resulting metrics. It is a good candidate for synthesis techiques as well, because it demonstrates robustness tradeoffs with control systems design parameters. Confidence intervals for the difference between two probabilities provide a tool that can be used in future optimization studies.

*Good overall robustness analysis tool*

Intuitive and simple

Makes use of engineering knowledge

Both stability and performance robustness metrics

Confidence intervals are easily interpreted

*Good candidate for control system synthesis*

Shows stability and performance
tradeoffs with design parameters

Confidence intervals for differences can be defined