SOLAR RESPONSE TO LUMINOSITY VARIATIONS

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The connection between solar luminosity and magnetic fields is now well-established. Magnetic fields under the guise of sunspots and faculae enhance or suppress heat transfer through the solar surface, leading to changes in the total solar luminosity. This raises the question of the effect that such surface heat transfer perturbations have on the internal structure of the sun. The problem has been considered previously by Foukal[1] and Spruit[2,3]. We here generalize the calculation of Spruit, removing the assumption of a constant heat diffusivity coefficient by treating the full mixing length heat transfer expression. Further, we treat the surface conditions in a simpler manner, and show that the previous conclusions of Foukal and Spruit are unaffected by these modifications.

The model treats the solar convection zone as a plane parallel layer of perfect gas, denoting H the heat flux, P the pressure, and T the temperature. We assume that the solar interior is unaffected by surface effects, and thus enforce a constant heat flux H_0 at the base of the convective layer. On the surface, we enforce a radiative boundary condition $H = \sigma T^4$, but allow σ to vary with time to model the time variations of the effective emissivity (caused by sunspots and faculae). In general, the total height of the layer will vary in response to the variations in σ while the total mass in the layer remains constant so that we define the location of the base and surface of the layer by the base and surface pressures P_0 and P_h. As mentioned, we neglect partial ionization effects by assuming a perfect gas equation of state, but leave γ (the adiabatic index) unfixed to preserve generality. Finally, we assume a constant gravitational field.

The model allows us to study various plage/sunspot scenarios through the use of their effective emissivity variation $\sigma(t)$. For the present work, we use $\sigma(t) = \sigma_0 + \delta\sigma u(t)$ where $u(t)$ is the unit step function, and $\delta\sigma$ is a small perturbation $\delta\sigma << \sigma_0$. With this choice of $\sigma(t)$, we make the following observations. For $t < 0$, we expect the fluid layer to be in a steady state with $H = H_0$ everywhere. At $t = 0$, the surface luminosity will jump from H_0 to $H_0(1 + \delta\sigma/\sigma_0)$. However, as $t \to \infty$ the layer will approach another steady state, with $H = H_0$ everywhere, so that the surface luminosity must relax from $H_0(1 + \delta\sigma/\sigma_0)$ back to H_0. It is the timescale of this relaxation that we are interested in. In what follows, it will be shown that each steady state corresponds approximately to an adiabat, so that the $\sigma(t)$ variation forces the fluid layer from one adiabat to another.

To begin the full solution, consideration of Fig. 1 leads us to the following set of five equations, which, in order, are the continuity equation, the hydrostatic equation, conservation of energy, mixing length convective heat transport, and the equation of state:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho v) = 0 \qquad (1)$$

$$\frac{\partial P}{\partial z} + \rho g = 0 \tag{2}$$

$$\frac{\partial H}{\partial z} + \frac{\partial}{\partial z}\left\{\frac{\gamma}{\gamma-1}Pv + \rho g v z\right\} + \frac{\partial}{\partial t}\left\{\frac{1}{\gamma-1}P + \rho g z\right\} = 0 \tag{3}$$

$$P = C_P \rho \sqrt{\frac{g}{T}} \frac{\ell^2}{4} \left\{(1-\frac{1}{\gamma})\frac{T}{P}\frac{\partial P}{\partial z} - \frac{\partial T}{\partial z}\right\}^{\frac{3}{2}} \tag{4}$$

$$P = \left(\frac{k}{\mu}\right)\rho T \tag{5}$$

where z is the depth coordinate, g is the solar gravitational constant, v is the fluid velocity, c_P is the specific heat per unit mass, ℓ is the mixing length and μ is the mass of an individual gas particle. We note that in equations (2) and (3) inertial terms second order in the perturbation have been neglected. In (4), we take the mixing length ℓ to be $\chi kT/\mu g$ where $kT/\mu g$ is the pressure scale height, and χ is a numerical factor of order unity.

To facilitate the application of the boundary conditions, we rewrite this set of equations using P as an independent variable in place of z. Having done this, we find that equations (1)-(5) reduce to a single equation:

$$\phi P^{2+\frac{1}{2\gamma}} \left(\frac{\partial G}{\partial P}\right) G^{-2} \left\{\left(3 - \frac{1}{2\gamma}\right)\frac{1}{P}\frac{\partial G}{\partial P} + \frac{3}{2}\frac{\partial^2 G}{\partial P^2}\right\} - \frac{\partial G}{\partial t} = 0 \tag{6}$$

where

$$\phi = \frac{\sqrt{3}\chi^2 g}{4} \tag{7}$$

$$G(P,z) = (\rho P^{-\frac{1}{\gamma}})^{-\frac{1}{3}} \tag{8}$$

To obtain the physical meaning of G, we note that G can be shown to be proportional to e^{KS} where S is the entropy, and K a constant.

In examining (6), we see that in steady state, the equation can be immediately solved to give:

$$G(P) = C_1 - C_0 P^{-(1-\frac{1}{3\gamma})} \tag{9}$$

where, for the boundary conditions on H shown in Fig. 1, we find

$$C_0 = \frac{1}{3}\left(\frac{4H}{\chi^2}\frac{\gamma-1}{\gamma}\right)^{\frac{2}{3}} \frac{3\gamma}{3\gamma-1} \tag{10}$$

$$C_1 = \left(\frac{k}{\mu}\right)^{\frac{1}{3}} \left(\frac{H}{\sigma}\right)^{\frac{1}{12}} P_h^{-\frac{1}{3}(1-\frac{1}{\gamma})} + C_0 P_h^{-(1-\frac{1}{3\gamma})} \tag{11}$$

For solar-like conditions, one can show that the second term in (9) is important only near the surface of the layer, yielding immediately the fact that the steady-state solutions lie approximately on adiabats.

We next use the steady state solutions (9) to linearize (6) as follows. For $t < 0$ and for $t \to \infty$, the layer is in steady state, as mentioned previously. We denote the initial steady state as $G^i_{ss}(P)$ and the final steady state as $G^f_{ss}(P)$, where G^i_{ss} and G^f_{ss} are given by (9) with the proper values of C_0 and C_1. An examination of (10) and (11) shows that the two steady-state solutions differ only in their value of C_1 i.e. they lie on different adiabats. We next define $\Delta G(P,t)$ as:

$$G(P,t) = G^f_{ss}(P) + \Delta G(P,t) \tag{12}$$

where ΔG is considered small. We then use (12) in (6) and retaining only terms first order in ΔG, we find:

$$\theta P^{1+\frac{2}{3\gamma}} \left(\frac{\partial^2 \Delta G}{\partial P^2} + \frac{2-\frac{1}{3\gamma}}{P} \frac{\partial \Delta G}{\partial P} \right) - \frac{\partial \Delta G}{\partial t} = 0 \tag{13}$$

where

$$\theta = \frac{3\sqrt{3}}{8} \chi^2 g \frac{\sqrt{C_0(1-\frac{1}{3\gamma})}}{C_1^2} \tag{14}$$

The values of C_0 and C_1 in (12) are those appropriate for $G^f_{ss}(P)$. In deriving (13), we have neglected the pressure dependent term in $G^f_{ss}(P)$ since it is small, thereby assuming polytropic steady state solutions. The boundary conditions on $\Delta G(P,t)$ can be shown to be:

$$\frac{\partial \Delta G}{\partial P}|_{P_0} = 0 \tag{15}$$

$$\Delta G|_{P_h} = \psi \frac{\partial \Delta G}{\partial P}|_{P_h} \tag{16}$$

$$\Delta G(P,0) = \kappa \tag{17}$$

$$\Delta G(P,t) \longrightarrow_{t\to\infty} 0 \tag{18}$$

where

$$\psi = \frac{1}{8} \left(\frac{k}{\mu}\right)^{\frac{1}{3}} \left(\frac{H}{\sigma}\right)^{\frac{1}{12}} \frac{P_h^{\frac{5}{3}}}{(1-\frac{1}{3\gamma})C_0} \tag{19}$$

$$\kappa = \left(\frac{k}{\mu}\right)^{\frac{1}{3}} \left(\frac{H}{\sigma}\right)^{\frac{1}{12}} P_h^{-\frac{1}{3}(1-\frac{1}{\gamma})} \frac{1}{12} \frac{\delta\sigma}{\sigma} \tag{20}$$

With (15)-(18), (13) is a well-defined boundary value problem whose solution can be written as an eigenfunction series as follows:

$$\Delta G(P,t) = \sum_m A_m M_m(P) e^{-t/\tau_m} \tag{21}$$

where

$$M_m(P) = P^{-\gamma\eta} \left\{ J_\nu(\xi_m P^\eta) + B_m Y_\nu(\xi_m P^\eta) \right\} \tag{22}$$

$$B_m = \frac{-J_{\gamma+1}(\xi_m P_0^\eta)}{Y_{\gamma+1}(\xi_m P_0^\eta)} \tag{23}$$

$$A_m = \frac{-(\delta\kappa/\eta\psi)P_h M_m(P_h)}{M_m^2(P_h)\left\{\eta_m^2 P_h^{2\eta} + \frac{2\nu}{\eta}\frac{P_h}{\psi} + \left(\frac{P_h}{\psi\eta}\right)^2\right\} - \left(\frac{2}{\pi}\right)^2 P_h^{-2\nu\eta}/Y_{\nu+1}^2(\xi_m P_0^\eta)} \tag{24}$$

$$\nu = \frac{3\gamma - 1}{3\gamma - 2} \tag{25}$$

$$\eta = \frac{3\gamma - 2}{6\gamma} \tag{26}$$

$$\xi_m = \frac{1}{\eta}\frac{1}{\sqrt{\theta\tau_m}} \tag{27}$$

The characteristic times τ_m are solutions to the eigenvalue equation:

$$\frac{P_h^{1-\eta}}{\psi\eta\xi_m}\left[J_\nu(\xi_m P_h^\eta) + B_m Y_\nu(\xi_m P_h^\eta)\right] + \left[J_{\nu+1}(\xi_m P_h^\eta) + B_m Y_{\nu+1}(\xi_m P_h^\eta)\right] = 0 \tag{28}$$

Having the solution for $\Delta G(P,t)$, we note the following connection to the heat flux.

$$H(P,t) = H_0 + \delta H(P,t)$$
$$\delta H(P,t) = \frac{3}{4}\frac{H_o P^{1-\eta\gamma}}{\eta\nu C_0}\frac{\partial \Delta G}{\partial P} \equiv H_0\left(\frac{\delta\sigma}{\sigma_0}\right)\sum D_m e^{-t/\tau_m} \tag{29}$$

The surface luminosity is then given by (29) evaluated at $P = P_h$.

As noted by Spruit, the solution (29) exhibits some general characteristics. The luminosity is shown to relax on two different timescales: one on the order of 10^5 years, and the other on the order of 50 days. The eigenvalue equation can be solved to yield an expression for the long timescale as follows:

$$\tau_0 = \frac{4}{9}\frac{\psi\eta\nu C_0 C_1^2}{gH(\eta^2 - \frac{1}{36})}\frac{P_0^{3\eta+\frac{1}{2}}}{P_h^{\eta+\frac{3}{2}}} \tag{30}$$

The next longest timescale is found from the first zero of

$$J_{\gamma+1}\left(\frac{P_0^\eta}{\eta\sqrt{\theta\tau_1}}\right) = 0 \tag{31}$$

Finally, we find D_0, the amplitude of the long timescale mode, to be::

$$D_0 = \left(1 + \frac{3\gamma}{3\gamma - 1}\frac{P_h}{\psi}\right)^{-1} \tag{32}$$

For typical values of $P_h = 10^6$ dynes, $P_0 = 5 \times 10^{13}$ dynes, $\gamma = \frac{3}{2}$ and, $\chi = 1$, we find $\tau_0 = 350{,}000$ years, $\tau_1 = 51$ days, $D_0 = .549$.

In summary, the model shows that following the application of a step function emissivity change, a fraction $1 - D_0$ of the luminosity change relaxes away after $\sim$ 50 days. This corresponds to the thermal diffusion time across the convection zone, adjusting the difference of the adiabatic temperature gradient and the actual temperature gradient to a value in correspondence with the surface change. In other words, the whole convection zone "feels" the perturbation on this timescale. The remaining fraction relaxes away on a timescale of 10^5 years, corresponding to the convective layer radiating away enough energy so that it can adjust to its new adiabat. These are the same results arrived at by Spruit and Foukal.

For variations of σ on timescales of 10-200 years, then, the only important relaxation is the 50 day one. If the amplitude of this relaxation is small, the luminosity follows the σ variation.

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References

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