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EIGENVALUE PLACEMENT AND STABILIZATION BY CONSTRAINED OPTIMIZATION

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ABSTRACT

A pole placement algorithm is proposed which uses constrained non-linear optimization techniques on a finite dimensional model of a linear n degree of freedom system. Low order feedback control is assumed where r poles may be assigned; r being the rank of the sensor coefficient matrix. It is shown that by combining feedback control theory methods with optimization techniques, one can ensure the stability characteristics of a system, and can alter its transient response.

INTRODUCTION

One common method of approaching the problems of controlling the vibration of a structure is to employ eigenvalue (pole) placement methods. Such solutions have attracted the attention of numerous authors over the past twenty-five years, including W. M. Wonham [6], E. J. Davison [3], S. Srinathkumar [5], A. N. Andry et al [1], [2] and many others.

In exploring pole placement in dynamical systems, an inadequacy of stability considerations in contemporary algorithms was noted and thus motivated this work. It appears that the problem has not been solved or even addressed in many approaches.

If a system is controllable, one has the ability to place a predetermined number of poles. Thus, when pole placement techniques are employed, there is a limit on the number of poles that may be assigned. As is well known, the rank of the sensor coefficient matrix determines how many poles may be placed exactly. These poles may be noted as the controllable eigenvalues of the system, while the remaining may be labelled uncontrollable.

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Thus, due to restrictions inherent to every system, every pole may not be desirably placed. Therefore, one does not have control over the full order of the system. When moving the allowable eigenvalues, those which are not placed will also be affected, with the possibility of generating an unstable state.

Since an unstable system is undesirable, the ability to place a predetermined number of poles, while forcing the system to remain stable would be quite desirable to the designer. Many pole placement methods yield satisfactory assignment of the desired modes, but unfortunately can drive the remaining eigenvalues unstable. Thus, requiring iteration of the algorithms, compromising the desired choice of eigenvalues or eigenvectors, until a stable response results. With the large number of modes required in modelling flexible structures, these methods become costly and time consuming.

Hence, a pole placement method is proposed which constrains the unspecified modes to be stable by taking advantage of constrained optimization techniques. It appears that no previous work has guaranteed stable unplaced poles or has assured the magnitude of relative stability.

Several numerical examples will be presented, and results will be compared with those of Srinathkumar [5].

PROPOSED SOLUTION

The systems studied in this paper are of the mechanical type, which are second order by nature, incorporating mass, stiffness and damping parameters, where only the class of discrete systems shall be investigated.

Assuming small motions about the equillibrium point implies linearization of the equations of motion, which become

$$[M]\underline{\ddot{q}}(t) + [D+G]\underline{\dot{q}}(t) + [S+H]\underline{q}(t) = \underline{F}(t) \tag{1}$$

The forcing function vector, $\underline{F}(t)$, may then be described as

$$\underline{F}(t) = [V]\underline{\dot{q}}(t) + [P]\underline{q}(t),$$

where [V] and [P] are the velocity and position feedback matrices, respectively. $\underline{q}(t)$ is the coordinate vector, while $\underline{\dot{q}}(t)$ and $\underline{\ddot{q}}(t)$ are the first and second time derivatives of this vector.

[M] is known as the mass or inertia matrix, [D] is called the damping matrix, and [S] is the stiffness matrix. The matrix [G] may be referred to as the gyroscopic or Coriolos matrix, and [H] is the circulatory matrix.

The [M], [D], [S], [G] and [H] matrices are assumed to be time-invariant, and therefore are represented by constant values, all being of nth order, where n represents the number of degrees of freedom of the system.

Using normal state space methods by letting

$$\underline{x}(t) = \begin{bmatrix} \dot{\underline{q}}(t) \\ \hline \underline{q}(t) \end{bmatrix},$$

the n-dimensional system becomes the following 2n-dimensional model:

$$\dot{\underline{x}}(t) = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \underline{x}(t) + \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} \underline{u}(t)$$

$$\underline{y}(t) = [C_1 \mid C_2]\underline{x}(t) \tag{2}$$

where [M] is assumed to have an inverse and $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \underline{u}(t)$ is a representation of the system's forcing function, $\underline{F}(t)$.

More simply, equation (2) may be expressed as follows:

$$\dot{\underline{x}}(t) = [A']\underline{x}(t) + [B]\underline{u}(t), \quad \underline{x}(0) = \underline{x}_0$$

$$\underline{y}(t) = [C]\underline{x}(t)$$

where $\quad \underline{u}(t) = [K]\underline{y}(t)$

$\underline{y}(t)$ is the output vector, [C] is a constant sensor coefficient matrix, and [K] is the feedback gain matrix. [B] may now be described as the constant coefficient matrix of actuator dynamics, and $\underline{u}(t)$ is the control vector. The following conditions hold:

i) $\underline{x}\epsilon R^{2n}$, $\underline{u}\epsilon R^{m}$, $\underline{y}\epsilon R^{r}$

ii) A', B, C are real, constant matrices of appropriate dimensions.

iii) rank $B = m \neq 0$, rank $C = r \neq 0$

By block diagram representation, the system described by equation (3) may be expressed as in Figure 1.

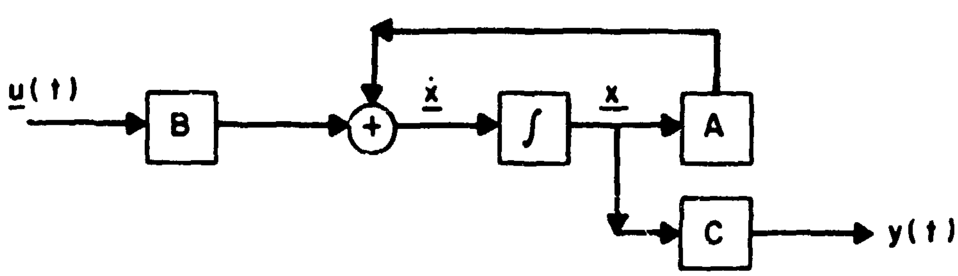

FIGURE 1

And a more revealing representation is shown in Figure 2.

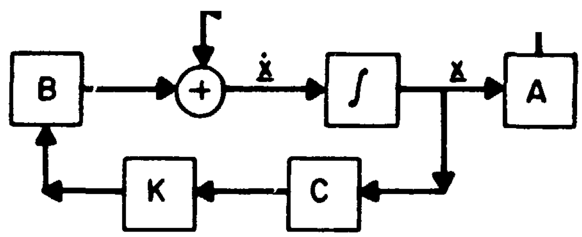

FIGURE 2

Equation (2) may be rewritten as follows:

$$\dot{\underline{x}}(t) = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \underline{x}(t) + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] [K][C_1 \mid C_2]\underline{x}(t)$$

or

$$\dot{\underline{x}}(t) = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \underline{x}(t) + \left[\begin{array}{c|c} B_1KC_1 & B_1KC_2 \\ \hline B_2KC_1 & B_2KC_2 \end{array}\right] \underline{x}(t)$$

$$\underline{y}(t) = [C_1 \mid C_2]\underline{x}(t) \qquad (3)$$

By comparison of equations (1) and (3), one may note that this implies:

$$[B_2] = [0],$$

thus

$$\dot{\underline{x}}(t) = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \underline{x}(t) + \left[\begin{array}{c|c} B_1KC_1 & B_1KC_2 \\ \hline 0 & 0 \end{array}\right] \underline{x}(t)$$

$$\underline{y}(t) = [C_1 \mid C_2]\underline{x}(t) \qquad (4)$$

If we define

$$[A] = \left[\begin{array}{c|c} -M^{-1}(D+G) + B_1KC_1 & -M^{-1}(S+H) + B_1KC_2 \\ \hline I_n & 0 \end{array}\right]$$

and describe equation (4) as follows:

$$\dot{\underline{x}} = [A]\underline{x}$$

$$\underline{y} = [C]\underline{x}$$

Then, the set of equations must satisfy the eigenvalue problem, i.e.,

$$[A]\underline{v}_i = \zeta_i \underline{v}_i \tag{5}$$

where $\left\{\zeta_i\right\}_{i=1}^{2n} \equiv$ the 2n eigenvalues

and $\left\{\underline{v}_i\right\}_{i=1}^{2n} \equiv$ the corresponding eigenvectors.

By substitution of equation (4) into equation (5),

$$\zeta_i \underline{v}_i = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \underline{v}_i + \left[\begin{array}{c|c} B_1KC_1 & B_1KC_2 \\ \hline 0 & 0 \end{array}\right] \underline{v}_i$$

$\underline{v}_i$ may then be defined to correspond to the above partitioning as follows:

$$\underline{v}_i = \left[\begin{array}{c} \underline{z}_i \\ \hline \underline{w}_i \end{array}\right]$$

yielding

$$\zeta_i \left[\begin{array}{c} \underline{z}_i \\ \hline \underline{w}_i \end{array}\right] = \left[\begin{array}{c|c} -M^{-1}(D+G) & -M^{-1}(S+H) \\ \hline I_n & 0 \end{array}\right] \left[\begin{array}{c} \underline{z}_i \\ \hline \underline{w}_i \end{array}\right] + \left[\begin{array}{c|c} B_1KC_1 & B_1KC_2 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c} \underline{z}_i \\ \hline \underline{w}_i \end{array}\right]$$

which implies

$$\zeta_i \underline{w}_i = \underline{z}_i$$

substituting,

$$\zeta_i^2 \underline{w}_i = -M^{-1}(D+G)\zeta_i \underline{w}_i - M^{-1}(S+H)\underline{w}_i + B_1KC_1\zeta_i \underline{w}_i + B_1KC_2\underline{w}_i \tag{6}$$

If we define $\{\lambda_1\}$ i = 1,2,...,r as the r eigenvalues to be placed, equation (6) may be expressed as

$$W\Lambda^2 = -M^{-1}(D+G)W\Lambda - M^{-1}(S+H)W + B_1KC_1W\Lambda + B_1KC_2W$$

where $$W = [\underline{w}_1 \,|\, \underline{w}_2 \,|\, \cdots\cdots \,|\, \underline{w}_r]$$

and $$\Delta = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_r)$$

By taking advantage of the generalized left inverse theorem,

$$[K] = [B_1^T B_1]^{-1}[B_1^T][W\Delta^2 + M^{-1}(D+G)W\Delta + M^{-1}(S+H)W][C_1 W\Delta + C_2 W]^{-1},$$

which is the equation describing the gain matrix needed to obtain those eigenvalues desired.

A single objective function was then determined from the set of equations described by equation (7), where the values of [K] were determined by minimizing that objective function. The constraints imposed on the system were that the real part of the eigenvalues of the closed loop system were all negative. These constraints were also modified, as was desired, to increase the stability margin.

NUMERICAL EXAMPLES

Example 1:

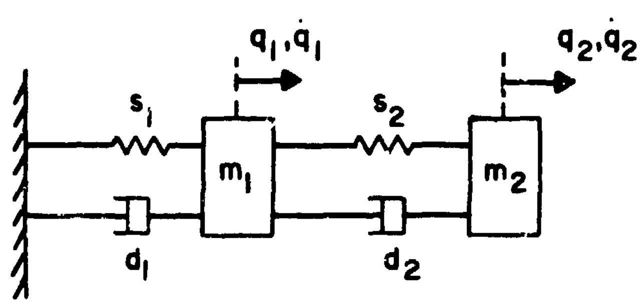

FIGURE 3

Specifications: $m_1 = m_2 = 1$
$S_1 = 4$
$S_2 = 1$
$d_1 = 2$
$d_2 = 1$

Eigenvalues of unforced system:

$$-1.666207 \pm 1.41334i$$
$$-.333783 \pm -.83265i$$

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Desired eigenvalues:

$$\lambda_1 = -4.0 + 0i$$
$$\lambda_2 = -3.0 + 0i$$

Resulting eigenvalues using the proposed method with no additional factor for relative stability:

$$-4.000000 + 0i$$
$$-1.335420 + 0i$$
$$-.249608 + 0i$$
$$-3.000000 + 0i$$

Resulting eigenvalues using the proposed method with added factor of relative stability:

$$-4.000000 + 0i$$
$$-2.999999 + 0i$$
$$-.572543 \pm 0.74353i$$

Resulting eigenvalues using Srinathkumar method:

$$9.1256 + 0i$$
$$-.8141 + 0i$$
$$-4.0000 + 0i$$
$$-3.0000 + 0i$$

Note that the method proposed here yields the desired eigenvalues and that the unspecified eigenvalues remain stable, whereas in the Srinathkumar method an unspecified eigenvalue is moved into the right half plane.

Example 2:

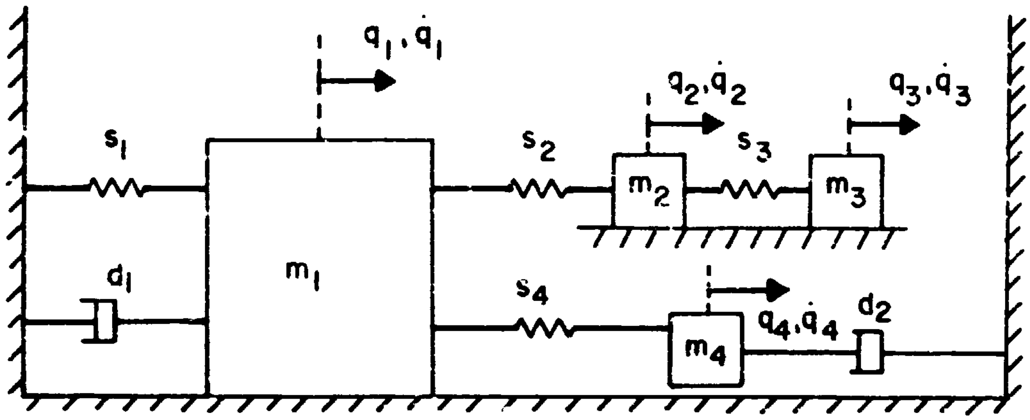

FIGURE 4

Specifications: $m_1 = 4$

$m_2 = m_3 = m_4 = 1$

$S_1 = S_2 = S_3 = S_4 = 1$

$d_1 = d_2 = .5$

Eigenvalues of unforced system:

$-.004055 \pm 1.647953i$
$-.170649 \pm 1.131418i$
$-.062364 \pm .355674i$
$-.075432 \pm .730441i$

$$[C] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Desired eigenvalues: $\lambda_{1,2} = -.4 \pm .5$

Resulting eigenvalues using the proposed method, where a factor for relative stability was added:

$$-.289342 \pm 1.378583i$$
$$-.145451 \pm 1.171345i$$
$$-.400007 \pm .500003i$$
$$-.197840 \pm .425944i$$

Example 3:

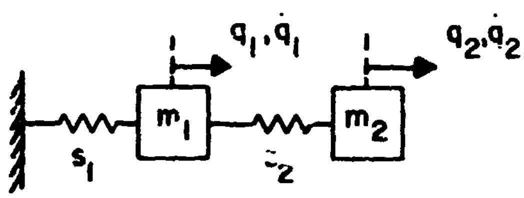

FIGURE 5

Specifications: $m_1 = m_2 = 1$

$S_1 = 3$

$S_2 = 1$

Eigenvalues of unforced system:

$$\pm 2.074313i$$
$$\pm .835000i$$

$$[C] = [1 \quad 1 \quad 0 \quad 0]$$

$$[B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Desired eigenvalue: $.5 + 0i$

Resulting eigenvalues using the proposed method, where factor for relative stability was added:

$$-.170373 \pm 1.809097i$$
$$-1.817157 + 0i$$
$$-.500003 + 0i$$

CONCLUSION

A pole placement algorithm has been proposed which used constrained non-linear programming techniques for a finite dimensional model of a linear n degree of freedom system. It has been shown that by constraining the eigenvalues of the full order system while simultaneously placing those allowable, one can ensure the stability characteristics of a system, and can alter its transient response.

Results of the Srinathkumar method were presented for Example 1, and showed how this method yielded the desired eigenvalues quite accurately, yet unfortunately forced the originally stable system unstable, therefore resulting in an undesirable response.

No previous work has guaranteed stable unplaced poles or has assured the magnitude of relative stability.

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