COMPARISON OF TWO ALGEBRAIC METHODS FOR CURVE/CURVE INTERSECTION

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Introduction

Most geometric modeling systems use either polynomial or rational functions to represent geometry. In such systems most computational problems can be formulated as systems of polynomials in one or more variables. Classical elimination theory can be used to solve such systems (Refs. 1,2,3,4,5). In this paper we summarize Cayley's method of elimination and show how it can best be used to solve the curve/curve intersection problem.

Summary of Elimination Using Cayley's Method

Let $P(x_1,\ldots,x_n)$, $Q(x_1,\ldots,x_n)$ be polynomials in the n variables $x_1,\ldots,x_n$, and assume that both P and Q are of degree $m > 0$ in x_n. If we consider P,Q as polynomials in the one variable x_n , with coefficients which are polynomials in $x_1,\ldots,x_{n-1}$, then we can write

$$P(x_n) = \sum_{k=0}^{m} a_k x_n^k \quad , \quad Q(x_n) = \sum_{k=0}^{m} b_k x_n^k \quad ,$$

where the a_k, b_k are polynomials in $x_1,\ldots,x_{n-1}$.

For $0 \le i,j \le m-1$, $m1=\max(0,i+j-m+1)$, and $m2=\min(i,j)$ define :

$$c_{ij} = \sum_{k=m1}^{m2} a_k b_{i+j-k+1} - a_{i+j-k+1} b_k \, .$$

The matrix $[c_{ij}]$ is called the Cayley (or Bezout) matrix and $\det[c_{ij}]$ is called the resultant of P and Q (denoted by Res(P,Q)).

Theorem : Res(P,Q) is a polynomial in $x_1,\ldots,x_{n-1}$. The following three conditions are equivalent :

1. Res(P,Q) = 0
2. P and Q have a common root
3. The mxm system of linear homogeneous equations given by $[c_{ij}]\,[x_n^j]^T = 0$ has a solution.

Furthermore , there (usually) exist polynomials $F(x_1,\ldots,x_{n-1})$, $G(x_1,\ldots,x_{n-1})$ such that if Res(P,Q) = 0 for some fixed $x_1,\ldots,x_{n-1}$ then the common root of P and Q is given by $x_n = -F/G$. The functions F and G are derived by Cramer's rule from the system $[c_{ij}]\,[x_n^j]^T = 0$.

Example.

$P(x) = (x-1)(x-2) = 2-3x+x^2 = 0$
$Q(x) = (x-1)(x+2) = -2+x+x^2 = 0$.

$$[c_{ij}] = \begin{bmatrix} a_0b_1-a_1b_0 & a_0b_2-a_2b_0 \\ a_0b_2-a_2b_0 & a_1b_2-a_2b_1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$$

Clearly $\det[c_{ij}] = 0$, and from $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = 0$

it follows that $-4+4x = 0$ and thus $x = 1$.

Curve/Curve Intersection Using Elimination

Let $C_1(s) = (x_1(s), y_1(s))$ and $C_2(t) = (x_2(t), y_2(t))$ be two rational parametric curves , both defined on [0,1]. There are two ways to use elimination to find the intersection points of C_1 and C_2.

Method 1. Implicitize one of the curves.

Steps :

a. Use elimination to convert $C_1(s)$ to its implicit representation $P(x,y) = 0$. In the process we obtain $s = -F(x,y)/G(x,y)$ This is an interesting problem in its own right.
b. Substitute $x_2(t)$, $y_2(t)$ into $P(x,y) = 0$ to get $Q(t) = 0$.
c. Find the roots t_i of $Q(t)$ within the range [0,1].
d. For each t_i , use $x_2(t)$, $y_2(t)$ to obtain the corresponding point (x_i, y_i) on C_2.
e. Use (x_i, y_i) and $s = -F/G$ to get s_i . If s_i is in [0,1] , then s_i , t_i , (x_i, y_i) is a solution (intersection point).

Example.

Let $C_1(s) = ((1-s^2)/(1+s^2), 2s/(1+s^2))$ and $C_2(t) = (t, t^2)$. C_1 is a circular , C_2 a parabolic arc , both in the first quadrant.

a. From $x=(1-s^2)/(1+s^2)$ and $y=2s/(1+s^2)$, it follows that
   $f(x,y,s) = (x-1)+(x+1)s^2 = 0$
   $g(x,y,s) = y-2s+ys^2 = 0$. Eliminating s , we obtain :

   $$[c_{ij}] = \begin{bmatrix} -2x+2 & -2y \\ -2y & 2x+2 \end{bmatrix} .$$

   Expanding $\det[c_{ij}]$ and setting it equal to zero yields $P(x,y) = x^2+y^2-1 = 0$. And from $(-2x+2)-2ys = 0$, we get $s = (1-x)/y$.
b. Substituting (t,t^2) into $P(x,y)$ yields $Q(t) = t^4+t^2-1 = 0$.
c. The only root in [0,1] is (approx.) $t = 0.786$.
d. From $C_2(t)$, we have $x = 0.786$, $y = 0.618$.
e. From $s = (1-x)/y$ we obtain $s = 0.346$.

Method 2. Subtract the coordinate functions.

Steps :

a. Form $P(s,t) = x_1(s)-x_2(t) = 0$ and $Q(s,t) = y_1(s)-y_2(t) = 0$.

b. Eliminate one of the variables , say s , to get Res(P,Q) = R(t) = 0 and s = -F(t)/G(t).
c. Find the roots t_i of R(t) in the range [0,1].
d. For each t_i , use s = -F/G to get s_i. If s_i is also in [0,1] , use $(x_2(t),y_2(t))$ to get (x_i,y_i).

Example.
Same as above.
a. $0 = P(s,t) = (1-s^2)/(1+s^2) - t = (1-t)-(1+t)s^2$
$0 = Q(s,t) = 2s/(1+s^2) - t^2 = -t^2+2s-t^2s^2$.

b.
$$R(t) = \begin{bmatrix} 2(1-t) & -2t^2 \\ -2t^2 & 2(1+t) \end{bmatrix} = t^4+t^2-1 = 0 .$$

And $2(1-t)-2t^2s = 0$ implies $s = (1-t)/t^2$.
c. Solving for t yields (approx.) t = 0.786 .
d. Substituting into $s = (1-t)/t^2$ yields s = 0.346 .

Comparison of the Two Methods

To our knowledge only method 1 has been mentioned in the CAD/CAM literature (Refs. 2,3). But method 2 is a more straightforward approach. Furthermore , it is computationally simpler , since the elements of the Cayley matrix are one variable instead of two variable polynomials. We implemented and tested both methods and found method 2 to be more efficient. We used six pairs of curves , representing mixtures of lines , circles , and cubic arcs. Several examples had multiple intersection points. For all six cases method 2 required less CPU time than method 1. The average time ratio of method 1 to method 2 was 3.13:1 , the least difference was 2.33:1 , and the most dramatic was 6.25:1 .

Conclusion

Both of the above methods can be extended to solve the surface/surface intersection problem. That is the direction of our current research.

References

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