An Intuitive Approach to Geometric Continuity for Parametric Curves and Surfaces

(Extended Abstract)†

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ABSTRACT

Parametric spline curves and surfaces are typically constructed so that some number of derivatives match where the curve segments or surface patches abut. If derivatives of up to order n are continuous, the segments or patches are said to meet with C^n, or n^{th} order *parametric continuity*. It has been shown previously that parametric continuity is sufficient, but not necessary, for geometric smoothness.

The geometric measures of *unit tangent* and *curvature* vectors for curves, and *tangent plane* and *Dupin indicatrix* for surfaces, have been used to define first and second order *geometric continuity*. In this work, we extend the notion of geometric continuity to arbitrary order n (G^n) for curves and surfaces, and present an intuitive development of constraints equations that are necessary and sufficient for it. The constraints result from a direct application of the univariate chain rule for curves, and the bivariate chain rule for surfaces. The constraints provide for the introduction of quantities known as *shape parameters*.

The approach we take is important for several reasons: First, it generalizes geometric continuity to arbitrary order for both curves and surfaces. Second, it shows the fundamental connection between geometric continuity of curves and geometric continuity of surfaces. Third, due to the chain rule derivation, constraints of any order can be determined more easily than derivations based exclusively on geometric measures.

1. Introduction

In recent years, computer-aided geometric design (CAGD) has relied heavily on mathematical descriptions of objects based on *parametric splines*. Spline curves are typically constructed by stitching together univariate parametric functions, requiring that some number of derivatives match at each *joint* (the points where the curve segments meet). If n derivatives agree at a given joint, the parametrizations there are said

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to meet with n^{th} *order parametric continuity* (C^n continuity for short). It has been previously demonstrated (see [2,7,8,11] for instance) that parametric continuity can be overly restrictive for many applications. To remedy this situation, another notion of continuity must be developed, one based on the geometry of the resulting curve or surface. We shall refer to this as *geometric continuity*.

It has recently come to our attention that many authors have independently defined this kind of continuity of first and second order (which we denote by G^1 and G^2, respectively) for curves and/or surfaces using geometric means. For curves, Fowler & Wilson[9], Sabin[13], Manning[11], Faux & Pratt[8], and Barsky[1] each independently defined first order continuity by requiring that the *unit tangent vectors* agree at the joints. To achieve second order continuity, both the unit tangent and *curvature* vectors were required to match. Nielson's ν-spline[12] possesses a similar kind of continuity. For surfaces, it is common to require matching of *tangent planes* for first order continuity (cf. Sabin[14] and Veron et al[15]). For surfaces of second order continuity, Veron et al and Kahmann[10] require continuity of *normal curvature* in every direction, at every point on the boundary shared by the constituent surface patches. As Veron et al and Kahmann each show, this is equivalent to requiring that the *Dupin indicatrix* of each patch agree at the boundary curve.

Although the geometric approaches described above are convenient and intuitive for first and second order continuity, a more algebraic development is better suited to the extension to continuity of higher order. The approach we take is based on the following simple idea:

> *P1: Don't base continuity on the parametrizations at hand; reparametrize, if necessary, to obtain parametrizations that meet with parametric continuity. If this can be done, the original parametrizations must also meet smoothly, at least in a geometric sense.*

The above concept is not a new one; similar principles have been discussed by Farin[7] and Veron et al[15]. What is new is the use of the principle to construct constraint equations (to be known as the *Beta constraints*) that are necessary and sufficient for geometric continuity of arbitrary order for both curves and surfaces. ‡

In this paper, we extend the notion of geometric continuity to arbitrary order n (G^n) and show (in a nonrigorous way) that the derivation of the Beta constraints results from a straightforward use of the univariate chain rule for curves and the bivariate chain rule for surfaces. This approach therefore provides new insight into the nature of geometric continuity and shows that geometric continuity of curves and surfaces need not be treated separately; the same basic principle of reparametrization applies to both. We also argue that, for first and second order continuity, the Beta constraints are equivalent to the geometric measures described above. However, due to chain rule derivation, the constraints are obtained with less effort using our method. For a more complete treatment, the reader is referred to Barsky & DeRose[3,5]

2. Geometric Continuity for Curves

We begin the study of geometric continuity for curves by examining the reparametrization process. Two parametrizations are said to be *GO-equivalent* if they have the same *geometry* (shape) and *orientation* (direction of tangent vector) at each point. Given a particular parametrization, all GO-equivalent parametrizations may be obtained by *functional composition*. More specifically, if $\mathbf{q}(u)$ and $\tilde{\mathbf{q}}(\tilde{u})$ are GO-equivalent, then they are related by $\tilde{\mathbf{q}}(\tilde{u}) = \mathbf{q}(u(\tilde{u}))$, for some appropriately chosen *change of parameter* $u(\tilde{u})$. Since $\mathbf{q}$ and $\tilde{\mathbf{q}}$ must have the same orientation, u must be an increasing function of $\tilde{u}$, implying that u must satisfy the *orientation preserving condition* $u^{(1)} > 0$ (in general, superscript (i) denotes the i^{th} derivative). A univariate parametrization is *regular* if the first derivative vector does not vanish. It is well known from differential geometry[6] that regularity is, in general, essential for the smoothness of the resulting curve. We will therefore restrict the discussion to regular parametrizations. We now give a more precise definition of G^n continuity:

‡ T. Goodman and L. Ramshaw have independently derived the univariate Beta constraints from the univariate chain rule.

Definition 1: *Let $\mathbf{r}(t), t \in [t_0, t_1]$ and $\mathbf{q}(u), u \in [u_0, u_1]$ be two parametrizations such that $\mathbf{r}(t_1) = \mathbf{q}(u_0)$ (see Figure 1). These parametrizations meet with G^n continuity at J if and only if there exist GO-equivalent parametrizations $\tilde{\mathbf{r}}(\tilde{t})$ and $\tilde{\mathbf{q}}(\tilde{u})$ that meet with C^n continuity.*

Definition 1 is simply a restatement of principle P1, but in practice one cannot examine all GO-equivalent parametrizations in an effort to find two that meet with parametric continuity. However, it is possible to find conditions on $\mathbf{r}$ and $\mathbf{q}$ that are necessary and sufficient for the *existence* of GO-equivalent parametrizations that meet with parametric continuity.

Due to the compositional structure of equivalent parametrizations, the derivation of the Beta constraints essentially reduces to an application of the chain rule. In particular, the chain rule is used to express derivatives of $\tilde{\mathbf{q}}$ in terms of derivatives of $\mathbf{q}$. For example, the Beta constraints for G^4 continuity for the situation shown in Figure 1 are:

$$\begin{aligned}
\mathbf{r}^{(1)}(t_1) &= \beta_1\,\mathbf{q}^{(1)}(u_0)\\
\mathbf{r}^{(2)}(t_1) &= \beta_1^2\,\mathbf{q}^{(2)}(u_0) + \beta_2\,\mathbf{q}^{(1)}(u_0)\\
\mathbf{r}^{(3)}(t_1) &= \beta_1^3\,\mathbf{q}^{(3)}(u_0) + 3\beta_1\beta_2\,\mathbf{q}^{(2)}(u_0) + \beta_3\,\mathbf{q}^{(1)}(u_0)\\
\mathbf{r}^{(4)}(t_1) &= \beta_1^4\,\mathbf{q}^{(4)}(u_0) + 6\beta_1^2\beta_2\,\mathbf{q}^{(3)}(u_0) + (4\beta_1\beta_3 + 3\beta_2^2)\,\mathbf{q}^{(2)}(u_0) + \beta_4\,\mathbf{q}^{(1)}(u_0).
\end{aligned} \tag{2.1}$$

where $\beta_1 > 0, \beta_2, \beta_3, \beta_4$ are arbitrary real numbers.

The right side of the first equation of (2.1) represents the derivative of $\tilde{\mathbf{q}}$ written in terms of the derivative of $\mathbf{q}$, where the substitution $u^{(1)} = \beta_1$ has been made. Thus, β_1 determines, to first order, the function $u(\tilde{u})$ in the neighborhood of the joint. Similarly, the right side of the second equation of (2.1) represents the second derivative of $\tilde{\mathbf{q}}$ expressed in terms of derivatives of $\mathbf{q}$, where $u^{(2)}$ has been given the value β_2 at the joint. This process is continued for the higher order constraints.

Although the first two equations of (2.1) were derived using the chain rule, they are identical to the constraints resulting from a geometric derivation of unit tangent and curvature vector continuity, respectively[2,11]. Thus, our approach reduces to previous definitions of G^1 and G^2 continuity.

In general, a property of the chain rule can be used to easily show that for n^{th} order geometric continuity for curves, n *shape parameters* $\beta_1, \ldots, \beta_n$ are introduced[3]. The quantities $\beta_1, \ldots, \beta_n$ are called shape parameters because they can be made available to a designer in a CAGD environment to change the shape of curves and surfaces. Since geometric continuity provides for the introduction of shape parameters, it may be desirable to generalize existing spline techniques to obtain their geometrically continuous analogues. For instance, the Beta-spline[1,2] is an *approximating* technique that possesses shape parameters; an *interpolating* technique is described in DeRose & Barsky[4]. Faux & Pratt[6] and Farin[7] use the extra freedom allowed by geometric continuity to place *Bézier control vertices*.

3. Geometric Continuity for Surfaces

A parametric surface patch is defined by a bivariate function such as $\mathbf{G}(u,v) = (x(u,v), y(u,v), z(u,v))$, where u and v are allowed to range over some region of the uv plane. Such a parametrization is *regular* if the first order partial derivatives are linearly independent. N^{th} order parametric continuity of two surface patches requires that all like partial derivatives of order up to n agree for each point of the boundary curve. Superscript (i,j) will be used to denote the i^{th} partial with respect to the first variable, and the j^{th} partial with respect to the second. Just as for curves, parametric continuity is sufficient for geometric smoothness, but can be overly restrictive.

The notion of reparametrization as a basis for the determination of continuity can readily be extended to surfaces by making a definition analogous to Definition 1. The *bivariate chain rule* can then be used to determine constraint equations. However, instead of shape parameters being introduced, a property of the bivariate chain rule shows that $n(n+3)$ *shape functions* are introduced when two surface patches are stitched together with G^n continuity. Referring to the situation of Figure 2 and using the bivariate chain rule, one can show that $\mathbf{F}(u,v)$ and $\mathbf{G}(s,t)$ meet with G^1 continuity if and only if

$$\begin{aligned} \mathbf{F}^{(0,1)} &= \beta_{00}\,\mathbf{G}^{(0,1)} + \beta_{01}\,\mathbf{G}^{(1,0)} \\ \mathbf{F}^{(1,0)} &= \beta_{10}\,\mathbf{G}^{(0,1)} + \beta_{11}\,\mathbf{G}^{(1,0)} \end{aligned} \tag{3.1}$$

holds for each point $\mathbf{P}$ of γ where the ***shape functions*** $\beta_{00}, \beta_{01}, \beta_{10}, \beta_{11}$ satisfy the orientation preserving condition $\beta_{00}\beta_{11} - \beta_{10}\beta_{01} > 0$. The shape functions determine how derivatives of $\mathbf{G}$ are related to derivatives of a GO-equivalent parametrization $(\tilde{\mathbf{G}})$ that meets $\mathbf{F}$ with first order parametric continuity. Although equations (3.1) were derived from the bivariate chain rule, they also have geometric significance. More specifically, they are necessary and sufficient conditions for ***tangent plane continuity*** between $\mathbf{F}$ and $\mathbf{G}$. Thus, the abstract algebraic approach of reparametrization and the chain rule agrees with geometric intuition for first order continuity of surfaces.

It can be shown that the constraints resulting from the chain rule approach are equivalent to requiring that the Dupin indicatrix of the patches match along the boundary curve. Thus, the chain rule approach agrees with geometric intuition for both G^1 and G^2 continuity. Moreover, the chain rule approach yields the second order constraints with less effort than the geometric approach. For higher order continuity, geometric intuition becomes more feeble, but the chain rule approach still applies.

4. Conclusion

We have defined n^{th} order geometric continuity for parametric curves and surfaces, and derived the Beta constraints that are necessary and sufficient for it. The derivation of the Beta constraints is based on a simple principle of reparametrization in conjunction with the univariate chain rule for curves, and the bivariate chain rule for surfaces. This approach therefore uncovers the connection between geometric continuity for curves and geometric continuity for surfaces, provides new insight into the nature of geometric continuity in general, and allows the determination of the Beta constraints with less effort than previously required.

The use of the Beta constraints allows the introduction of n shape parameters for curves, and $n(n+3)$ shape functions for surfaces. The shape parameters and shape functions may be used to modify the shape of a geometrically continuous curve or surface, respectively. However, geometric continuity is only appropriate for applications where the particular parametrization used is unimportant since parametric discontinuities are allowed.

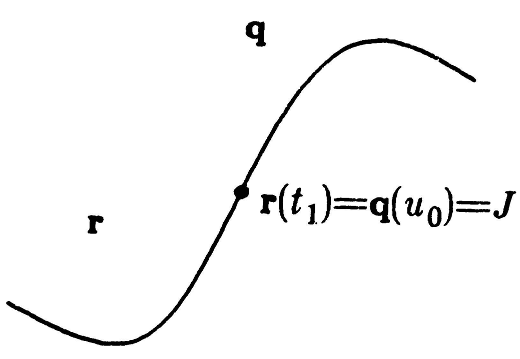

Figure 1: *The parametrizations* $\mathbf{r}$ *and* $\mathbf{q}$ *meet at the common point* $\mathbf{r}(t_1) = \mathbf{q}(u_0)$.

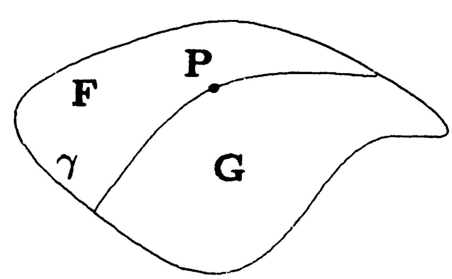

Figure 2: *The surface patches created by* $\mathbf{F}$ *and* $\mathbf{G}$ *meet at the boundary curve* γ.

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