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Adapting Shape Parameters for Cubic Bézier Curves

David Isacoff
and
Michael J. Bailey
CADLAB, Purdue University
134 Potter Engineering Center
West Lafayette, IN 47907

Bézier curves are a popular and well established tool in Computer Aided Geometric Design. One of the major drawbacks of the Bézier method, however, is that the curves often bear little resemblance to their control polygons. As a result, it becomes increasingly difficult to obtain anything but a rough outline of the desired shape. One possible solution would be to manipulate the curve itself instead of the control polygon. The following paper introduces into the standard cubic Bézier curve form two shape parameters, γ_1 and γ_2. These parameters give the user the ability to manipulate the curve while the control polygon retains its original form, thereby providing a more intuitive feel for the necessary changes to the curve in order to achieve the desired shape.

A 4^{th} order (3^{rd} degree) Bézier curve [BEZI72] is defined as

$$\mathbf{C}(t) = \sum_{i=0}^{3} p_i \phi_i(t) \qquad [1]$$

for $0 \leq t \leq 1$
where

$$\phi_i(t) = \binom{3}{i} t^i (1-t)^{3-i}$$

and the p_i's are vertices of the Bézier control polygon. Expanding [1] gives:

$$\mathbf{C}(t) = \phi_0(t)p_0 + \phi_1(t)p_1 + \phi_2(t)p_2 + \phi_3(t)p_3 \qquad [2]$$

In matrix form, [CLAR81] the cubic Bézier becomes:

$$C(t) = [t^3\ t^2\ t\ 1]\begin{bmatrix}-1 & 3 & -3 & 1\\ 3 & -6 & 3 & 0\\ -3 & 3 & 0 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}p_0\\ p_1\\ p_2\\ p_3\end{bmatrix} \quad [3]$$

$$= \mathbf{T\,M\,P}$$

where

T is the primitive polynomial basis

M is the basis change matrix and

P is the geometry vector

Two important properties of the Bézier curve are

$$\sum_i \phi_i(t) = 1 \qquad 0 \leq t \leq 1 \quad [4]$$

$$\phi_i(t) > 0 \qquad 0 \leq t \leq 1 \quad [5]$$

The first property is necessary for translational and rotational invariance. This means the curve will be independent of the choice of the coordinate system. In addition to the first property, the second property is necessary for the curve to lie within the convex hull generated by the vertices of its control polygon.

Evaluating **C**(t) at its endpoints (t = 0, t = 1) we get

$$\mathbf{C}(0) = p_0 \quad [6]$$

$$\mathbf{C}(1) = p_3$$

In a similar manner when we evaluate $\mathbf{C}'(t)$ at its endpoints (t = 0, t = 1) we get

$$\mathbf{C}'(0) = 3(p_1 - p_0) \quad [7]$$

$$\mathbf{C}'(1) = 3(p_3 - p_2)$$

By examining $\mathbf{C}'(0)$ and $\mathbf{C}'(1)$ we can see that the cubic Bézier representation expresses the tangents at the endpoints in terms of difference vectors between two geometric points multiplied by a constant. Therefore equation [7] can be rewritten as

$$\mathbf{C}'(0) = \gamma(p_1 - p_0) \quad [8]$$

$$C'(1) = \gamma(p_3 - p_2)$$

where $\gamma = 3$ in the case of the cubic Bézier curve.

Now suppose we allow a different γ for $\mathbf{C}'(0)$ and $\mathbf{C}'(1)$. For example

$$C'(0) = \gamma_1(p_1 - p_0) \qquad [9]$$

$$C'(1) = \gamma_2(p_3 - p_2)$$

How does this new change affect our original $\mathbf{C}(t)$? What are the new $\phi_i(t)$ needed to maintain the two previously stated properties?

The effect on $\mathbf{C}(t)$ can be observed by looking at the new basis change matrix $\mathbf{M}_\gamma$. Substituting into equation [3] we get

$$C(t) = \mathbf{T}\,\mathbf{M}_\gamma\,\mathbf{P}$$

where

$$\mathbf{M}_\gamma = \begin{bmatrix} 2-\gamma_1 & \gamma_1 & -\gamma_2 & \gamma_2-2 \\ 2\gamma_1-3 & -2\gamma_1 & \gamma_2 & 3-\gamma_2 \\ -\gamma_1 & \gamma_1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Expanding yields

$$C(t) = \psi_0(t)p_0 + \psi_1(t)p_1 + \psi_2(t)p_2 + \psi_3(t)p_3$$

where

$$\psi_0(t) = (1-t)^2(1 + 2t - \gamma_1 t)$$

$$\psi_1(t) = \gamma_1(t-1)^2 t$$

$$\psi_2(t) = \gamma_2(1-t)t^2$$

$$\psi_3(t) = t^2(\gamma_2 t - 2t - \gamma_2 + 3)$$

Note that $\sum_i \psi_i(t) = 1 \qquad 0 \le t \le 1$

and $\qquad \psi_i(t) > 0 \qquad 0 \le t \le 1, \text{ if } 0 < \gamma_1, \gamma_2 \le 3$

The restrictions of γ_1 and γ_2 are necessary in order to maintain the convex hull property. If $\gamma_1, \gamma_2 > 3$ convexity may still be maintained, but is not guaranteed as it is with the standard Bézier curves (Figure 1). The reason the convex hull property is desirable is that it gives the designer an initial feel for the shape of the curve. Once the rough outline has been drawn, however, this criterion may be temporarily sacrificed. In Figure 2 we can see the rough outline of an airfoil. Figure 3 demonstrates a typical refinement process and Figure 4 shows the completed design. After the final design is reached, or at any intermediate step, it is possible to generate the new Bézier control polygon for the existing curve, thus producing a natural method of step-wise refinement.

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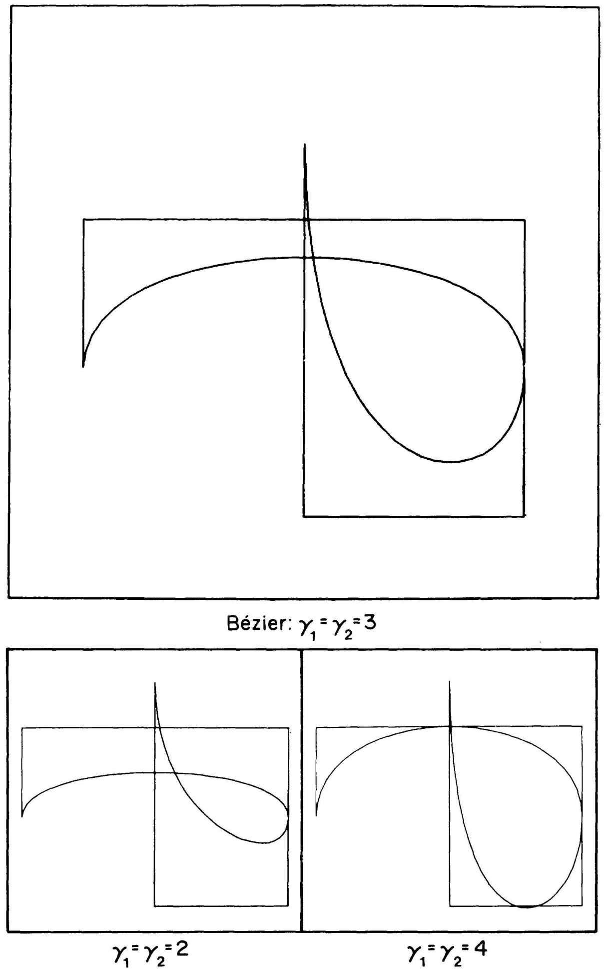

Figure 1

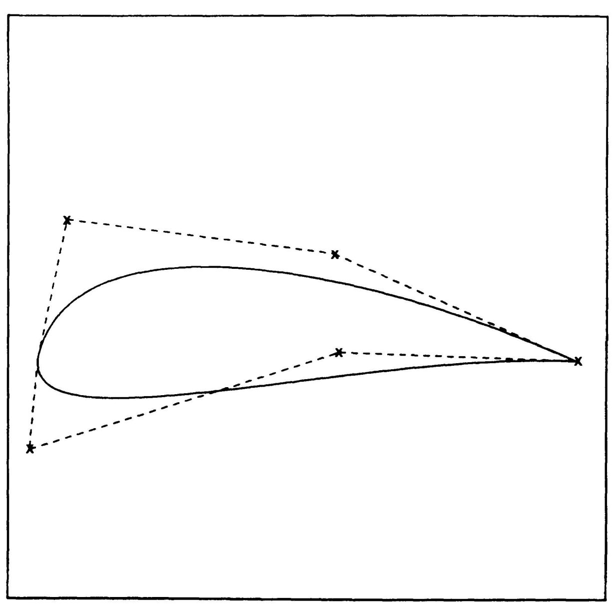

Figure 2: Rough Outline Using Standard Bézier Curves

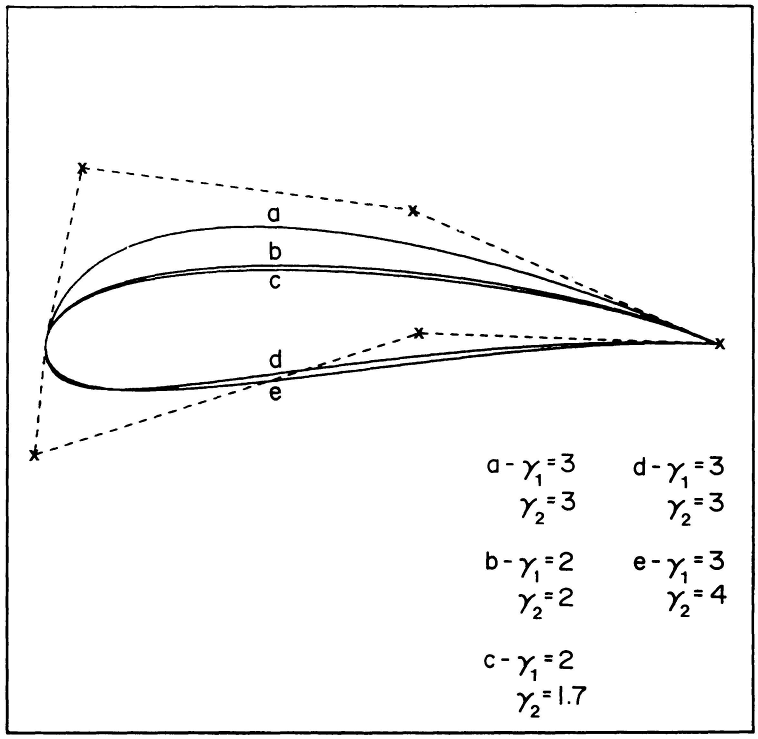

Figure 3: Refinement Process

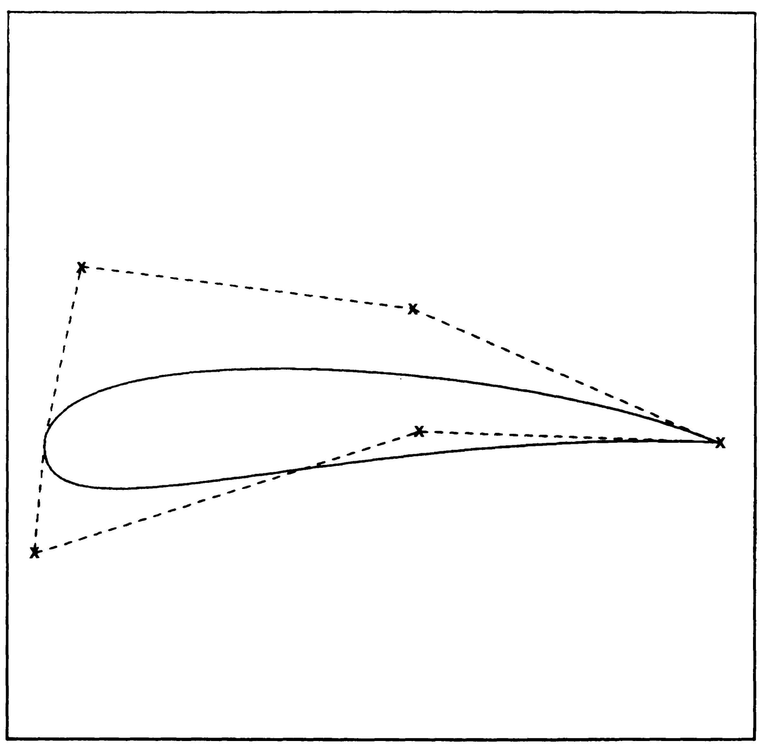

Figure 4: Completed Design with Original Control Polygon