Convex Interpolating Splines of Arbitrary Degree

Abstract

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1. In curve fitting problems that arise in science or engineering we often demand that our approximating function be shape preserving in the sense that the approximation is convex when the data are convex or that the approximation is monotone and convex when the data are monotone and convex. In this paper we construct such shape preserving approximations by interpolating the data with polynomial splines of arbitrary degree. We formulate a regularity condition on the data which insures the existence of such a shape preserving spline, we present an algorithm for its construction, and we bound the uniform norm of the error which results when the algorithm is used to produce an approximation to a given $f \in C[a,b]$.

2. Let $\Delta_n : a = x_0 < x_1 < \ldots < x_n = b$ denote an arbitrary but fixed partition of the interval $[a,b]$ with knots x_i, let $h_i = x_{i+1} - x_i$, $i = 0,1,\ldots,n-1$, and let $h = \max h_i$. Let $Sp(k, \ell, \Delta_n)$ denote the space of polynomial splines of degree k and deficiency $k - \ell$ and assume that $k = 3,4,\ldots$ and $\ell = 1,\ldots,[(k - 1)/2]$.

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Given real data $(x_0,f_0), \ldots, (x_n,f_n)$ we seek $s \in Sp(k, \ell, \Delta_n)$ such that

(P) $\quad s(x_i) = f_i, \quad i = 0,1,\ldots,n$ and s is convex on $[a,b]$.

The existence of a solution to (P) depends on the values of

$$\sigma_i = (f_{i+1} - f_i)/h_i, \quad i = 0,1,\ldots,n-1.$$

To construct s we set

(1) $\quad s(x) = s_i(x), \quad x_i \le x \le x_{i+1}, \quad i = 0,1,\ldots,n-1$

where

(2) $$s_i(x) = f_i\phi_0(t) + f_{i+1}\phi_1(t) + h_i[p_i\Psi_0(t) - p_{i+1}\Psi_1(t)], \qquad t = (x - x_i)/h_i,$$

where $p_0,p_1,\ldots,p_n$ are parameters to be determined later and

$$\phi_0(t) = \int_t^1 M_k^m(s)ds, \qquad \phi_1(t) = \int_0^t M_k^m(s)ds,$$

$$\Psi_0(t) = t\int_t^1 M_{k-1}^m(s)ds, \qquad \Psi_1(t) = (1 - t)\int_0^t M_{k-1}^{m-1}(s)ds,$$

where M_i^j denote the B-spline of degree $i - 1$ with knots $0 = t_0 = \ldots = t_j < t_{j+1} = \ldots = t_i = 1$. Clearly $s \in Sp(k, \ell, \Delta_n)$ with $\ell = \min\{k-m-1, m\} \le [(k - 1)/2]$.

<u>Theorem 1</u>. The spline function s given by (1) - (2) solves the problem (P) if and only if

(3) $$\frac{(k - m)p_i + m\, p_{i+1}}{k} \le \sigma_i \le \frac{(k-m-1)p_i + (m + 1)p_{i+1}}{k}$$

for $i = 0,1,\ldots,n-1$. If in addition $p_0 \ge 0$, then s is also nondecreasing on $[a,b]$.

3. The parameters p_i which appear in (3) can be constructed as follows. We initially define

$$\ell_i = \frac{k\sigma_i - (m+1)\sigma_{i+1}}{k-m-1}, \quad i = 0,\ldots,n-2, \quad \ell_{n-1} = \ell_n = -\infty,$$

$$u_0 = -\infty, \quad u_i = \frac{k\sigma_{i-1} - (k-m-1)p_{i-1}}{m+1}, \quad i = 1,\ldots,n,$$

$$v_0 = \infty, \quad v_i = \frac{k\sigma_{i-1} - (k-m)p_{i-1}}{m}, \quad i = 1,\ldots,n,$$

and then for each $i = 0,1,\ldots,n$ set $c_i = \max\{u_i, \ell_i\}$, $d_i = \min\{v_i, \sigma_i\}$.

Theorem 2. Let $\sigma_1 \geq \sigma_0$ and let $\sigma_n = +\infty$. If

$$(m+1)(\sigma_{i+1} - \sigma_i) \geq (k-m-1)(\sigma_i - \sigma_{i-1}), \quad i = 1,\ldots,n-2,$$

then $c_i \leq d_i$, $i = 0,\ldots,n$ and any choice of $p_i \in [c_i, d_i]$, $i = 0,\ldots,n$ makes (1) - (2) a solution of (P).

4. The error which results when the above procedure is used to construct a polynomial spline approximation to a continuous function f may be bounded as follows:

Theorem 3. Let $f \in C[a,b]$ be nondecreasing and convex on $[a,b]$, and let s be the spline interpolant produced by using (1) - (3) with $f_i = f(x_i)$, $i = 0,\ldots,n$ and with $p_0 \geq 0$. Then

$$\|f - s\|_\infty \leq \gamma_{k,m}\,\omega(f;h)$$

where $\omega(f;\cdot)$ is the modulus of continuity of f and

$$\gamma_{k,m} = 1 + A\cdot k/m \quad \text{and} \quad \frac{1}{2} - \binom{k-1}{m}2^{-k} \leq A \leq 1.$$

Moreover, for $k = 2q - 1$ and $m = q - 1$ $(q > 1)$

$$\frac{7}{4} \leq \gamma_{3,1} < \gamma_{5,2} < \cdots \leq 2.$$