https://ntrs.nasa.gov/search.jsp?R=19850020254 2020-03-22T19:38:25+00:00Z

MULTIVARIATE SPLINE ALGORITHMS FOR CAGD

Wolfgang Boehm
Angewandte Geometrie und GDV
T. U. Braunschweig

ABSTRACT

The theory of multivariate splines was developed in the early 1980s, and since then has generated increasing interest. Researchers in computer-aided geometric design (CAGD) hoped to get a new, useful tool for the representation and handling of surfaces. This interest in multivariate splines is based on three fruitful ideas:

Schoenberg's geometric definition of splines (Schoenberg, 1966)

Geometric recursion and subdivision

The Bernstein-Bézier representation

Multivariate splines are easily developed from the geometric definition of splines, given by Schoenberg in 1966: A polyhedron P that spans R^m is affinely mapped into R^s, $s \leq m$. All points $p \in P$ that are mapped onto a single point $x \in R^s$ from an m-s dimensional polyhedron Q(x). The volume V(x) of Q is called a spline function. It is a piecewise polynomial of order m-s and "smooth". Note that one can easily find different polyhedra defining the same spline.

The image of a vertex of P will be called a knot, while the image of an edge of P will be called a knot line, connecting two knots. These knot lines correspond to the segment boundaries.

Two special polyhedra present themselves for the definition of B-splines: a simplex S and a box or parallelepiped B, where the edges of S project into an irregular grid, while the edges of B project into the edges of a regular grid, as shown in figure 1. More general splines may be found by forming linear combinations of these B-splines, where the three-dimensional coefficients are called the spline "control points".

Note that univariate splines are simplex splines, where s = 1, whereas splines over a regular triangular grid are box splines, where s = 2.

Two simple facts render the development of the construction of B-splines:

Any "face" of a simplex or a box is again a simplex or box but of lower dimension.

Any simplex or box can be easily subdivided into smaller simplices or boxes.

The first fact gives a geometric approach to Mansfield-like recursion formulas that express a B-spline in B-splines of lower order, where the coefficients depend on x. By repeated recursion, the B-spline will be expressed as B-splines of order 1; i.e. piecewise constants. Considering the corresponding "nets" of control points, one gets de Boor - like algorithms for the calculation of a given linear combination of B-splines at x.

In the case of a simplex spline, the second fact gives a so-called "insertion algorithm" that constructs the new control points if an additional knot is inserted. In the case of a box spline, this fact gives the so called "subdivision" algorithm, which constructs a "refinement" of the control net (see figure 2).

If more than one vertex of a simplex is mapped into one knot, the corresponding spline function will be degenerate. In particular, if the knots form a simplex of R^s, the spline will be a "truncated Bernstein polynomial". In this case the Mansfield-like recursion formula for simplex splines degenerates to the well-known recursion formula for Bernstein polynomials, while the de Boor - like algorithm degenerates to the algorithm of de Casteljau, where the control points are called Bézier points. Clearly, the Bernstein expansion of a B-spline corresponds to a suitable simplicial decomposition of the simplex or box, such that the Bernstein representation of a B-spline can easily be constructed (see figure 3). Furthermore, especially if the B-spline is a truncated Bernstein polynomial, one gets the subdivision algorithm for Bernstein polynomials as well as a refinement of the Bézier net.

Fortunately, two of the above algorithms seem to fall in the class of non-tensor-product box splines:

The approximation of the spline by a repeated refinement for global representation

The construction of the Bézier net for application of the full Bernstein-Bézier method for local representation

It should be mentioned that both algorithms start from the control net and use - in the known cases - repeated "filling and/or averaging" procedures, as shown in figure 4 for the case of a triangular grid:

A rhombic scheme is filled with data from the previous step,

A new scheme is formed by line averaging these data.

REFERENCES

Boehm, W. (1983): Subdividing Multivariate Splines, CAD 15, 345-352.

Boehm, W. (1984): Calculating with Box Splines, CAGD 1, 149-162.

Dahmen, W., and Micchellli, C. (1983): Multivariate Splines - A New Constructive Approach. R. Barnhill and W. Boehm, Surfaces in CAGD, edited by North-Holland, Amsterdam.

Dahmen, W., and Micchelli, C. (1984): Subdivision Algorithms for the Generation of Box Spline Surfaces, CAGD 1, 115-129.

Prautzsch, H. (1984): Unterteilungsalgorithmen fuer multivariate Splines - ein geometrischer Zugang. Ph.D. Thesis, Technische Universitaet Braunschweig.

Prautzsch, H. (1985): Generalized Subdivisioin and Convergence, CAGD, to appear.

Schoenberg, I. J. (1966): On Spline Functions. University of Wisconsin MRC-TSR-625, Madison.

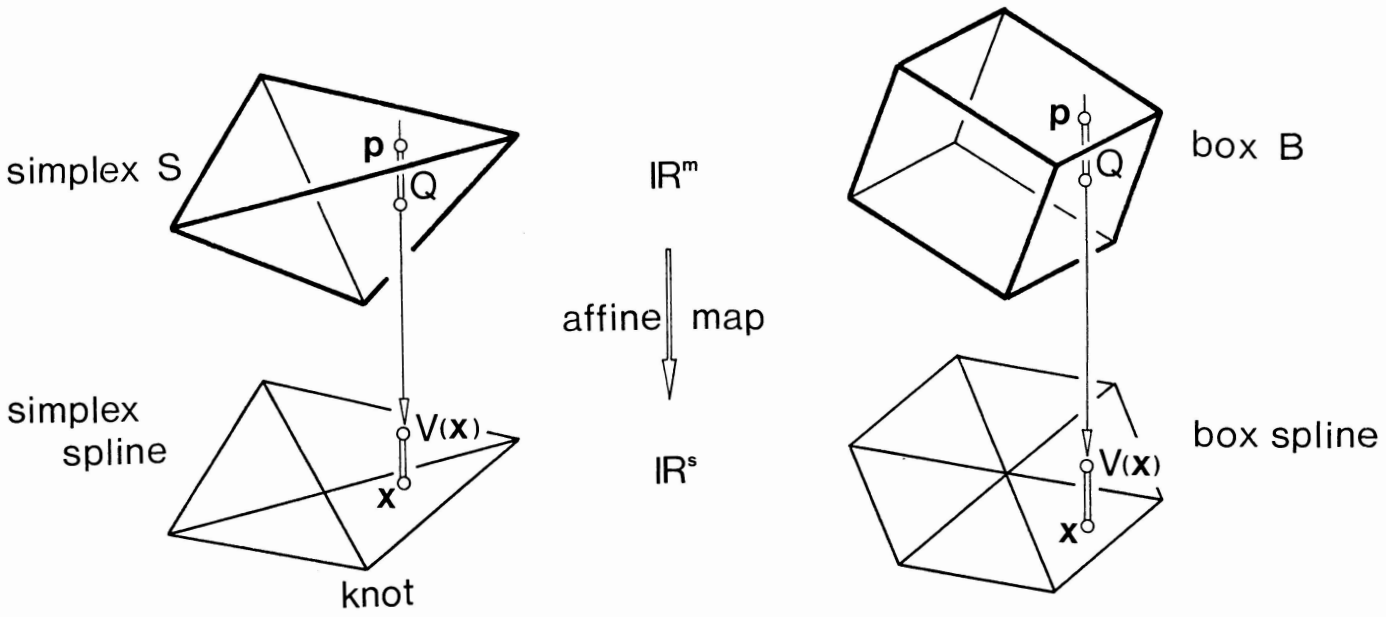

FIGURE 1

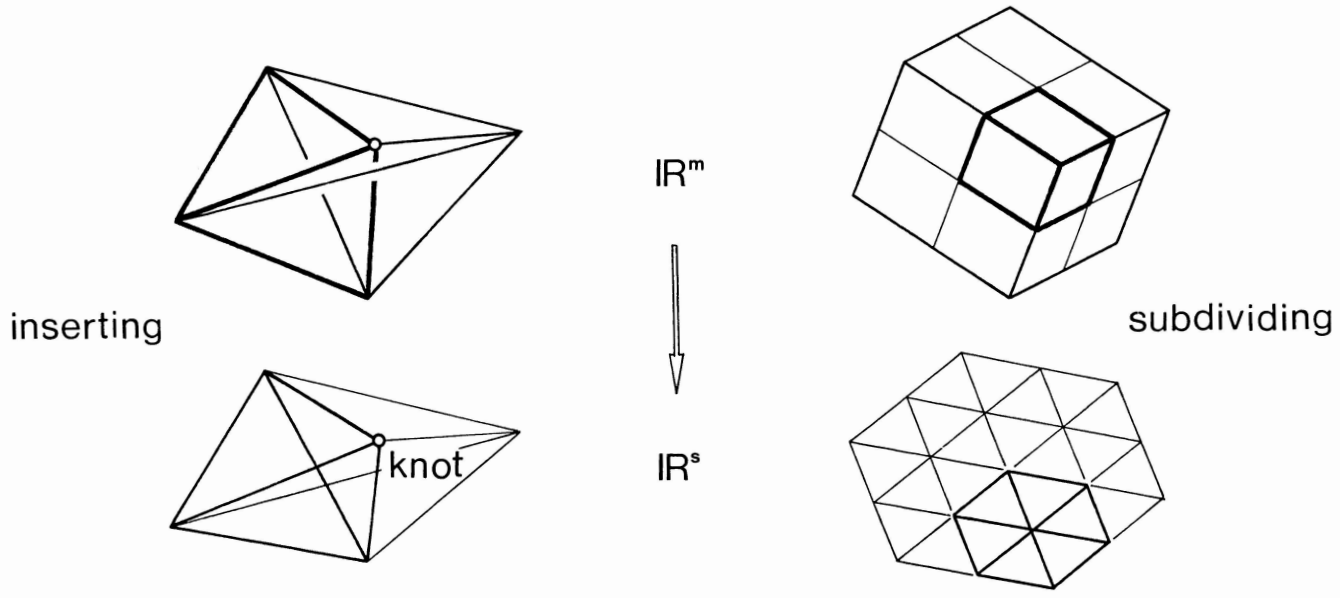

FIGURE 2

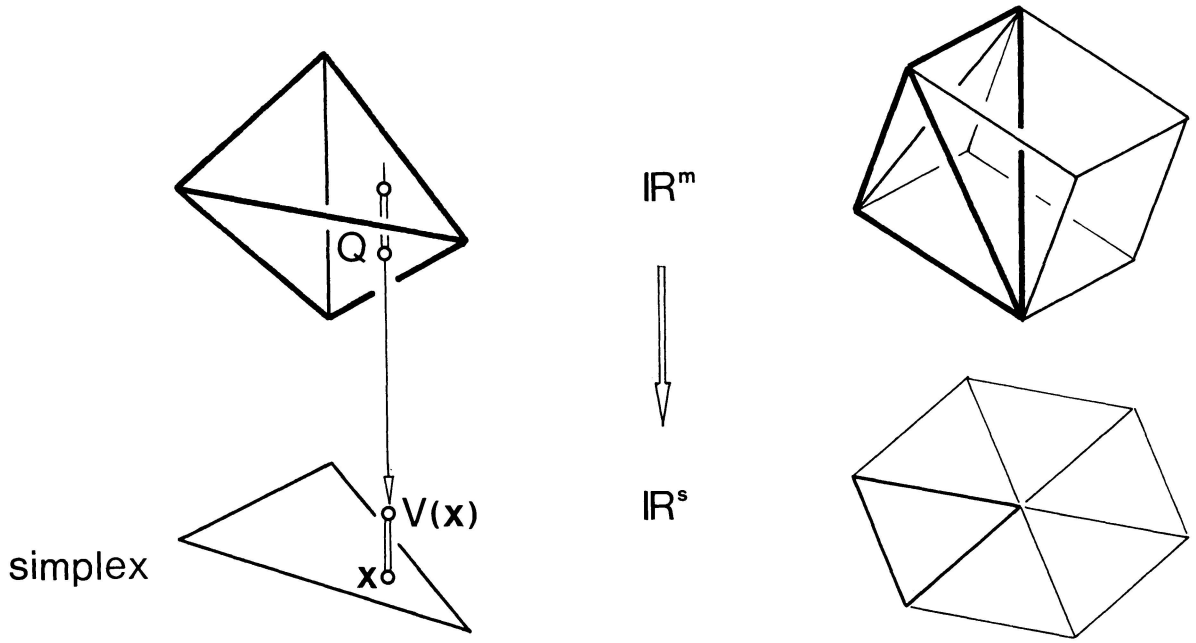

FIGURE 3

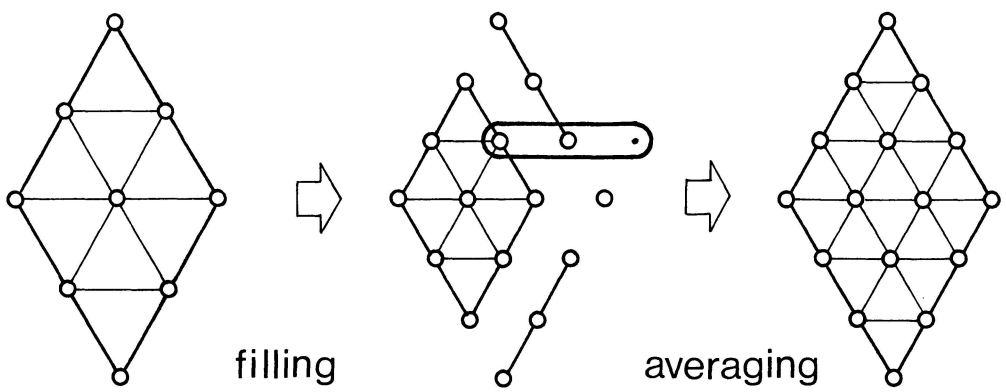

FIGURE 4