THE CONSTRUCTION OF CURVES AND SURFACES USING NUMERICAL OPTIMIZATION TECHNIQUES

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INTRODUCTION:

Numerical optimization techniques are playing an increasing role in curve and surface construction. Often difficult problems in curve and surface construction, especially when some aspect of shape control is involved, can be phrased as a constrained optimization problem. Recently, efficient and robust optimization codes that allow both linear and non-linear constraints to be imposed on construction problems have become available (e.g., (1)). These codes enable a variety of such problems to be solved.

In this paper we explore four such classes of problems: parametric curve fitting with non-linear shape constraints; explicit surface fitting with linear shape constraints; surface fitting to scattered data giving rise to ill-posed problems; and, finally, variable knot problems. In each of these problems there is a non-linear aspect: either the shape of the curve or surface is important for manufacturing or engineering reasons or the shape affects the convergence of numerical algorithms which use the curve or surface or the placement of knots affects the accuracy of the fits.

In all cases the class of functions used is that of parametric spline curves and tensor or direct product spline surfaces. The reason for choosing this class is that splines provide flexible models that are easily evaluated and stored. Furthermore, the B-spline representation of splines leads to convenient expressions for shape control over regions.

We look first at the problem of constructing a curve satisfying shape constraints.

CURVE CONSTRUCTION WITH SHAPE CONSTRAINTS:

We begin our discussion with a quick review of shape for functions and curves including a critique of current methods. We then discuss the use of optimization methods to construct shape controlled curves.

In (2) the question of shape for parametric curves is studied in detail. There it is shown that the problem of constructing parametric curves $C(t) = (x(t),y(t))$ for which $C(t_i) = P_i$ for given planar points P_i and in addition requiring that the curve satisfy the inequality

(1) $x''(t)y'(t)-x'(t)y''(t) \geq 0$

for all t gives rise to optimization problems with non-linear constraints.

The first thing to be done is to reduce the continuous constraints defined by (1) to a finite set of discrete constraints. This is most conveniently done in the manner of (1) by settling on cubic splines as candidates for $x(t)$ and $y(t)$. By representing the spline curve as a B-spline series

(2) $C(t) = \sum Q_i B_i(t)$

the constraints (1) may be replaced by discrete constraints involving the B-spline coefficients Q_i. We describe this procedure and apply it to several examples.

The next two problems involve the construction of explicit tensor or direct product spline surfaces using optimization methods.

AN EXPLICIT SURFACE WITH LINEAR SHAPE CONSTRAINTS:

The previous example dealt with the problem of shape control for parametric curves and surfaces and gave rise to an optimization problem with non-linear constraints. We now turn to a class of problems involving the construction of explicit surfaces $F(x,y)$ which approximate given surfaces but where the approximation is complicated by having a region where shape constraints need to be imposed.

The surfaces to be constructed are direct product spline surfaces of the type

(3) $\sum \sum a_{ij} B_i(x) C_j(y)$

and the type of approximation is least squares fitting. This is all straightforward. However, these approximations are to be used to describe properties of certain gases. These gas laws often have a region as in Figure 1 where the property is independent of one of the variables, e.g., the partial of temperature with respect to enthalpy is zero. Ignoring these 'flat' regions when doing the fitting gives rise to undesirable oscillations of the function. These oscillations severely affect the performance of the numerical integrators which will use this model.

However, it is possible using optimization methods to consider applying the constraints to the fitted function and thus to take advantage of the improved shape properties.

There are two problems to be considered here. The first is that of defining the region of constraints. Since the region has highly curved boundaries we approach this problem using the method of curved knot lines (3). The second problem consists of applying the constraints themselves. This we do by constraining the B-spline coefficients of the spline surface. We describe some experiments and show the results of using these methods.

ILL-POSED SURFACE PROBLEMS:

Another class of problems to which we will apply optimization techniques is that of surface fitting to scattered data. Again, consider the problem of fitting a direct product spline surface as described by (4) to data whose x-y coordinates are as depicted in Figure 2. There are good reasons for attempting a fit using direct product splines rather than using more traditional scattered data interpolation methods, e.g., those described in (5). The most important of these is simplification of the required data structures and evaluations.

On the other hand, there are problems. For instance, it will often happen that fitting with the direct product form (3) leads to ill-posed problems due to the fact that there are frequently elements $B_i(x)C_j(y)$ which are unsupported by the data. That is, the linear system which arises from (3) will be singular and these singularities must be dealt with. The proper handling of these singular systems has important consequences for the shape of the surface. For example one could solve the system by requiring that among all solutions the norm of the coefficient vector be minimized. This however, may lead to undesirable shape properties in the resulting surface. We will explore methods of handling these shape problems.

VARIABLE KNOT PROBLEMS:

The final class of problems that are candidates for optimization is the variable knot problems, especially those in two and three dimensions. For these problems, the proper placement of the knots is often of critical importance to the approximation. We will look at using optimization techniques to obtain best positions for the knots.

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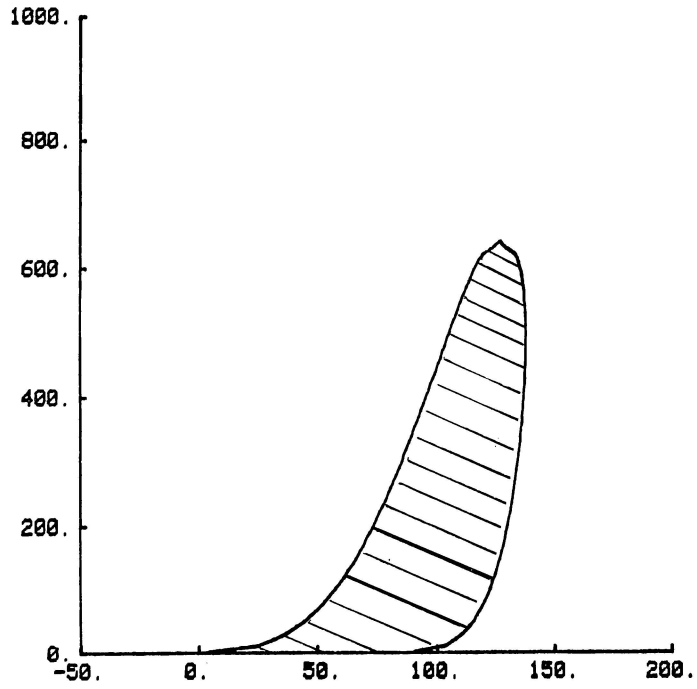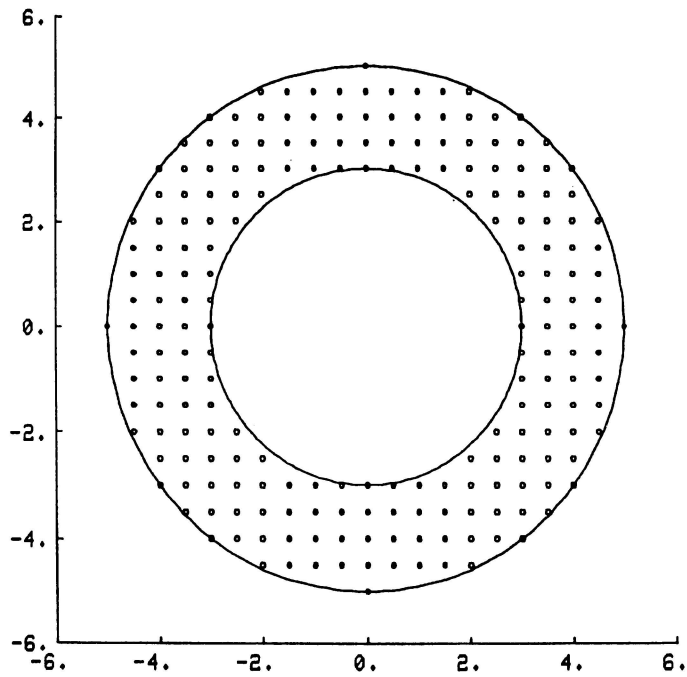
Figure 1

Constrained region for FREON 22.

Figure 2

Domain leading to an ill-posed problem.