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A FOUR-NODE BILINEAR ISOPARAMETRIC ELEMENT IN ROCKWELL NASTRAN

C. LIAO and R. E. ALLISON

ROCKWELL INTERNATIONAL
NORTH AMERICAN AIRCRAFT OPERATIONS
EL SEGUNDO, CALIFORNIA

SUMMARY

Development and evaluation of the Rockwell NASTRAN four-node quadrilateral (QUAD4) element is presented. The element derivation utilizes bilinear isoparametric techniques both for membrane and bending characteristics. The QUAD4 element coordinate system, membrane properties, lumped mass matrix, and treatment of warping are based upon the COSMIC/NASTRAN QDMEM1 element while the bending characteristics are based upon a paper by T. J. R. Hughes. The effects of warping on the bending stiffness, consistent mass, and geometric stiffness are based upon a paper by R. H. MacNeal. Numerical integration is accomplished by Gaussian quadrature on a 2 x 2 grid. Practical user support features include variable element thickness, thermal analysis and layered composite material definitions.

INTRODUCTION

Rockwell NASTRAN is the NASA/COSMIC released NASTRAN with Rockwell developed technical and efficiency enhancements incorporated. A total of nine Rockwell divisions fund the NASTRAN Group Service activities which include user consultation, development, maintenance, and validation of the production program. Rockwell NASTRAN is installed on IBM and CDC computing systems at three geographical locations. The program is being used by the participating divisions which are located in California, Oklahoma, Ohio, Michigan and Pennsylvania.

The Rockwell QUAD4 has been developed in order to provide our users with a state-of-the-art general quadrilateral element. The improved efficiency and greater accuracy provided by this element eliminate the need of any of the other COSMIC/NASTRAN quadrilateral elements. Practical user support features incorporated in the development include varying element thickness, thermal strains, and laminated composite material inputs. The element derivation utilizes bilinear isoparametric techniques both for membrane and bending characteristics with numerical integration being accomplished by Gaussian quadrature on a 2 x 2 grid.

The QUAD4 element coordinate system, membrane properties, lumped mass matrix and treatment of warping are based upon the COSMIC/NASTRAN QDMEM1 element while the bending properties are based upon a recent paper by T. J. R. Hughes (ref. 1). The effects of warping on the bending stiffness, consistent mass, and geometric stiffness are based upon a paper by R. H. MacNeal (ref. 2). The theory adopted from reference 1 appears to minimize or preclude some of the complications alluded to in reference 2. In particular, no special local Cartesian system or selective integration procedure is required to achieve a reasonably good element behavior.

General theoretical background of the element stiffness matrix is presented in equations 1 through 35 of the theoretical background section. Derivation of the equivalent thermal applied load vector is presented in equations 38 through 41.

The evaluation of element test results as proposed by reference 3 are presented in table 1. The test results for static analysis of various structures, mechanical loadings, and thermal analysis are presented in tables 2 through 8. The results for the real eigenvalue test case are presented in table 9. The transverse central deflection computed for the three composite material test cases using the MSC/QUAD4 and the Rockwell/QUAD4 element is presented in table 10.

THEORETICAL BACKGROUND

The relationship between forces and strains (including thermal terms) is described by the following matrix where the vectors $\{\varepsilon^t\}$ and $\{\chi^t\}$ are thermal generated strains and curvatures, respectively.

$$\left\{ \begin{matrix} f \\ m \\ q \end{matrix} \right\} = \begin{bmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & C \end{bmatrix} \left\{ \begin{matrix} \epsilon_m - \epsilon^t \\ \chi - \chi^t \\ \gamma \end{matrix} \right\} \tag{1}$$

where

$$\{ f \} = \left\{ \begin{matrix} f_x \\ f_y \\ f_{xy} \end{matrix} \right\}, \quad \text{membrane forces per unit length} \tag{2}$$

$$\{ m \} = \left\{ \begin{matrix} m_x \\ m_y \\ m_{xy} \end{matrix} \right\}, \quad \text{bending moments per unit length} \tag{3}$$

$$\{ q \} = \left\{ \begin{matrix} q_x \\ q_y \end{matrix} \right\}, \quad \text{transverse shear forces per unit length} \tag{4}$$

$$\{ \epsilon \} = \left\{ \begin{matrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{matrix} \right\}, \quad \text{membrane strains in means planeh} \tag{5}$$

$$\{ \chi \} = \left\{ \begin{matrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{matrix} \right\}, \quad \text{curvatures} \tag{6}$$

$$\{ \gamma \} = \left\{ \begin{matrix} \gamma_x \\ \gamma_y \end{matrix} \right\}, \quad \text{transverse shear strains} \tag{7}$$

The terms A, B and D are defined by the following integrals:

$$A = \int G_e \quad dz \tag{8}$$

$$B = \int (-z)\, G_e \quad dz \tag{9}$$

$$D = \int z^2 \; G_e \; dz \tag{10}$$

and

$$C = H_s \; G_3 \tag{11}$$

The limits on the integration are from the bottom surface to the top surface of the plate. The matrix of material moduli, [Ge] , has the following form for orthotropic materials:

$$[G_e] = \begin{bmatrix} \dfrac{E_1}{1-\nu_1\nu_2} & \dfrac{\nu_1 E_2}{1-\nu_1\nu_2} & 0 \\ \dfrac{\nu_2 E_1}{1-\nu_1\nu_2} & \dfrac{E_2}{1-\nu_1\nu_2} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{12}$$

Here, $\nu_1 E_2 = \nu_2 E_1$, is required that the matrix of elastic moduli be symmetric. The $[G_3]$ is a 2 x 2 matrix of elastic coefficients for transverse shear. H_s, the effective thickness for transverse shear, has a default value of $H_s / H = 5/6$, which is the correct value for a homogeneous plate with an actual membrane thickness of H.

Figure 1 depicts a plate composed of the eight laminas. For this case, A, B and D are defined as follows:

$$A = \sum_{k=1}^{n} G_e^k (h_k - h_{k-1}) \tag{13}$$

$$B = \frac{1}{2} \sum_{k=1}^{n} G_e^k (h_k^2 - h_{k-1}^2) \tag{14}$$

$$D = \frac{1}{3} \sum_{k=1}^{n} G_e^k (h_k^3 - h_{k-1}^3) \tag{15}$$

Let A^c and N_a denote the area and shape functions, respectively, of an element, where n is the number of element nodes. For the case of a homogeneous, isotropic, linearly elastic plate of thickness H, the element stiffness matrix, K^e, may be defined as follows.

$$K^e = K^e_b + K^e_s \tag{16}$$

$$K^e_b = \int_{A^e} R^{b\,T} D \; R^b \; dA \quad \text{bending stiffness} \tag{17}$$

$$K^e_s = \int_{A^e} R^{s\,T} C \; R^s \; dA \quad \text{shear stiffness} \tag{18}$$

where

$$R^b = \left[R^b_1 , R^b_2 \ldots R^b_n \right] \tag{19}$$

$$R^s = \left[R^s_1 , R^s_2 \ldots R^s_n \right] \tag{20}$$

The formulation of the element stiffness matrix follows the procedure defined in reference 4 and 5. Then R^b's can be written in the following form:

$$R^b_a = \begin{bmatrix} 0 & 0 & N_{a'2} \\ 0 & N_{a'1} & 0 \\ 0 & N_{a'2} & N_{a'1} \end{bmatrix} \qquad 1 \leq a \leq n \tag{21}$$

The shear stiffness is obtained by the technique mentioned in reference 1. The detailed procedures are discussed next.

Geometric and kinematic data are defined in figure 2, and the direction vectors have unit length (e.g. $\|e_{11}\| = 1$, etc.). Let W_a and θ_a denote the transverse displacement and rotation vector, respectively, associated with node a. Throughout, a subscript b will equal a+1 modulo 4.

The definition of the element shear strains may be described in the following steps.

(1) For each element side, define a shear strain component at the midpoint, in a direction parallel to the side.

$$\bar{g}_a = (W_b - W_a)/l_a - \bar{e}_{a1} \cdot (\bar{\theta}_b + \bar{\theta}_a)/2 \tag{22}$$

(2) For each node, define a shear strain vector. (See figure 3 geometric interpretation of this process.)

$$\bar{\gamma}_b = \gamma_{b1}\bar{e}_{b1} + \gamma_{b2}\bar{e}_{b2} \tag{23}$$

$$\gamma_{b1} = (1-\alpha_b^2)^{-1}(g_{b1} - g_{b2}\alpha_b) \tag{24}$$

$$\gamma_{b2} = (1-\alpha_b^2)^{-1}(g_{b2} - g_{b1}\alpha_b) \tag{25}$$

$$\alpha_b = \bar{e}_{b1} \cdot \bar{e}_{b2} \tag{26}$$

$$g_{b1} = g_b \tag{27}$$

$$g_{b2} = -g_a \tag{28}$$

(3) Interpolate the nodal values by way of the bilinear shape functions (N_a's)

$$\gamma = \sum_{a=1}^{4} N_a \gamma_a \tag{29}$$

For the transverse shear strain interpolations derived in the previous section, R^s takes on the following form:

$$R_b^s = R_{b1}^s R_{b2}^s R_{b3}^s \qquad 1 \le b \le 4 \tag{30}$$

$$R_{b1}^s = l_a^1 \bar{G}_a - \bar{l}_b^1 \bar{G}_b \tag{31}$$

$$R_{b2}^s = (e_{b2}^1 \bar{G}_a - e_{b1}^1 G_b)/2 \tag{32}$$

$$R_{b3}^s = (e_{b2}^2 \bar{G}_a - e_{b1}^2 \bar{G}_b)/2 \tag{33}$$

$$\bar{G}_a = (1-\alpha_a^2)^{-1} N_a (\bar{e}_{a1} - \alpha_a \bar{e}_{a2}) - (1-\alpha_b^2)^{-1} N_b (\bar{e}_{b2} - \alpha_b \bar{e}_{b1}) \tag{34}$$

$$\bar{e}_{bl} = \begin{Bmatrix} e^1{}_{bl} \\ e^2{}_{bl} \end{Bmatrix} \text{ , etc.} \tag{35}$$

The element stress resultants may be obtained from the following relations:

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \end{Bmatrix} = - D\, R^b d^e \qquad \text{bending moments} \tag{36}$$

$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \quad C\, R^s d^e \qquad \text{shear resultants} \tag{37}$$

where

d^e = element displacement vector

Finally, thermal expansion is represented by a vector of thermal strains

$$\left\{\epsilon^t\right\} = \begin{Bmatrix} \epsilon_x^t \\ \epsilon_y^t \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \cdot \\ \\ {}_{xy} \end{Bmatrix} (T - T_0) = \left\{ \;{}^t \right\} (T-T_0) \tag{38}$$

where α^t = thermal expansion coefficients

T = Temperature at any point in the element

T_0 = reference temperature of the material

An equivalent elastic state of stress that will produce the same thermal stress is

$$\{\sigma_t\} = [G]\{\epsilon^t\} \tag{39}$$

An equivalent set of generalized loads P applied to grid points of the element is obtained by

$$P = \int_A \left\{\epsilon\right\}^T \left\{\sigma_t\right\} h \, d A \tag{40}$$

The equivalent thermal moment vector is defined as

$$M_t = -\int_z [G] \left\{\alpha^t\right\} \bar{T}' z \, d z \tag{41}$$

where T' is the thermal gradient at a cross-section of the plate.

NUMERICAL EXAMPLES

The test problems have been selected from reference 3. The elements tested included the COSMIC/QUAD2, the MSC/QUAD4 and the Rockwell/QUAD4. The test runs for QUAD2 and QUAD4 were performed on an IBM 3081 computer at the Rockwell Western Computing Center while the MSC/QUAD4 result were obtained by utilizing version 63 of MSC/NASTRAN on the Rockwell Scientific Computing Center CDC/CYBER equipment.

The grading system for finite elements proposed by reference 3 is:

Grade	Range
A	2% ≥ Error
B	10% ≥ Error > 2%
C	20% ≥ Error > 10%
D	50% ≥ Error > 20%
F	Error > 50%

The structures analyzed to evaluate the test elements included a patch test plate (figure 4), a straight cantilever beam (figure 5), a curved cantilever beam (figure 6), a rectangular plate with different aspect ratios (figure 7), a Scordelis-Lo roof (figure 8), and a simply supported plate (figure 9) for normal modes and layered composite analysis.

Table 1 presents the summary of grading results for the tested elements. The result for each of the individual test cases are reported in tables 2 through 10. The patch test results presented in Table 2 are reported in the form of percentage error of the computed stresses. The results reported in tables 3 through 7 are shown in normalized form where the computed displacement data has been divided by the theoretical value. The most disturbing failure of the QUAD2 element is its inability to get a passing grade for the straight beam in-plane shear and twist cases. QUAD2 also failed in the curved beam and Scordelis-Lo roof problems. Neither of the QUAD4's or the QUAD2 could achieve a passing grade for the straight beam in-plane shear with trapezoidal shaped elements. In general, our published results agree, but there are some differences from those reported in reference 3. In particular, the results of the twist case for all element configurations of the straight cantilevered beam problem do not agree with the results presented in reference 3. We believe that this was due to a problem with version 63 of MSC/NASTRAN as installed on our CDC equipment at the time we were making our test case runs.

CONCLUSION

In this paper, we have examined the behavior of the new four-node quadrilateral element implemented in Rockwell NASTRAN. The element has been shown to behave well for a variety of plate problems and has retained simplicity in the formulation. The formulation enabled straightforward generation of a linear triangular bending element, which has also been successfully implemented in Rockwell/NASTRAN.

REFERENCES

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2. MacNeal, R. H.: A Simple Quadrilateral Shell Element. Comp. & Struct., Vol. 8, 1978, pp. 175-183.

3. MacNeal, R. H. and Harder, R. L.: A Proposed Standard Set of Problems to Test Finite Element Accuracy. Proc. AIAA/ASME Struct. Dyn. Conf., Palm Springs, May, 1984.

4. Rockwell NASTRAN Theoretical Manual Level 17.500, NA-79-323, June 29, 1979, pp. 8.19-1-8.19-18.

5. Rockwell NASTRAN Programmer's Manual Level 17.500, NA-79-325, Sept. 10, 1979, pp. 5.8-33-5.8-44.

6. Rockwell NASTRAN Demonstration Manual Level 17.500, NA-79-324, June 29, 1979.

Table 1 Summary of Test Results

Test	Table	Element Shape	RI/ QUAD4	QUAD2	MSC/ QUAD4
1 Patch Test, Membrane	2(a)	Irregular	A	A	A
2 Patch Test, Bending	2(b)	Irregular	A	A	A
3 Straight Beam, Extension	3(a,b,c)	All	A	A	A
4 Straight Beam, In-Plane Shear	3(a)	Regular	B	F	B
5 Straight Beam, In-Plane Shear	3(c)	Irregular	D	F	D
6 Straight Beam, Out-of-Plane Shear	3(b,c)	Regular	A	B	A
7 Straight Beam, Out-of-Plane Shear	3(b,c)	Irregular	B	B	B
8 Straight Beam, Twist	3(a,b,c)	All	B	D	D
9 Curved Beam, In-Plane Shear	4	Regular	C	F	C
10 Curved Beam, Out-of-Plane Shear	4	Regular	B	D	C
11 Rectangular Plate (N=4)	5,6(a)	Regular	A	A	A
12 Scordelis-Lo Roof (N=4)	7	Regular	A	D	A

Failed Test Grade (D's and F's)

Table 2 Patch Test Results (Figure 4)
Max. Errors (%) of Stress

(a) Membrane Plate

	RI/QUAD4	QUAD2	MSC/QUAD4
$\sigma_x = \sigma_y$	4.2	4.2	0.0
τ_{xy}	1.0	1.0	0.0

(b) Bending Plate

	RI/QUAD4	QUAD2	MSC/QUAD4
$m_x = m_y$	4.2	4.2	0.0
m_{xy}	0.9	0.9	0.0
$y = \sigma_y$	4.2	4.2	0.0
τ_{xy}	1.0	1.0	0.0

Table 3. Results for Straight Cantilever Beam (Fig. 5)
Normalized Tip Displacement in Direction of Load

Tip Loading Direction	RI/QUAD4	QUAD2	MSC/QUAD4
(a) Rectangular Elements			
Extension	0.996	0.992	0.996
In-Plane Shear	0.904	0.032	0.904
Out-of-Plane Shear	0.980	0.971	0.986
Twist	0.941	0.567	0.702
(b) Trapezoidal Elements			
Extension	0.996	0.993	0.996
In-Plane Shear	0.071	0.016	0.071
Out-of-Plane Shear	0.964	0.963	0.958
Twist	0.884	0.605	0.705
(c) Parallelogram Elements			
Extension	0.996	0.992	0.996
In-Plane Shear	0.808	0.144	0.795
Out-of-Plane Shear	0.978	0.961	0.977
Twist	0.849	0.615	0.705

Table 4 Results for Curved Beam (Fig. 6)
Normalized Tip Displacement in Direction of Load

Tip Loading Direction	RI/QUAD4	QUAD2	MSC/QUAD4
In-Plane Vertical	0.835	0.025	0.835
Out-of-Plane	0.956	0.597	0.868

Table 5 Results for Rectangular Plate Simple Supports (Fig. 7)
with Concentrated Load

Normalized Transverse Deflection at Center

Mesh Size(N)*	RI/QUAD4	QUAD2	MSC/QUAD4
(a) Aspect Ratio = 1.0			
2	0.992	1.035	0.960
4	0.995	1.011	1.017
8	1.033	1.083	1.045
(b) Aspect Ratio = 5.0			
2	0.844	0.493	0.870
4	0.928	0.685	0.962
8	0.986	0.845	1.005

* only one quadrant is discretized

Table 6 Results for Rectangular Plate Clamped Supports (Figure 7) With a Uniform Load

Normalized Lateral Deflection at Center

(a) Aspect Ratio = 1.0

Mesh Size (N)	RI/QUAD4	QUAD2	MSC/QUAD4
2	0.961	1.024	1.008
4	0.993	1.019	1.032
8	1.016	1.057	1.040

(b) Aspect Ratio = 5.0

Mesh Size (N)	RI/QUAD4	QUAD2	MSC/QUAD4
2	1.124	0.873	1.314
4	0.962	1.001	1.016
8	1.002	1.019	[illegible]

Table 7 Results For Scordelis-Lo Roof (Figure 8)

Normalized Vertical Deflection at Midpoint of Free Edge

Mesh Size (N)	RI/QUAD4	QUAD2	MSC/QUAD4
2	1.309	0.881	1.313
4	1.017	0.690	1.021

Table 8 Comparison of Analytical, QUAD4, and QUAD1 NASTRAN DEMO 1-11-1 (Reference 6)

Category	Max. Analytical	CQUAD1	CQUAD4
Displacement	6.2898×10^{-1}	6.2895×10^{-1}	6.317195×10^{-1}
Moment m_y	1.4770×10^{2}	1.4888×10^{2}	1.4832200×10^{2}
Stress τ_y	7.764618×10^{3}	7.792977×10^{3}	7.779586×10^{3}

Table 9 Natural Frequency Comparison, cps on NASTRAN/DEMO 3-1-2 (4x4)

Mode No.	Theoretical	RI/QUAD4	QUAD1
1	0.9069	0.8823	0.9021
2	2.2672	2.3376	2.2837
3	4.5345	4.3515	4.7179

Table 10 Transverse Central Deflection of Simply Supported Composite Square Plate Under a Uniform Pressure (Figure 9)

No. of Plies	Type of Laminate	RI/QUAD4(IBM)	MSC/QUAD4(CDC)
2	90°/0°	5.63410×10^{-3}	5.589612×10^{-3}
3	0°/90°/0°	5.55896×10^{-3}	5.423785×10^{-3}
4	90°/0°/90°/0°	5.58961×10^{-3}	5.666982×10^{-3}

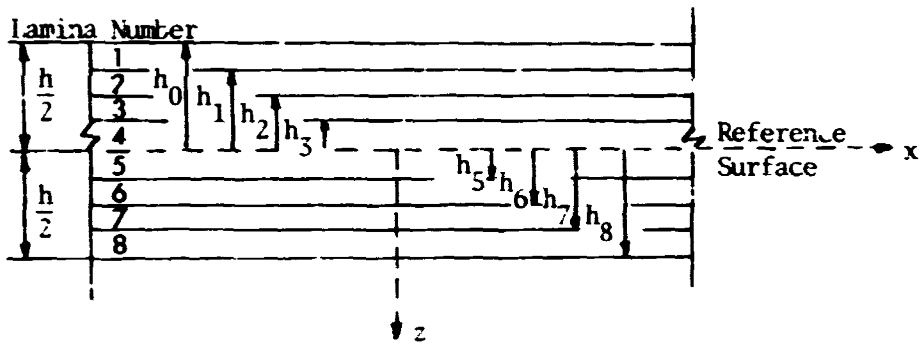


Figure 1. Laminated Plate

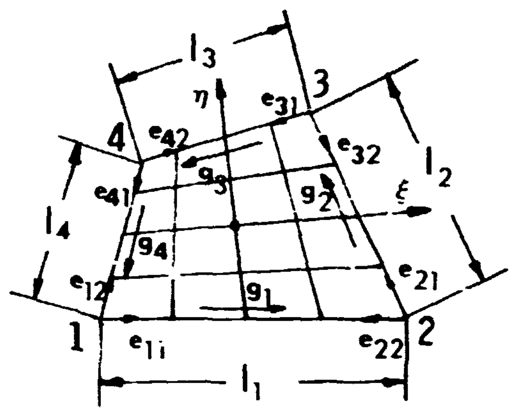
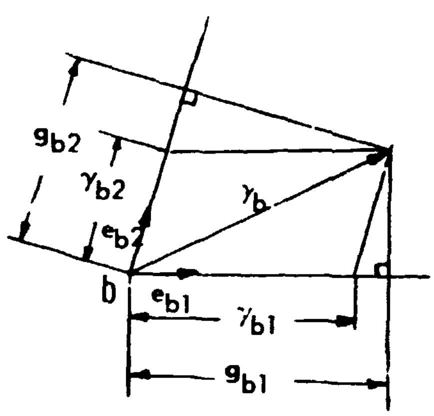
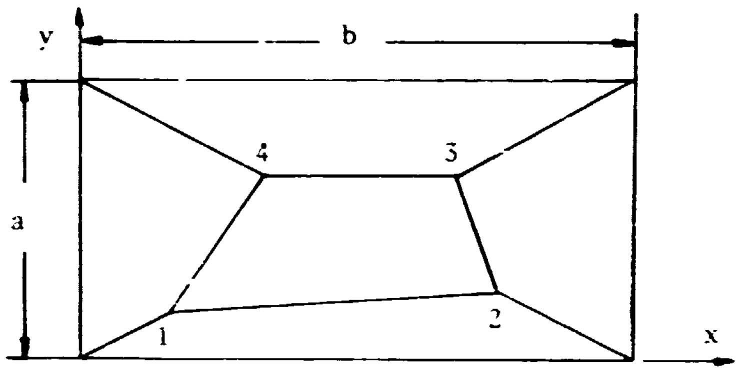


Figure 2. Geometric and Kinematic Data for the Four-Node Quadriliateral Element



a	b
1	2
2	3
3	4
4	1

Figure 3. Definition of Nodal Transverse Shear Strain Vector



$a = .12; \quad b = .24; \quad t = .001$

$E = 1.0 \times 10^{6}; \quad \nu = 0.25$

Location of Inner Nodes:

	x	y
1	.04	.02
2	.18	.03
3	.16	.08
4	.08	.08

Boundary Conditions:

(a) Membrane

$$u = 10^{-3}\ (x + y/2)$$

$$v = 10^{-3}\ (y + x/2)$$

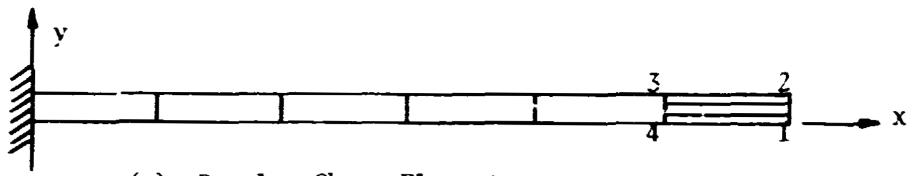
(b) Bending

$$w = 10^{-3}\ (x^{2}+xy+y^{2})/2$$

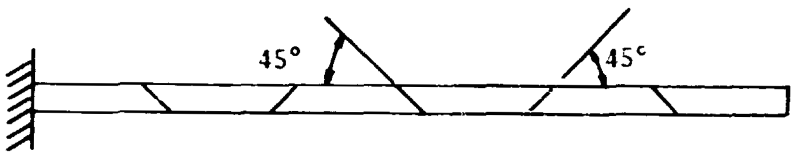
$$\theta_{x} = 10^{-3}\ (y+x/2)$$

$$\theta_{y} = 10^{-3}\ (-x-y/2)$$

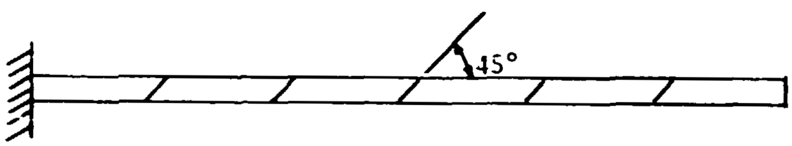
Figure 4. Patch Test for Plates



(a) Regular Shape Elements



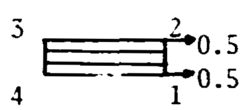
(b) Trapezoidal Shape Elements



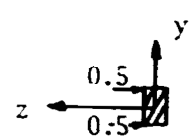
(c) Parallelogram Shape Elements

Length = 6.0; Height = 0.2; Thickness = 0.1
E = 1.0 x 10^7; ν = 0.3; Mesh = 6 x 1
Loading: Unit forces at free end

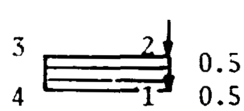
Extension



Out-of-plane shear



In-plane shear



Twist

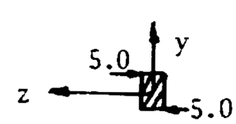
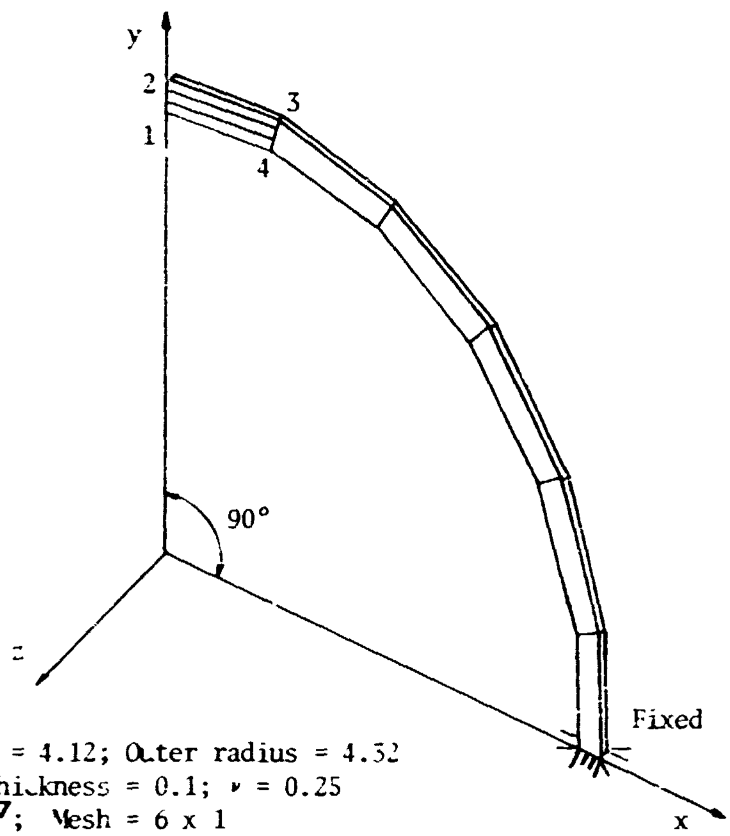
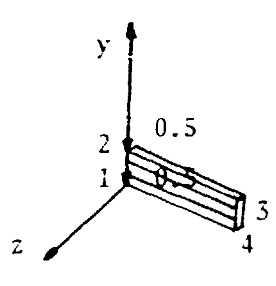


Figure 5. Straight Cantilever Beam



Inner radius = 4.12; Outer radius = 4.32
Arc = 90°; Thickness = 0.1; ν = 0.25
E = 1.0 x 10^7; Mesh = 6 x 1
Loading: Unit forces at tip

In-plane shear



Out-of-plane shear

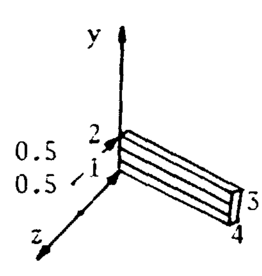
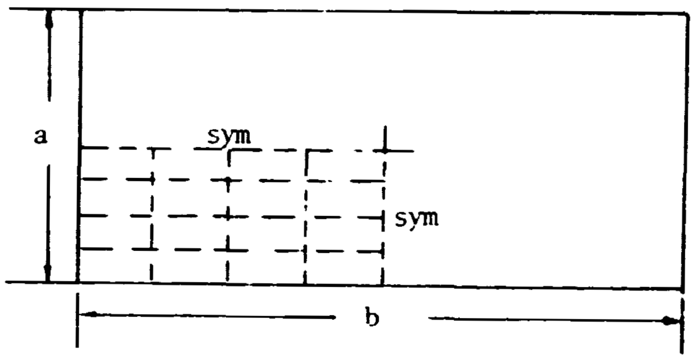
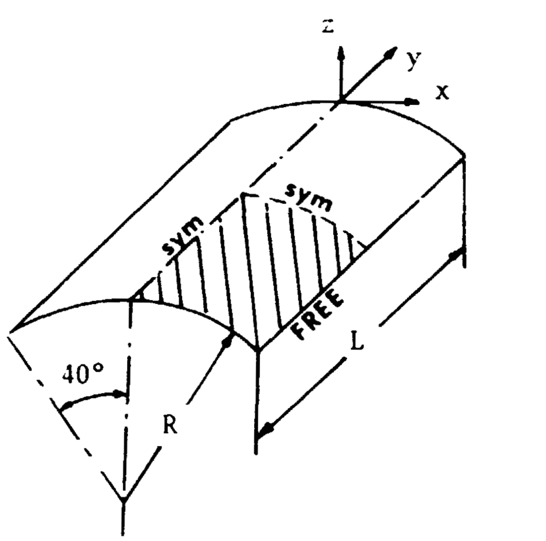


Figure 6. Curved Beam



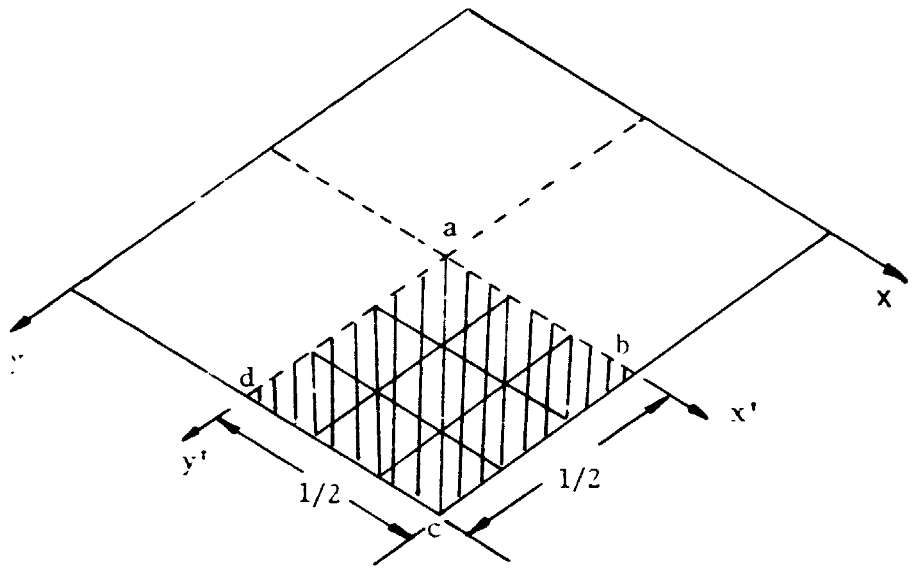
$a = 2.0$; $b = 2.0$ or 10.0; $\nu = 0.3$
Thickness = 0.001; $E = 1.7472 \times 10^7$
Boundaries = simply supported or clamped
Mesh = N x N (on 1/4 of plate)
Loading: Uniform pressure $q = 10^{-4}$ or
Central load $p = 4.0 \times 10^{-4}$

Figure 7. Rectangular Plate



Radius = 25.0; Length = 50.0; Thickness = 0.25
$\nu = 0.0$; Loading = 90.0 per unit area in -z direction
$E = 4.32 \times 10^8$; $U_x = U_z = 0$ on curved edges
Mesh = N x N on shaded area

Figure 8. Scordelis-Lo Roof



$l = 10.0$; $T = 0.2$; $\nu = .25$

$E_1 = 20. \times 10^6$; $E_2 = .5 \times 10^6$; $G = .25 \times 10^6$

Loading Condition: 0.5 psi uniform pressure

Case 1 :	2 piles, material angle of fiber		90°/0°
Case 2 :	3 piles,	" "	0°/90°/0°
Case 3 :	4 piles,	" "	90°/0°/90°/0°

Figure 9. Simply Supported Square Plate for a Layered Structure