ACCURACY OF THREE DIMENSIONAL SOLID FINITE ELEMENTS

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SUMMARY

This paper presents the results of a study to determine the accuracy of the three dimensional solid elements available in NASTRAN for predicting displacements. Of particular interest in the study is determining how to effectively use solid elements in analyzing thick optical mirrors, as might exist in a large telescope. Surface deformations due to thermal and gravity loading can be significant contributors to the determination of the overall optical quality of a telescope. The study investigates most of the solid elements currently available in either COSMIC or MSC NASTRAN. Error bounds as a function of mesh refinement and element aspect ratios are addressed. It is shown that the MSC solid elements are, in general, more accurate than their COSMIC NASTRAN counterparts due to the specialized numerical integration used. In addition, the MSC elements appear to be more economical to use on the DEC VAX 11/780 computer.

INTRODUCTION

Optical mirrors for spaceborne telescopes often have thicknesses which are as much as 10 percent of their diameter. Because of this, and also due to the need to obtain accurate predictions of optical surface deformations in the presence of temperature gradients that are often nonlinear, these mirrors are modeled with three dimensional, solid, finite elements. In addition to thermal deformation being a significant design driver for these mirrors, deformation due to "gravity release" must also be considered. When fabricated and tested, the mirror is in a 1-g environment which is "released" once the telescope reaches the 0-g environment in orbit.

A common design practice is to fabricate these mirrors from low coefficient of thermal expansion (CTE) materials but to mount them to conventional metal housings with much larger CTE's. In order to avoid stressing the mirror when it is subject to simple bulk temperature changes, the mirror is often mounted to its housing through the use of (nearly) kinematic mounts that may connect to the mirror in only three locations.

The purpose of this study is to determine how to effectively model such mirrors for thermal and gravity loading using the available solid elements in NASTRAN. In a study such as this there are three significant questions to consider.

1. How fine a mesh must be used to obtain displacements of the optical surface within a certain error range?

2. Do the individual solid finite elements exhibit sensitivity to aspect ratios significantly different than 1.0?

3. Which of the available elements are the most economical to use (accuracy vs cost in computer resources)?

Questions 1 and 2 above can best be answered in an investigation that completely separates the two effects. That is, the mesh refinement studies should be done using elements with an aspect ratio of 1.0. Then, once a fine enough mesh has been reached such that the errors are small, the effects of aspect ratio can be investigated by keeping the mesh size constant and varying the overall dimensions of the problem, thus resulting in each element aspect ratio changing. Obviously, in order to accomplish this latter step there must be a theoretical (or some other good comparison) solution to the problem with which to compare the finite element model results since at each step a problem of different dimensions (and therefore different theoretical solution) is being modeled. This technique has been used in previous studies for two dimensional membrane elements, [1], [2].

With the above considerations in mind, the sample problem used in the study (described in more detail below) is a cubic slab of equal dimension in the X-Y plane and whose thickness varies between one-twentieth and one-half of the X-Y plane dimensions (see Figure 1). The slabs were constrained in a kinematic fashion simulating the type of restraint often used in actual practice. For the thick slab, a mesh refinement can be made of elements having aspect ratios of 1.0 without requiring an exorbitant number of elements in the X-Y plane when a reasonable number is used through the thickness. For a linear temperature gradient through the slab there exist theoretical solutions. Thus the thick slab, although it does not look much like a mirror, is a good candidate for the mesh refinement studies when the temperature gradient loading is used. Also, since the theoretical solution exists for arbitrary thickness of the slab, the sample problem is also suitable for an element aspect ratio study. Keeping the mesh refinement constant (that is the number of elements in the model), simply by varying the slab thickness, the element aspect ratios must change.

For the gravity loading the situation is not as good. There is no theoretical solution to compare to so it is not practical to attempt the aspect ratio study with this loading. However, if the various elements used in the study show a trend toward convergence to an answer as the mesh is refined, the mesh refinement studies can provide useful information with this loading.

PROBLEM DETERMINATION

Figure 1 shows the geometry, coordinate system, boundary conditions, and basic material information used in this study. The constraints are kinematic and the problem is symmetric about the x=0 plane. That is, the x displacement is zero along the x=0 plane. Using this constraint, only half the slab was modeled for the study.

Both COSMIC and MSC elements were tested in this study. This included hexahedral elements with up to two midside nodes and wedge shaped elements with up to one midside node. Figure 2 displays the different types examined.

The mesh geometry is shown in Figure 3. All models had half the number of elements through the thickness as in the other two dimensions. Only the diamond pattern shown in Figure 3 was used for wedge shaped elements and the triangular shape was always in the x-y plane as recommended in [4]. When an element aspect ratio was desired, the value of t was changed while the number of elements through the thickness remained unchanged. This created shorter or squat elements.

Temperature Gradient Loading

The temperatures applied to the test model varied only in the z direction. A linear gradient was created which ranged from $+\frac{1}{2}$ to $-\frac{1}{2}$ degrees centigrade. The equation for this gradient is:

$$T = \frac{1}{2}\left(1 - \frac{z}{t}\right)$$

$$T = \text{temperature}, \quad t = \text{thickness}$$

An exact theoretical solution exists for a linear temperature gradient [3] and in this case the equations are:

$$u = \bar{u} + u_0 - y\theta_{z_0} + z\theta_{y_0}$$
$$v = \bar{v} + v_0 - z\theta_{x_0} + x\theta_{z_0}$$
$$w = \bar{w} + w_0 - x\theta_{y_0} + y\theta_{x_0}$$

where

$$u, v, w = \text{displacements}, \quad \alpha = CTE$$

$$\bar{u} = \alpha\left(\frac{1}{2} - \frac{z}{t}\right)x$$
$$\bar{v} = \alpha\left(\frac{1}{2} - \frac{z}{t}\right)y$$
$$\bar{w} = \alpha\left[\left(\frac{1}{2} - \frac{z}{t}\right)z + \frac{1}{2t}(x^2 + y^2 + z^2)\right]$$

Using the boundary conditions as shown in Figure 1 we get:

$$u = \alpha\left(\frac{1}{2} - \frac{z}{t}\right)x$$
$$v = \alpha\left(\frac{1}{2} - \frac{z}{t}\right)y$$
$$w = \alpha\left[\left(\frac{1}{2} - \frac{z}{t}\right)z + \frac{1}{2t}(x^2 + y^2 + z^2)\right]$$

Gravity Loading

Since many telescopes operate in a zero gravity environment, surface deformations due to their original one G environment are important in evaluating their optical quality. The load used in this study was

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applied in the -z direction with the supports at the bottom surface. No theoretical solution is available for this case.

A thin slab (i.e. t=.0504M (2")) model was used for the gravity loading. Since this created elements with aspect ratios of 5, only elements which showed no susceptibility to aspect ratio errors in the temperature load case were used.

RESULTS

Temperature Gradient Loading

For the linear temperature gradient loading the results of the mesh refinement and aspect ratios studies are shown in Figures 4 and 5. Figure 4 shows the error in displacement at point A as a function of mesh size (N) for each of the nine solid elements in the study. The number of elements in any one model is N x N x N/2 in the x, y, z directions, respectively, for the complete problem and are half this amount for the one-half of the slab actually modeled due to symmetry. In order to keep the element aspect ratio 1.0 for this part of the study, the slab dimensions were chosen to be .508 x .508 x .254M (20 x 20 x 10").

The linear temperature gradient, coupled with the stress-free mounting used in the problem, produces a linear strain variation through the thickness of the slab. It is to be expected, therefore, that isoparametric elements of high enough order, when integrated using Gauss quadratur with a sufficient number of points, will yield exact results (even when the problem contains only one element through the thickness). This is true for the COSMIC quadratic and cubic elements, CIHEX2 and CIHEX3. In addition, it should also be true for the MSC elements CHEXA (20 node) and CPENTA (15 node) which are also quadratic isoparametric elements but which use reduced integration for selected terms in the stiffness matrix. Reduced integration is used in conjunction with many isoparametric elements in an effort to improve on the overly stiff behavior of some of the lower order of these elements [5]-[8]. When used in situations in which there is primarily bending behavior, the lower order isoparametric elements have a "parasitic" shear introduced due to the allowable modes of deformation as defined by the polynomial displacement field. This parasitic shear can be removed by selectively reducing the order of numerical integration used in generating the element stiffness matrices. The result is an element which no longer demonstrates interelement displacement compatibility but which, paradoxically, often performs better. Table 1 lists the order of the numerical integration used for the isoparametric elements used in this study. The exact details of the reduced integration for the MSC elements is not made clear in their documentation but is believed to be such that the shear terms are represented by only one point for the linear displacement polynomial elements.

Figure 4 shows that the MSC linear isoparametric elements give exact behavior as well as the higher order elements which were expected to be exact. The linear elements, like their COSMIC counterpart CIHEX1, are based on products of linear shape functions which are capable of

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representing all constant states of strain and some linear states as well. However, the linear normal strains in the x, y directions are not represented by these elements so that the exact behavior of the 8 node CHEXA MSC element is not fully understood. It should be anticipated however, that the 8 node CHEXA would be better for this temperature loading than the COSMIC CIHEX1 due to the reduced integration used.

In order to determine the sensitivity of these elements to element aspect ratios different than 1.0, only the finest mesh for each element was run with the slab thickness decreasing in successive runs from .254 M (10") to .0254 M (1"). This resulted in element aspect ratios changing from 1.0 to 10 (that is, the element dimensions in the x-y plane, in ratio to its thickness changed from 1.0 to 10). Figure 5 shows how the displacement errors are affected by these changes in element aspect ratio. The only isoparametric element which shows sensitivity to aspect ratio is the COSMIC linear element CIHEX1.

Since the slab thickness has decreased from .254 M (10") to .0254 M (1") while the element aspect ratio increased from 1 to 10, the slab deflections are due more to bending now than they were when the thickness was .254 M (10"). As mentioned previously, the CIHEX1 has a parasitic shear due to bending so that it might be expected that the deflection errors would worsen as the slab exhibits more bending. The other isoparametric elements avoid this problem, to some degree or another, by either having higher order displacement polynomials and/or reduced integration.

Gravity Loading

As with the temperature loading study, the gravity loading study began with the thick slab in order to preserve the 1.0 element aspect ratio during the mesh refinement portion of the study. As mentioned previously, there is no theoretical solution to a thick slab loaded by a body force when constrained in the kinematic fashion used in this study. Thus, Figure 6, which shows the mesh refinement results, has actual displacement (rather than displacement error) plotted versus mesh size. The results shown on Figure 6 indicate no convergence has taken place for the range of mesh sizes used. Apparently, the reason for no convergence is due to the concentrated loads at the support points (along either side of the slab). The mesh for this kind of a problem really needs to be more refined in these areas rather than just using a uniform mesh for the whole problem. In addition, the "point" support must be distributed over a finite surface area to avoid the singularity (in displacement and stress) that exists, in the limit, as the mesh size (N) becomes infinitely large. However, in a three dimensional problem, this mesh refinement is very difficult to do without the aid of some automatic mesh generation program (which was not available for a this study).

The difficulty in convergence evidenced by the results in Figure 6 was thought to be compounded by the fact that the slab is quite thick (one half of the in-plane dimensions). Since the overall study is really aimed at how to model moderately thick mirrors it was decided to change

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the slab to one that had a thickness of one-tenth the in-plane dimensions but keeping the element mesh arrangement the same (N x N x N/2). Since this would result in elements with aspect ratios of 5, only those elements that exhibited no aspect ratio sensitivity in the thermal load study were used. This included the 1 and 2 mid-side node COSMIC isoparametric elements and all of the MSC isoparametric elements.

Figure 7 shows the results of the mesh refinement study using the thin slab (.504 x .504 x .0504 M (20 x 80 x 2")). Again it is obvious that convergence has not been reached by any of the element types even though the finest mesh for the MSC 20 node hexahedral was 10 x 10 x 5 and contained 4064 degrees of freedom for the half model. This run took nearly six hours of CPU time on the VAX 11/780. An attempt to run this element for the 12 x 12 x 6 mesh failed due to the size of a matrix multiply requiring more memory than was available. Table 3 shows how large these problems can become (for the half model) using the N x N x N/2 uniform mesh. The run that failed contained nearly 6700 degrees of freedom.

From the lack of convergence for this more practical problem it is obvious that a uniform mesh size is very impractical to use for a loading and constraint system like those encountered for some large mirrors. The local deformation in the vicinity of the support points requires refinement. Figure 8 illustrates this by showing the displacement at point B which is directly above one of the support points (which carries one half the weight of the slab). These displacements obviously are much smaller than those at point A, however, they are still growing significantly as the mesh is refined. In an effort to see if the convergence problem is really associated with the concentrated loading at the support points, the relative deflection between points A and B was determined. Figure 9 shows this relative displacement and it appears to be converging indicating that the front surface (i.e. the z = t surface) shape is defined but that the whole slab is still "sinking" over the point support under B (and its symmetric counterpart at y = z = 0 and x = 1/2). Figure 9 is really only useful in demonstrating that the mesh really needs refining (and the point support needs to be distributed over a finite area) in the vicinity of the supports in order to obtain useful information about the deflections of the slab.

Timing Study

Table 2 shows all of the elements tested and the computer time required to generate their stiffness matrices on the DEC VAX 11/780. The COSMIC isoparametric elements with one and two midside nodes take an appreciable amount of time and limit the practical problem sizes which can be run.

In reality, it is the total computer costs for a given accuracy that is of most importance. One of the original goals of this study was to address this issue. However, due to the difficulty with obtaining convergence for the gravity loading, a "comparison" answer was not able to be found with which to guage the absolute accuracy of the

gravity loading results. This precluded obtaining a plot of accuracy versus cost for this loading. However, based on the solution times for the data of Figures 7-9, it appears that the MSC hexahedra with one midside node is the best element in terms of accuracy versus cost.

Conclusions

Most of the available isoparametric solid elements in both COSMIC and MSC NASTRAN appear to be well suited for thermal deformation analyses of kinematically supported mirror-type structures. An exception to this is the COSMIC 8 noded hex element (CIHEX1) which exhibits large errors when the element aspect ratio deviates from 1.0; the element should not be used for aspect ratios above 2.0.

For gravity loading, the situation is much less clear. Use of a uniform mesh in three dimensional problems is an attractive choice due to the complexity of envisioning nonuniform meshes in more than two dimensions. However, the study indicates that with point supports, an extremely fine mesh would have to be used to get convergence for any of the elements used and the "point" support must be distributed over some fixed finite area as the mesh becomes more and more refined. It appears that mesh refinement in the vicinity of the supports is the only realistic way to model these structures and use of some automatic mesh generation program to accomplish this is a necessity.

From the standpoint of economy, the MSC 20 node hex element (CHEXA) appears to be by far the best of all of the available elements.

References

[1] Guidelines for Finite Element Idealization, ASCE National Structural Engineering Convention, New Orleans, April 1975, Preprint 2504.

[2] W. R. Case and J. B. Mason, "NASTRAN Finite Element Idealization Study," Sixth NASTRAN Users' Colloquium, Cleveland, Ohio, October 1977, pp. 383-404.

[3] B. A. Boley and J. H. Weiner, Theory of Thermal Stresses, John Wiley and Sons, Inc., London, 1960, pp. 272-275.

[4] MSC/NASTRAN Applications Manual, Section 5.13, The MacNeal-Schwendler Corp., Los Angeles, CA (1978).

[5] O. C. Zienkiewicz, R. L. Taylor and J. M. Too, "Reduced Integration Techniques in General Analysis of Plates and Shells," Int. J. Num. Meth. Engng. 3, 275-290 (1971).

[6] S. F. Pawsey and R. W. Claugh, "Improved Numerical Integration of Thick Shell Finite Elements," Int. J. Num. Meth. Engng. 3, 575-586 (1971).

[7] B. M. R. Irons and T. K. Hellen, "On Reduced Integration in Solid Isoparametric Elements When Used in Shells with Membrane Modes," Int. J. Num. Meth. Engng. 1179-1182 (1975).

[8] MacNeal, R. H., "A Simple Quadrilateral Shell Element," Computers and Structures, Vol. 8, pp. 175-183, 1978.

Table 1

Gauss Quadrature Numerical

Integration Mesh Used For

Isoparametric Elements

NASTRAN Version	Isoparametric Element	Number of Nodes	Displacement Polynomial	Gauss Quadrative Mesh Points
COSMIC	CIHEX1	8	linear	2x2x2 = 8
	CIHEX2	20	quadratic	3x3x3 = 27
	CIHEX3	32	cubic	3x3x3 = 27
MSC	CHEXA	8	linear	2x2x2 = 8[1]
	CHEXA	20	quadratic	3x3x3 = 27[1]
	CPENTA	6	linear	6[1]
	CPENTA	15	quadratic	9[1]

[1] All MSC elements use a numerical integration for selected terms (shear) at a number of Gauss points <u>reduced</u> from those indicated.

Table 2

Element Generation

NASTRAN Version	Element	EMG + EMA time per element (C.P.U. sec)[2]
COSMIC		
	CHEXA2	2.1
	CWEDGE	1.0
	CIHEX1	1.2
	CIHEX2	10–13
	CIHEX3	23–29
MSC		
	CHEXA (8 nodes)	.8
	CHEXA (20 nodes)	4.5
	CPENTA (6 nodes)	.4
	CPENTA (15 nodes)	1.8

*These times were taken from models run on a VAX 11/780 using COSMIC versions 17.6 and April 1983 and MSC version 63.

Table 3

Number of Degrees of Freedom for Specific Model Sizes

	HEX or WEDGE 0 midside nodes	HEX 1 midside node	WEDGE 1 midside node	HEX 2 midside nodes
N				
2	27	80	92	133
4	117	383	455	649
6	305	1052	1268	1799
8	627	2231	2711	3835
10	1119	4064	4964	7009
12	1817	6695	8207	11,573

APPENDIX - NOTATION

AR_e = ELEMENT ASPECT RATIO (le/te)

ℓ = PROBLEM SLAB LENGTH

ℓ_e = ELEMENT LENGTH

E = YOUNGS MODULUS

t = PROBLEM SLAB THICKNESS

t_e = ELEMENT THICKNESS

u, v, ω = DISPLACEMENTS

a, α = COEFFICIENT OF EXPANSION

ρ = DENSITY

υ = POISSON RATIO

FIG. 1
TEST PROBLEM
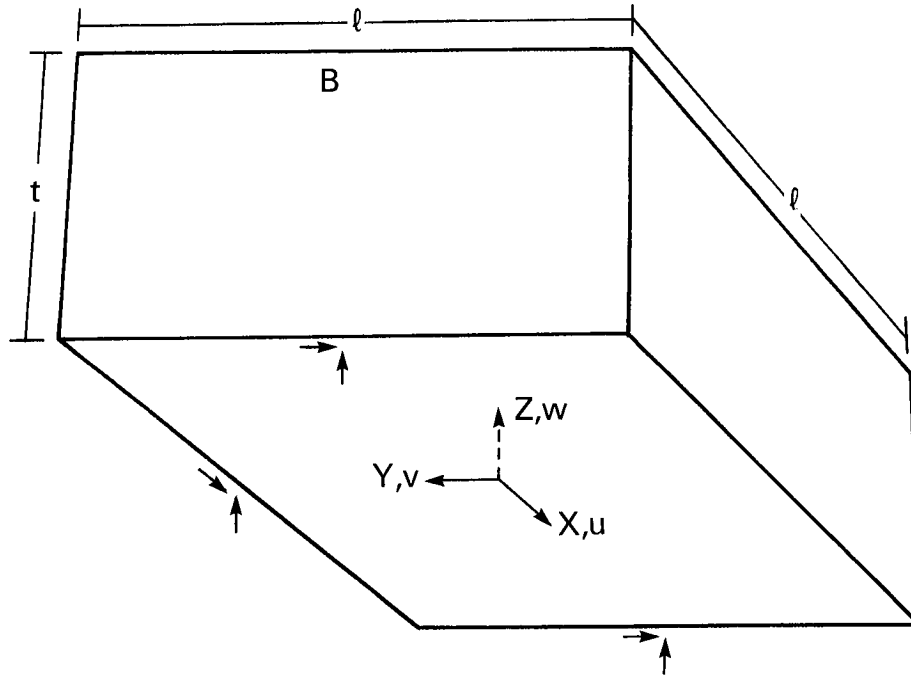
↑ INDICATES DEGREE OF FREEDOM CONSTRAINED FOR
   KINEMATIC MOUNTS

$v = w = 0$ AT $Y = Z = 0$, $X = \pm \ell/2$

$u = w = 0$ AT $X = Z = 0$, $Y = \ell/2$

$A = (0.,0.,t)$

$B = (\ell/2,0.,t)$

MATERIAL INFORMATION (ALUMINUM)

$E = 6.89 \times 10^{10}$ N/M^2 $(10 \times 10^6$ LB/IN$^2)$

$\rho = 2.71 \times 10^3$ KG/M^3 $(.098$ LB/IN$^3)$

$a = 22.7 \times 10^{-6}$ /K

$\upsilon = .33$

FIG. 2
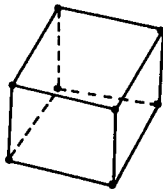
ELEMENT TYPES

CHEXA2 10 TETRAHEDRA (COSMIC)
CIHEX1 LINEAR ISOPARAMETRIC (COSMIC)
CHEXA LINEAR ISOPARAMETRIC (MSC)
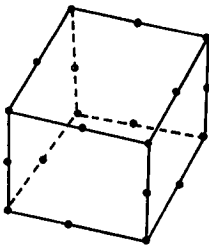
8 NODES

CIHEX2 QUADRATIC ISOPARAMETRIC (COSMIC)
CHEXA QUADRATIC ISOPARAMETRIC (MSC)

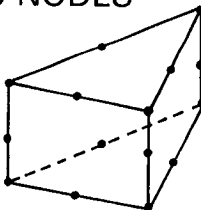**20 NODES**
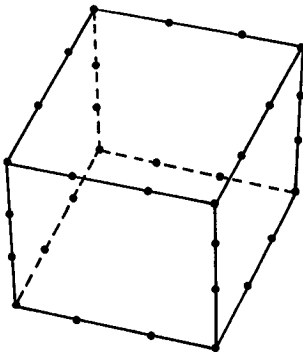
32 NODES CIHEX3 CUBIC ISOPARAMETRIC (COSMIC)
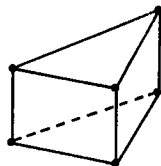
CWEDGE 3 TETRAHEDRA (COSMIC)

6 NODES CPENTA LINEAR ISOPARAMETRIC (MSC)

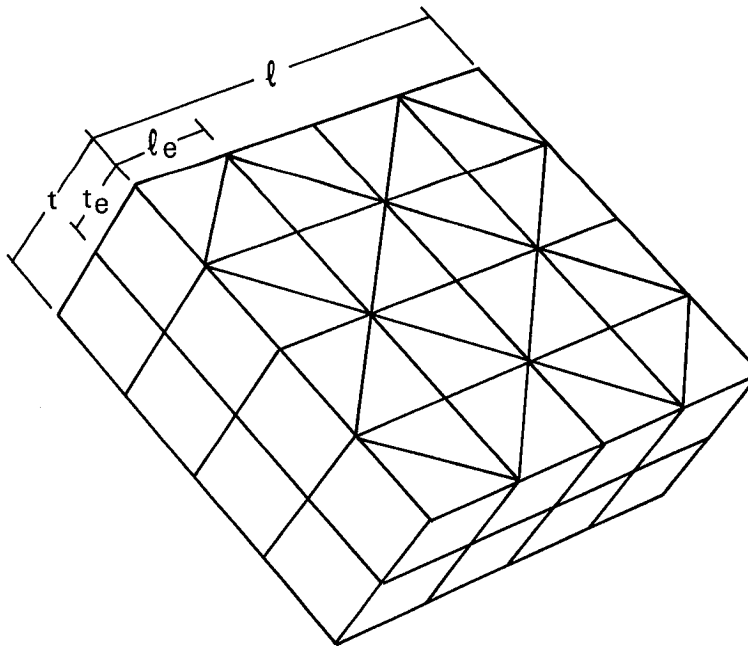**15 NODES** CPENTA QUADRATIC ISOPARAMETRIC (MSC)

FIG. 3

MESH GEOMETRY

$$AR_e = \ell_e / t_e$$
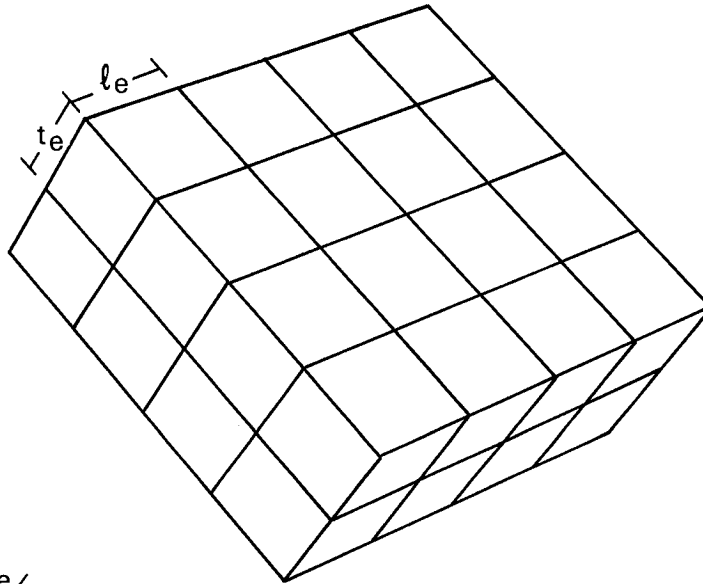
$$N = \ell / \ell_e$$

HEXAHEDRAL PATTERN

WEDGE PATTERN

FIG. 4
Z DISPLACEMENT ERROR AT (A)
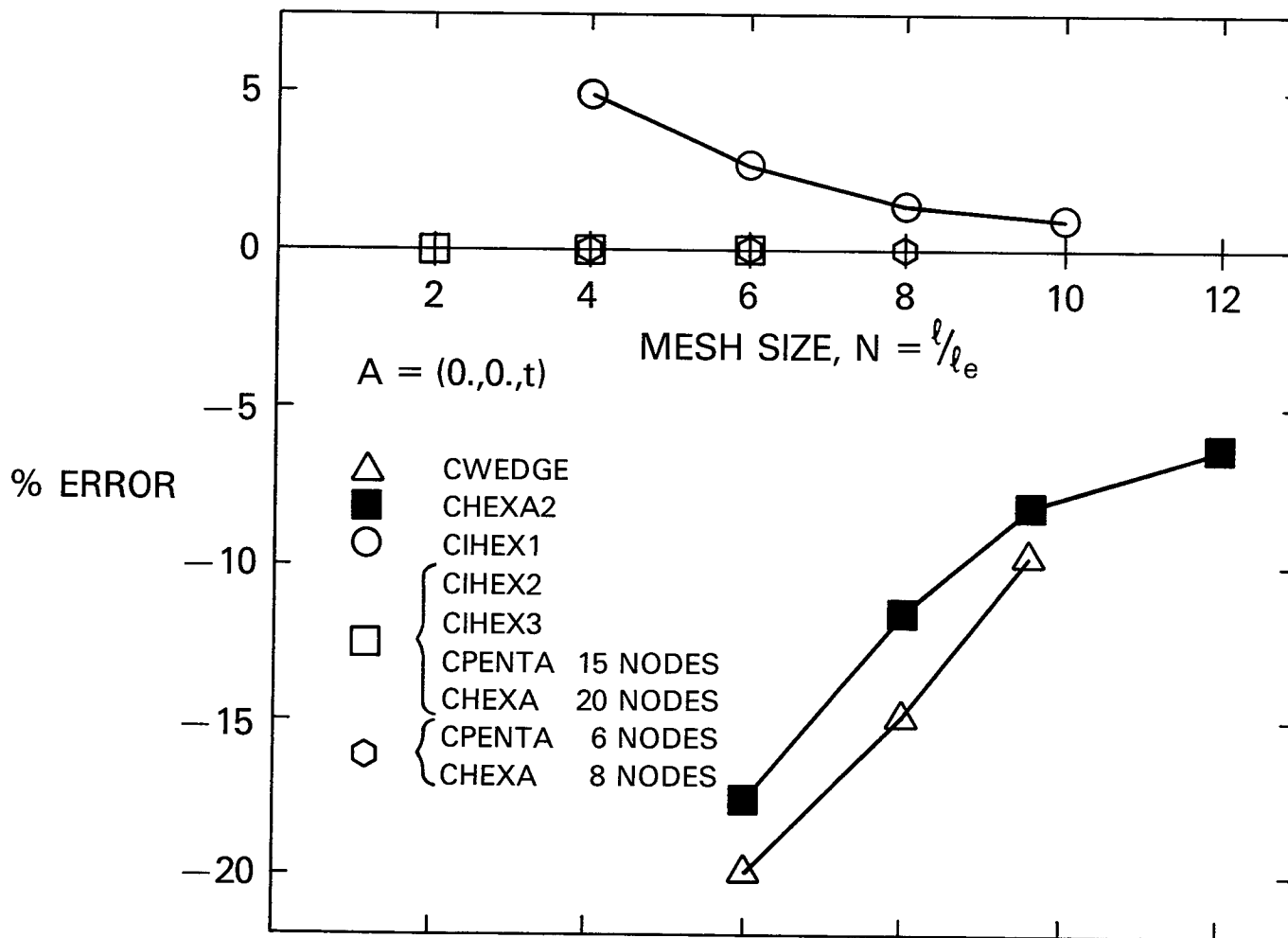## THICK SLAB—LINEAR TEMPERATURE GRADIENT
(MESH SIZE STUDY)

% ERROR

MESH SIZE, $N = \ell/\ell_e$

$A = (0.,0.,t)$

△   CWEDGE
■   CHEXA2
○   CIHEX1
       ⎧ CIHEX2
       | CIHEX3
□ | CPENTA   15 NODES
       ⎩ CHEXA   20 NODES
       ⎧ CPENTA   6 NODES
⬡   ⎩ CHEXA   8 NODES

FIG. 5
Z DISPLACEMENT ERROR AT (A)
LINEAR TEMPERATURE GRADIENT
(ASPECT RATIO STUDY)

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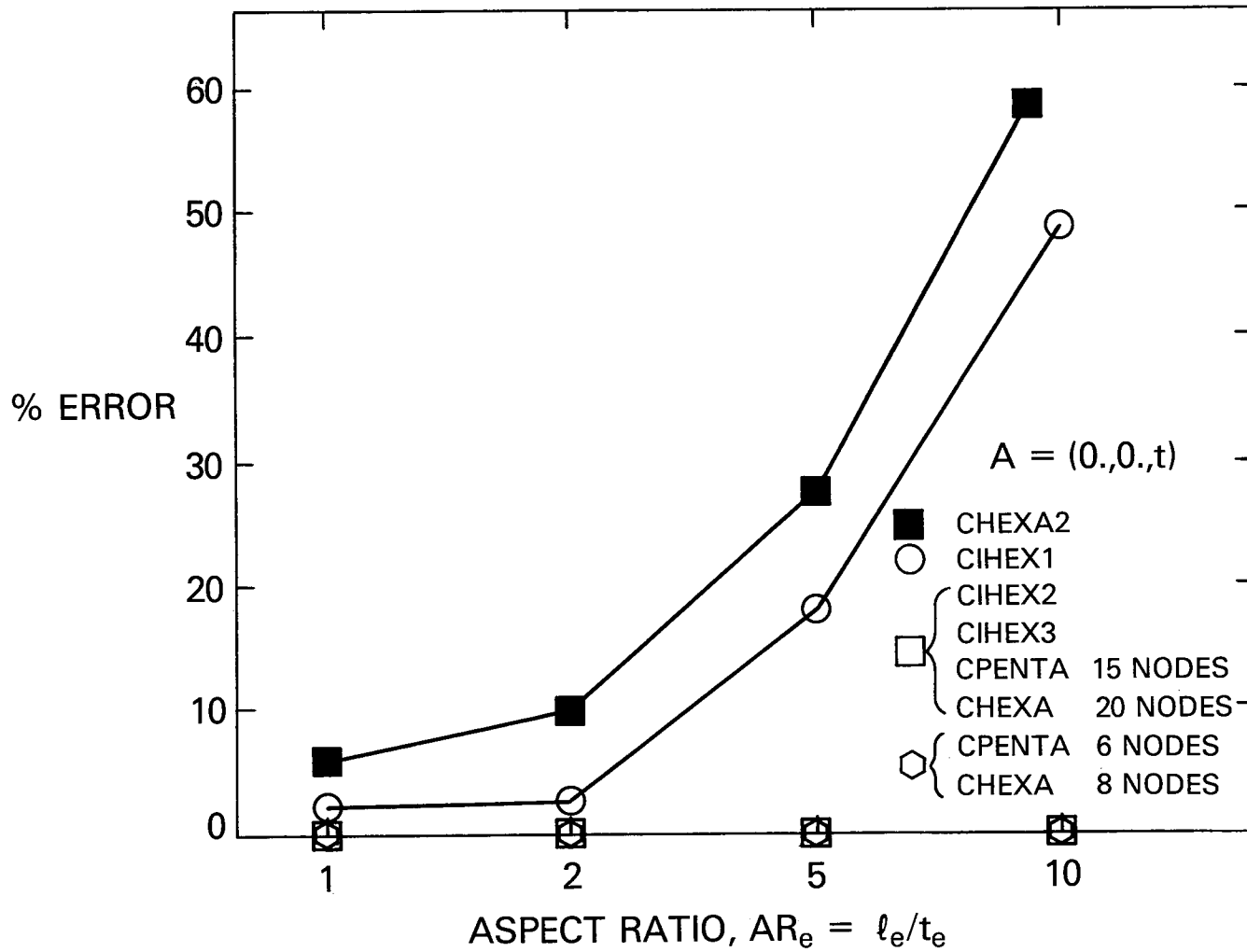
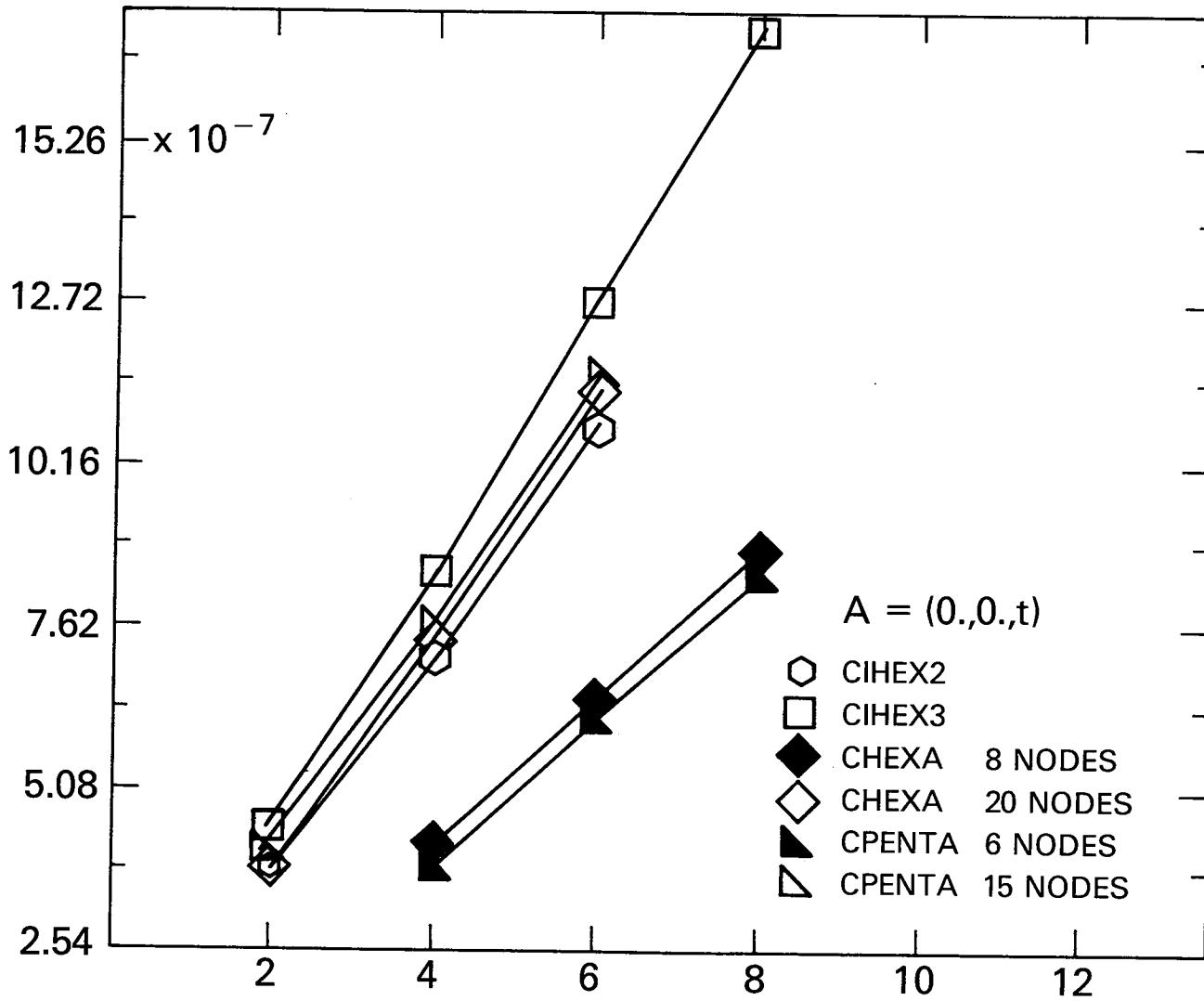# FIG. 6
Z DISPLACEMENT AT (A)
THICK SLAB — GRAVITY

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FIG. 7
Z DISPLACEMENT AT (A)
THIN SLAB—GRAVITY LOADING
(MESH SIZE STUDY)

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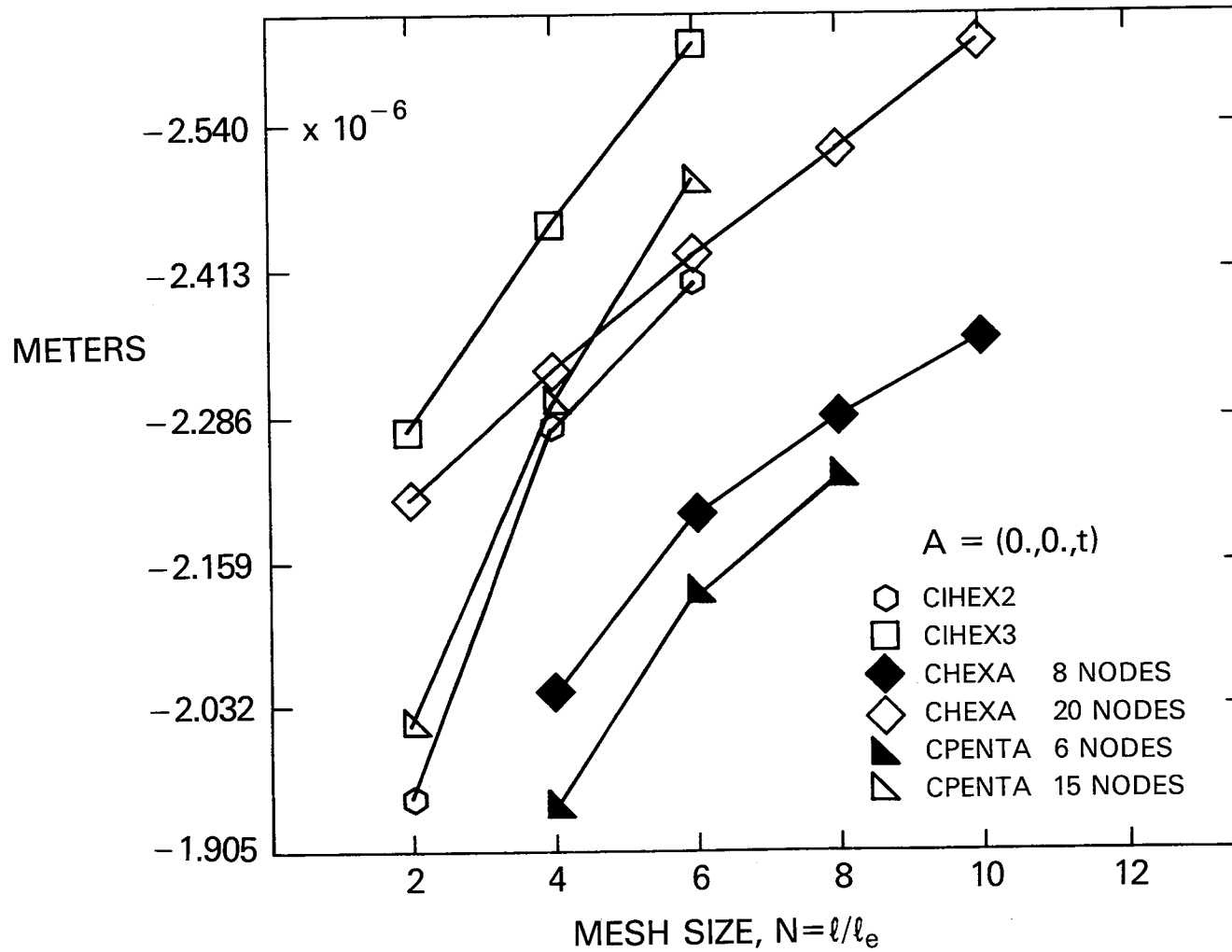
FIG. 8

Z DISPLACEMENT AT (B)
THIN SLAB—GRAVITY LOADING
(MESH SIZE STUDY)

$B = (-10.,0.,t)$

○ CIHEX2
□ CIHEX3
◇ CHEXA   20 NODES
△ CPENTA 15 NODES
▱ { CHEXA   8 NODES
    CPENTA 6 NODES

METERS

-33.02 — $\times 10^{-6}$
-27.94
-22.86
-17.78
-12.7
-7.62
-2.54
0

2   4   6   8   10   12
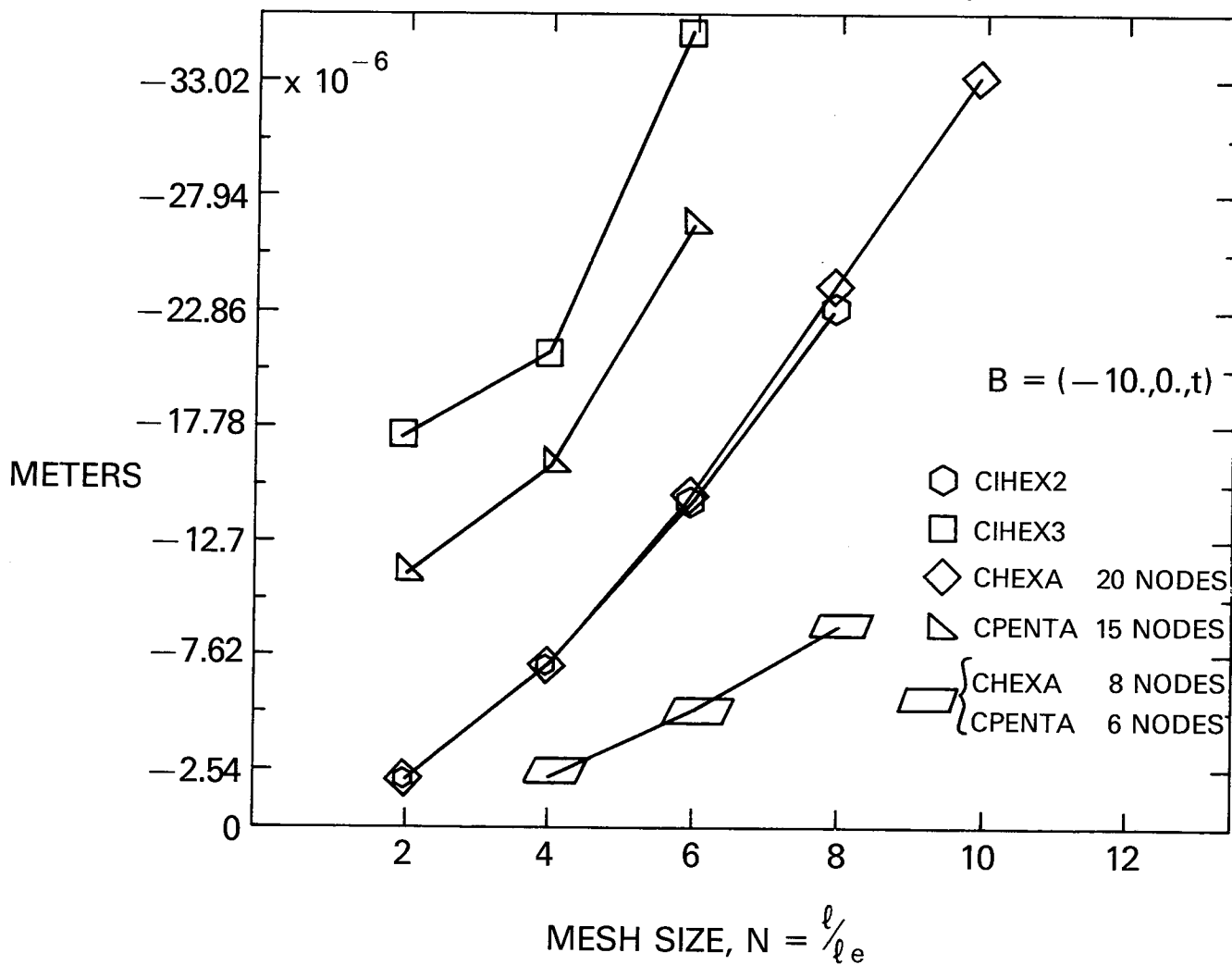
MESH SIZE, $N = \ell/\ell_e$

FIG. 9
RELATIVE DISPLACEMENT (A-B)
THIN SLAB—GRAVITY LOADING
(MESH SIZE STUDY)