An Approach to Optimization of Low-Power Stirling Cryocoolers*

D. B. Sullivan, R. Radebaugh, D. E. Daney, and J. E. Zimmerman
National Bureau of Standards
Electromagnetic Technology Division
Boulder, CO 80303

ABSTRACT

We describe a method for optimizing the design (shape of the displacer) of low-power Stirling cryocoolers relative to the power required to operate the systems. A variational calculation which includes static conduction, shuttle, and radiation losses, as well as regenerator inefficiency, has been completed for coolers operating in the 300 K to 10 K range. While the calculations apply to tapered displacer machines, comparison of the results with stepped-displacer cryocoolers indicates reasonable agreement.

INTRODUCTION

The general wisdom of intercepting heat transfer from a warm to a cold region by refrigeration at intermediate temperature levels has long been appreciated. The Carnot coefficient, $T_{amb}/T-1$, which governs the power required to refrigerate at a temperature, T, below the ambient temperature, T_{amb} is at the heart of this thought. Cryocoolers are commonly designed to fit an *a priori*, and often highly subjective, choice of refrigeration power at one or more low temperatures. One measure of performance is the efficiency of the cryocooler at the specified refrigeration power, that is, the ratio of the actual drive power to the ideal drive power as calculated using the Carnot coefficient. It is implicit in this approach that if such a machine is used in an application requiring very much less than the specified refrigeration power, the total system may be grossly inefficient. There is now a need to provide refrigeration for a number of very low-power cryoelectronic devices which require almost negligible refrigeration power (less than a milliwatt) at temperatures below about 8 K. In this case optimal interception of heat flow is of paramount importance and efficiency becomes a meaningless measure of performance. In this paper we consider such optimization for a class of low-power Stirling cryocoolers which use plastic materials in a gap-type regenerator [1-3]. In the course of this study we would like to propose alternate measures of performance for such machines which could provide for a rational comparison of different concepts.

* This work was supported in part by the Office of Naval Research under contract number N00014-82-F-0040.

As a preliminary problem, we consider the optimum refrigeration distribution for leads and support structures which connect room temperature to low temperature regions. This problem leads naturally to the consideration of the geometrical configuration of a Stirling cryocooler which provides for optimum cooling (that is, minimum drive power), which is the thrust of this work.

OPTIMAL REFRIGERATION DISTRIBUTION FOR CONDUCTION LOSSES

The problem considered here is that of interception of heat flowing along a support rod or electrical lead with the minimum amount of work performed by the refrigerator. For simplicity we assume a uniform cross section of area, A, for the heat conducting element and a thermal conductivity, k, which is independent of temperature. The optimization takes the form of a variational calculus problem [4] in which the direct output is the axial temperature distribution which results in the minimum work performed by the refrigerator. From this temperature distribution we can readily calculate the continuous refrigeration distribution which our imaginary refrigerator must provide.

The conduction heat flow is simply

$$\dot{Q} = kA \frac{dT}{dx} \quad . \tag{1}$$

The net work, w, done by the refrigerator in cooling from ambient temperature, T_{amb} at x to some lower temperature, T_o, at x = 0 is given by

$$w = \int_0^a \left(\frac{T_{amb}}{T} - 1\right) \frac{d\dot{Q}}{dx}\, dx = \int_0^a \left(\frac{T_{amb}}{T} - 1\right) kA \frac{d^2T}{dx^2}\, dx \ , \tag{2}$$

We consider variations of T and find the T(x) for which w is stationary, that is, at an extremum. Application of the calculus of variations [4] leads to the differential equation

$$T \frac{d^2T}{dx^2} - \left(\frac{dT}{dx}\right)^2 = 0 \ , \tag{3}$$

which has the solution

$$T = T_o e^{\alpha x} , \qquad (4)$$

where

$$\alpha = \frac{1}{a} \ln (T_{amb}/T_o). \qquad (5)$$

This exponential temperature distribution is the optimum one for this situation (constant k and uniform cross section). The distribution of refrigeration is calculated using equation 2.

If the thermal conductivity has the form $k = k_n T^n$, (n is a positive integer) the variational method leads to a differential equation

$$T \frac{d^2T}{dx^2} + \frac{1}{2} (n-2) \left(\frac{dT}{dx}\right)^2 = 0 \qquad (6)$$

which except for n = 0 has the solution

$$T = (T_o^{n/2} - \beta x)^{2/n} , \qquad (7)$$

where

$$\beta = (T_{amb}^{n/2} - T_o^{n/2})/a . \qquad (8)$$

This type of calculation need not be restricted to elements of uniform cross section. For example, the optimum temperature distribution for heat flow along a truncated cone (big end at higher temperature) can be shown by a similiar approach to be derived from the differential equation

$$\frac{d^2T}{dx^2} + (\frac{n}{2} -1) \frac{1}{T} \left(\frac{dT}{dx}\right)^2 + \frac{2}{x} \frac{dT}{dx} = 0 , \qquad (9)$$

where n is the power of the temperature in the thermal conductivity (as before).

The conduction loss is given rigorously by the equation

$$\dot{Q}_{con} = kA \frac{dT}{dx} = \pi k(r + t)^2 \frac{dT}{dx} \quad , \tag{10}$$

where the thermal conductivity, k, is a function of temperature, T, and the cross-sectional area, A, is taken to be a function of the position, x, along the displacer, since we wish to consider variations of the displacer shape and thus of its area. In the second form of this equation, the area has been expressed in terms of the displacer radius, r, and the sleeve wall thickness, t.

For plastic displacers operating at low speed, Radebaugh and Zimmerman [5] have shown that a conduction-like form, devoid of gas properties, is appropriate for the description of shuttle heat loss.

$$\dot{Q}_{shu} = r\, S^2 \sqrt{\pi \dot{n} k C_v / 2}\ \frac{dT}{dx} = \beta \sqrt{k C_v}\ r \frac{dT}{dx} \quad . \tag{11}$$

Here r, the radius of the displacer, is a function x. S is the stroke, $\dot{n}$ is the frequency and k and C_v the thermal conductivity and specific heat of the displacer and sleeve walls, both being explicit functions of temperature. The factor β includes all of the constants in the equation. The assumptions are that (i) solid thermal properties are only weak functions of temperature, (ii) thermal resistance of the gas gap is negligible, (iii) the effect of gas-pressure cycling is negligible, i.e., heat capacity of the gas in the gap is negligible and regeneration is complete, (iv) interaction of the axial conduction effects in the displacer and wall are negligible, and (v) motion of the displacer is sinusoidal. The authors showed good experimental agreement with this equation for temperatures down to 102^{o} K. Assumption (iii) concerning the heat capacity of the gas might be expected to break down as the temperature falls below 10^{o} K where regeneration problems arise.

For the sake of this calculation we wish to consider an alternate, distributed form for the radiation loss [6]. We assume that 4 or 5 radiation shields will be attached along the length of the machine to intercept the radiation at higher temperatures where refrigeration efficiency is better and that liberal use of multiple super-insulation layers will be made between radiation shields. If Δx is the spacing (axially) between grounded radiation shields and n is the number of super-insulation layers per unit length (again axially), then we write

$$\dot{Q}_{rad} = (2\varepsilon\sigma A_s)\ (\Delta x/(n\Delta x+1))T^3 \frac{dT}{dx} = \gamma T^3 \frac{dT}{dx}\ , \qquad (12)$$

where A_s is surface area of the radiation shields (assumed to be the same for all shields), σ is the Stephan Boltzman constant and ε the emissivity of the superinsulation layers (taken to be 0.5 for the usual aluminized plastic material. In the abbreviated form γ collects together all of the constant terms. To be sure, this is a very rough sort of estimate, but it can be left so for several reasons. First, with sufficient shielding radiation losses can be held to a level which is significantly below other losses (10 to 20%) and second, radiation losses can be made negligible at the low temperature end where most of the real refrigeration problems arise. Finally, since we are approximating a discrete process (4 or 5 grounded shields) with a differential equation we should not take the exact form too seriously since the heat loss is, in fact, not distributed continuously along the displacer.

Radebaugh [7] has expressed the regenerator inefficiency for low-speed, plastic-displacer Stirling coolers in a form which is suitable for our purposes.

$$\dot{Q}_{reg} = (mC_p)^2\ \frac{\dot{n}}{4r}\ (2\pi kC_v/\dot{n})^{-1/2}\ \frac{dT}{dx} = \delta(mC_p)^2/(r\sqrt{kC_v})\ . \qquad (13)$$

Here m is the total helium mass flow into the expansion space and C_p is the specific heat of helium at constant pressure. Again, δ is a collection of all the constants. The assumptions here are many: (i) temperature variation at the surface is sinusoidal, (ii) surface temperature is the gas temperature, (iii) gas volume in the regenerator is negligible, (iv) sinusoidal heat flow is only in the radial direction, (v) the expansion and compression space are isothermal, and (vi) various heat flows to the expansion space are independent of one another. Assumptions (i) and (v) are of greatest concern. The mass flow is only approximately sinusoidal, but the assumption greatly simplifies the form of the equation. Radebaugh shows that assumption (v) yields errors about a factor of 2 at 5 K and these reduce rapidly as temperature is increased.

A final loss term accounting for the non-ideal properties of helium gas might also be included. Like the regenerator inefficiency the non-ideal gas term (alternatively referred to as the enthalpy deficit) also depends on the gas flow and thus the calculations of the two terms should be similar. There is an important qualitative difference between the two terms however. The steeply decreasing heat capacity of many regenerator materials at low temperatures and the consequent steeply rising regenerator efficiency makes it difficult to achieve temperatures below 5 or 6 K (although a temperature approaching 3 K was achieved by operating a regenerative machine at sub-atmospheric pressure [3].

According to our preliminary calculations, the non-ideal gas properties of helium do not particularly limit the temperature that can be achieved, and in fact it is conceivable that they might be used to advantage in certain situations. We will assume ideal-gas behavior for the present calculation, but the non-ideal behavior could be added later as a refinement of the model.

The differential refrigeration, dW, is

$$dW = \begin{pmatrix}\text{Pressure}\\ \text{Change}\end{pmatrix}\begin{pmatrix}\text{Differential}\\ \text{Volume}\end{pmatrix}\begin{pmatrix}\text{Cycle}\\ \text{Rate}\end{pmatrix} \qquad (14)$$

$$= (P_u - P_\ell)\,(2\pi S r dr)\,(\dot{n}) \quad ,$$

where P_u and P_ℓ are the upper and lower pressures, respectively. Integrating, the refrigeration is found to be

$$W = Br^2 \ , \qquad (15)$$

where $B = \pi S\dot{n}\,(P_u - P_\ell)$ is a constant. W is thus a function of the radius, r, and r is a function of the position, x, along the displacer. This form is valid where the displacer follows a square wave motion. Where sinusoidal motion is used, the Schmidt analysis should be applied, but for simplicity we simply reduce equation 14 by the multiplicative factor $2/\pi$.

Also, equation 14 applies only to the case where there is no pressure change during the displacement parts of the thermodynamic cycle. The ideal cycle (consisting of two isothermal and two isobaric processes) is commonly called the Ericsson cycle. In the ideal Stirling cycle (two isothermal and two constant-volume processes), there are pressure changes during all parts of the cycle and consequently a significant amount of refrigeration occurs during displacement as well as during expansion by the piston [8]. We have not tried to take account of this effect since it is not yet clear whether the Stirling or the Ericsson cycle (or some other idealization) is a better approximation to the actual operation of the machines to which these calculations are to be applied.

HEAT CONTENT OF THE WORKING FLUID

In equation 13, the term mC_p, which is a function of x, represents the heat content of all the gas which moves down past the point x on the displacer. It clearly depends on the geometry of the displacer and thus is not known until the displacer shape is known. It can be expressed as

$$mC_p = NC_{pm} = \frac{PV}{RT} C_{pm} = \frac{5}{2} \frac{PV}{T} \quad . \tag{16}$$

Here N is the number of moles of gas and $C_{pm} = \frac{5}{2} R$ is the specific heat per mole for helium gas. The volume, V, and temperature are functions of x and we choose to fix the pressure at P_u, a point to which we will return. Then we may write

$$mC_p = \frac{5}{2} P_u \int_0^V \frac{dV}{T} = \frac{5}{2} P_u \left[\frac{V_o}{T_o} + S \int_0^x \frac{2\pi r}{T} \frac{dr}{dx} dx \right] \tag{17}$$

$$= \left[\frac{r_o^{\,2}}{T_o} + 2 \int_0^x \frac{r}{T} \frac{dr}{dx} dx \right] \quad ,$$

where $\lambda = 5\pi SP_u/2$, r_o is the radius of the displacer at the bottom end, and $V_o = \pi r_o^{\,2} S$ is the expansion volume at the bottom end. That the displacer should be blunt (of finite radius) at the bottom end can be understood by noting that some finite heat is always transferred all the way to the bottom end and thus some finite refrigeration is required at that point. Conduction through the sleeve wall is just one of the ways in which heat arrives at the bottom end.

In general the maximum pressure, P_u, is not coincident with the minimum volume in a Stirling machine, so, in principle, some pressure between P_u and P_ℓ should be used in equation 17. However, the regenerator loss term is calculated under the assumption that the temperature variation is sinusoidal and this also is not quite right. This last consideration suggests that equation 13 will underestimate the regenerator loss. Thus, we choose to use P_u in calculating mC_p since this increases the regenerator inefficiency term and compensates to some degree for the effect of non-sinusoidal temperature variation.

GENERAL APPROACH TO OPTIMIZATION

The approach which we have chosen involves variation of solutions to arrive at the optimum solution from an initial, rather arbitrary guess.

We select a power series to represent the temperature distribution along the displacer.

$$T(x) = T_0 + T_1x + T_2x^2 + T_3x^3 + \ldots\ldots T_7x^7 \ . \qquad (18)$$

If the top end temperature is constrained to be ambient temperature, T_{amb}, (e.g. 300^o K) and the bottom temperature (at x = 0) is fixed at T_o (e.g. 10 K), the normalization of x to the length, L, of the refrigerator yields the constraint

$$T_1 + T_2 + T_3 + \ldots . T_7 = T_{amb} - T_o \ , \qquad (19)$$

since x = 1 at the top end. A simple initial guess is that the temperature distribution is linear, that is, $T_1 = T_{amb} - T_o$ and the other coefficients are zero. This also fixes the initial guess for the temperature derivative, dT/dx.

Referring to figure 1 for nomenclature, the radius, r_o, of the displacer at the cold end (x = 0) can be found by equating the refrigeration term (equation 15) to the sum of the loss terms (equations 10, 11, 12, and 13). Since mC_p (given by equation 17) is known at the bottom in terms of r_o we can insert that into the regenerator loss term. The result is

$$Br_o^2 = \left[\pi k(r_o + t)^2 + \beta\sqrt{kC_v}\,r_o + \gamma T^3 + \delta\lambda^2 r_o^3/\sqrt{kC_v}\right]\frac{dT}{dx} \ . \qquad (20)$$

In terms of the initial guess for T(x), a consistent value for r_o can be obtained from this cubic equation since all elements in it are known.

In general we cannot use this procedure to obtain r at other positions along the displacer because the value for mC_p at points above the bottom end depends on the shape of the displacer which is as yet undetermined. However, assuming that the regenerator term is not the overwhelming one, an iterative process (by computer) can be used to arrive at a displacer shape which provides for consistency between mC_p as a function of x and the balance of refrigeration and losses. The balance equation is now

$$Br^2 = \left[\pi k(r+t)^2 + \beta\sqrt{kC_v}\,r + \gamma T^3 + \delta\ (mC_p)^2/(r\sqrt{kC_v})\right]\frac{dT}{dx} \ . \qquad (21)$$

Here we must use equation 17 for mC_p. We break the range of x into increments of Δx and, for $x = \Delta x$ calculate a value for r from this equation using the value of mC_p at $x = 0$. mC_p at $x = \Delta x$ can now be calculated from equation 17. We proceed to subsequent increments using the same procedure (that is, using the value of mC_p from the previous increment) until the entire range of x is covered. This procedure generates a first approximation for r(x) and for mC_p as a function of x. The values for mC_p are then shifted down one increment (Δx) since they are really associated with the values of r at those points. In the next iteration we use this information to improve upon the first approximation, that is, a new r(x) (and thus mC_p) is calculated using the old mC_p. The process is repeated until convergence is achieved, at which point we have a self consistent solution (mC_p and displacer shape). In practice this convergence is rather rapid, requiring less than 10 iterations.

For this particular guess at the optimum solution, the net power required to operate the machine can now be calculated using equation 2 where dQ/dx is the sum of the derivatives of the loss terms. Since this sum equals the derivative of the refrigeration power, that could equally well be used to calculate the net power (this is a convenient internal check on the consistency of the calculation).

Having calculated the net power to run this particular refrigerator, we can proceed to search for the optimum design, that is, the temperature distribution and displacer shape which minimize the drive power. The process involves variation of the coefficients in the expansion of T (see equation 18) consistent with the constraint given by equation 19. The variations which yield reductions in the power requirement are followed until further variations of any of the coefficients yield no further decrease.

We now assert (without proof) that this is the optimum solution. On general grounds one can argue that this could represent a local minimum (a sort of metastable solution) and that a solution of still lower power requirement might yet exist. In fact, we can offer no general proof that this is not true, but uniqueness of the solution can be tested by trying a variety of initial guesses for T(x). These will produce a different history in the search for the optimum solution, but arrival at the same solution will add confidence to the validity of the final result. A final point should be made. The problems on conduction loss considered in section 2.0 all had unique solutions and these were produced by a more rigorous method. While the refrigerator problem is more complex, the loss terms (Q's) in most reasonable situations (certainly in all cases which we have considered) are rather simple monotonically increasing functions of postion along the displacer and thus we don't expect the general character of the solutions to be significantly different from those derived in section 2.0.

As mentioned at the beginning of this section (3.0), additional support members or electrical leads can be included if necessary. The implicit assumption of this assertion is that the only loss for these elements will be by conduction and that the temperature profile for them will be identical with that for the cooler itself. In general, this will require a number of thermal tie points between these elements and the cooler. With this condition, the additional loss can be added directly to equations 20 and 21. The cross sectional area for such members could be constant or some fixed function of x. Another reasonable modification which we have incorporated is the allowance for a hollow displacer (to cut down on conduction loss). For simplicity of description this has not been included explicitly, but the process involves a simple modification of the conduction term in equations 20 and 21. Actually we assume that the hollow space is filled with helium gas and include conductive but not convective heat transfer in the gas (i.e. convection eliminated by filling with glass fiber or some cellular material). Finally, if we add a constant power to these two equations we can provide for dissipation at the bottom end. In effect, the solution thus obtained will produce a refrigeration power equal to this constant.

The method used to attack this refrigerator problem parallels the simpler problem described in section 2.0. In both cases we consider variations in T for which the work integral, equation 2, is stationary with the constraint that temperature is fixed at the hot and cold ends. The primary difference of a general nature is that the system is really taken through the variations in the one case where an identification of the stationary solution based on the formal variational calculus is used in the other.

DESCRIPTION OF THE COMPUTER CODE

A simplified version of the flow chart which describes the computer code is shown in figure 2. Care should be used in literal use of this flow chart since there are implied actions which have been omitted for simplicity. These will be mentioned in the step by step description of the program. The numbers below refer to the segments of the program as shown in the figure. (1) The T(I) referred to here are the seven coefficients in the expansion of temperature as a power series in x (equation 18). Thus the index I which identifies the particular coefficient runs from 1 to 7. J is an integer which indexes the particular passage through the calculation for a particular value of I. It is reset to 0 for each change of I. (2) In this step the radius of the displacer at the bottom end is calculated using equation 20. J is also advanced at this step. (3 and 4) The procedures for these steps are described rather completely in section 3.3. The pertinent equations are 17 and 21. (5) The power integral to be calculated is given by equation 2. In this equation dQ/dx is the sum of the derivatives of the loss terms given by equations 10, 11, 12, and 13. In general these are expressed not only as functions of r, k, and C_v but also of dr/dx, dK/dT, dC_v/dT, dT/dx, and d^2T/dx^2. These are readily generated from information already available at this point. The power integral includes the power required for refrigeration in the bottom expansion volume. (6) J = 1 means this is the first pass for a particular value of I and thus, before proceeding, a variation must be made in T so as to have two values of power for comparison. The process for varying T, depicted by the box here, involves addition of a fixed increment to one coefficient. Actually, to fulfill the requirement of equation 19 (boundary condition for temperature), an addition to one coefficient must be balanced by an equal reduction in one or more of the other coefficients. This will be discussed in more detail shortly. (7 and 8) Here the power just calculated in step 5 is compared with the value calculated in the previous pass. If this new value is smaller, the variation is pursued for another pass. If the new power is larger either the current round of variations has been played out for this particular value of I or this is the second pass and the variation should be reversed to see whether reduced power is found for the opposite variation. If J = 2 (i.e. second pass), it should be clear that the reversed variation must take one back beyond the first pass selection. We do this by reversing the increment and doubling its value of this one pass. Also, once the variation is reversed, the program should retain that sign for subsequent passes so that a yes answer at step 7 results in a further variation in the profitable direction. (9) Once the variations for a particular coefficient I are completed (no further reduction in power) the program moves on. In making this step the power (and the coefficient value associated with it) must be the lowest value just found. This simply means that one must go back to the previous pass since the logic has forced exit from step 8 after an increase in the power. In step 9, a test is done to see if all coefficients have been varied. If not the next one is selected and the process restarted. (10) If all of the coefficients have been varied, this set is compared with the set from the previous pass (the initial set if this is the first time through to this point). If the two sets are not identical, the whole process is rerun to generate another set of coefficients. This process is repeated until the coefficients do not change with further variations. (11) At this point a further refinement of the result can be achieved by a reduction of the increment used in the variational process. The integration increment used in step 5 could also be reduced to improve that process. Then, with the last set of temperature coefficients as a starting point, the entire procedure could be repeated until some preset measure of convergence is met. We have found that the computer costs at this stage are not negligible and have chosen to run through the whole process no more than 2 or 3 times. In general, the degree of convergence has generally been quite acceptable where a variational increment of 1.5 K and 50 steps per integral was used. (12) The final step involves output and is highly subjective in form. We have chosen to generate plots of T, r, mC_p, Q, and W (the power integral) as functions of x. The Q and W output can be broken down into the separate loss terms to show the relative importance of the different losses along the displacer.

The rate of convergence on the solution depends upon the exact form of the variation in step 6. We have tried only two. First (to satisfy equation 19), we have simply added to one coefficient and subtracted from the adjacent one. While this works sometimes, it can lead to difficulty and we have achieved better results including faster convergence with another simple procedure. The increment which is added to one coefficient is simply divided by 7 and that amount is subtracted from every coefficient. This procedure more clearly emphasizes the one coefficient. The program could be improved (faster convergence) by more careful consideration of the form of the variation.

At times it has been difficult to arrive at a satisfactory initial guess for the coefficients. Without the regenerator term almost any simple guess works (e.g. T_1 = 290 K and all others are zero). But when the regenerator term is included this is not the case. The solution of equation 20 for the bottom end radius becomes highly dependent upon dT/dx at x = 0. If this derivative is too large there is no real, positive solution to the cubic equation 20 which simply means that the selected temperature derivative is inconsistent with any realizable refrigerator. For sufficiently small values of dT/dx at x = 0, two real, positive solutions are found and we always select the smaller of the two. The program can potentially run into unrealistic situations on its own. This happens particularly where the initial temperature profile is dramatically removed from the final one or as the calculation is pushed into regions where realizable results simply aren't expected for other reasons. For example, as the length of the displacer (an externally fixed constant) is reduced, the power requirement increases without bound and it becomes increasingly difficult to find any initial temperature profile which does not cause problems. Another type of problem arises when dT/dx goes to zero at any point along the displacer. Here, equations 20 and 21 become indeterminant, that is, no solution can be found for r (or r_0). The net effect of these considerations is a need to build certain tests into the program to assure proper exit and diagnostics when an unrealistic solution is encountered.

RESULTS OF THE CALCULATION

Based on earlier experimental work [1-3] on coolers of this type (stepped rather than tapered displacers), we have selected a set of parameters for a 'standard' low-power cryocooler. These parameters are given in Table I and represent the base from which we vary an individual parameter to study its effect. The remaining figures in the paper refer to this standard set of parameters. The displacer is taken to be hollow (as mentioned in section 3.3) so the table also includes a displacer wall thickness as well as a sleeve wall thickness.

The thermal conductivity is that of spun glass epoxy (designated G-10 by the National Electronic Manufacturers Association), which we represent by

$$k = 5.7 \times 10^{-2} + 5.03 \times 10^{-3}T - 2.02 \times 10^{-5}T^2 + 3.6 \times 10^{-8}T^3 ,$$

where the units are W/K·m. Similarly, the specific heat of the G-10 material is represented by

$$C_v = -1.36 \times 10^4 + 4.4 \times 10^3 T + 17.9T^2 - 8.72 \times 10^{-2} T^3 + 1.3 \times 10^{-4} T^4,$$

where the units are J/km^3.

Figure 3 shows the results for $T_o = 5$ K, a lower temperature than might be justified by the assumptions, but which emphasizes the effects of inclusion of the regenerator loss term. The upper two boxes show the optimized temperature profile and displacer shape in cases both including and excluding regenerator loss. In the bottom box we present the work integral (equation 2) for each loss term as a function of position along the displacer. Each point on one of these curves represents the total drive power required to take care of the particular loss term below the associated value of x. The curves in this lower box are associated with the profiles which include regenerator loss in the upper two boxes.

Several points should be noted. First, while regenerator losses are only significant at the cold end, radiation losses can be effectively intercepted at higher temperature. The inclusion of regenerator loss forces an elongated narrow end to the displacer which might be approximated by two conical sections. The temperature profile is similarly distorted. The importance of shuttle heat transfer is clear and its similarity to conductive heat transfer is perhaps surprising.

Note the slight irregularity at the upper end of the displacer for the case which excludes regenerator loss. We often find distortions (both positive and negative) in this region which we attribute to the lack of emphasis given this part of the displacer by the calculation. That is, the contribution to the power integral at the upper end is almost negligible and thus the variations simply emphasize the regions of the displacer which contribute significantly to the work integral. Presumably, such distortion would be eliminated with complete convergence, but with little further change in the power requirement. This is born out in a study of several iterations with successively smaller variational increment (see section 3.4).

Figures 4, 5, 6, and 7 display the effects of altering the stroke, length, frequency, and bottom-end dissipation for $T_o = 10$ K. The curves represent the total power for each loss element (the value of the respective parts of the work integral at the top of the cooler). The refrigerator has been optimized for every value of the independent variable (e.g. stroke) where all the other parameters are maintained at the standard values shown in Table 1.

Several general comments can be made. Except where length is varied, the radiation loss is independent of the other parameters. This simply means that modest variations of the temperature profile have no marked effect on radiation loss. The linear increase of radiation loss with length (figure 5) seems reasonable since radiation surface area is also linearly dependent on length. Regenerator loss is strongly dependent on the expansion volume at the bottom end. For example, as stroke is increased (figure 4) the regenerator loss increases as expected. The increases of regenerator loss with decreasing stroke in the limit of short stroke can be understood as arising from the precipitous increase in bottom end radius in this region. Remember that these curves represent a continuous set of optimized refrigerators, not a single refrigerator of fixed geometry. The regenerator loss dependence on expansion volume is clearly seen in equation 17 where the heat content of the fluid depends on how much gas is present. This plays an important role in the loss term (equation 13).

Shuttle heat transfer and conduction loss dominate these figures with only minor contributions from radiation and regeneration. The strong cross- over of the two leading terms with variation of stroke (figure 4) produces the most notable minimum in total power. The minimum appears to be at about 3 mm rather than the 5 mm value of the standard parameter set. For alterations of length (figure 5) and frequency (figure 6), the most notable effect is a sharp rise in power for small values of these parameters. Conduction and shuttle losses generally increase with increase in the size (radius) of the displacer. This is consistent with their respective dependences on displacer cross sectional area and surface area. The exception to this rule occurs where decrease in stroke results in decreasing shuttle loss even though the displacer size is rising rapidly.

For modest heat dissipation at the cold end, the power requirement rises almost linearly as seen in figure 7. Actually the only loss term which is not essentially linear over this range is regenerator loss. This is the only case where the drive power required for the cold end expansion space becomes really significant, rising to 0.36 watts at a refrigeration power of 10 mW from 0.03 watts with no dissipation. As discussed in section 3.4, the optimization includes the power required for this bottom end cooling, but it is not shown in figures 4, 5, 6, and 7 because it is so small.

Figure 8 shows a comparison between this theory and a stepped displacer cryocooler [3] which fits all of the standard set of parameters (Table I). The quantitative agreement is very good considering the number and nature of the approximations described in section 3.1 and the fact that the comparison is between a tapered and a stepped displacer. We have measured the pressure-volume (P-V) diagram for this 4-stage machine, with the helium pressure adjusted to give 10 K at the cold end. The area of the P-V diagram multiplied by the operating speed gave 2 watts as the mechanical power actually performed on the working fluid. Correcting for non-isothermal compression reduced this to a net power of 1.3 watts. The analytical result is 0.48 watts, surprisingly good agreement considering that the stepped displacer is not the optimum shape and that it is approximately twice as large as the calculated optimum.

CONCLUSIONS AND DISCUSSION

This last comparison with experiment adds confidence to the results of the calculations and suggests that they can provide reliable guidance in the design of such cryocoolers. The insight into the balance between the different loss terms in different parts of the refrigerator may also prove useful. It is reassuring to find that each loss term varies in a manner which can be qualitatively rationalized. The fact that regenerator losses rise sharply in the region below 10 K is consistent with real refrigerator experience. Thus, the model may eventually prove useful in evaluating the real benefits of new regenerator materials prior to tedious experimental tests.

The model could be improved in a number of ways. First, further thought given to the variational method could lead to more rapid convergence and hence savings of computer time. The effects of non-sinusoidal motion and helium mass flow should be more exactly included in the shuttle and regenerator terms to improve their accuracy. The non-ideal properties of the working fluid should be included, a proposition which may not be too difficult since a mechanism for including gas effects (the regenerator term) has already been included. There are, of course, other refinements which might be made since the loss terms rest upon numerous assumptions. Careful study of these assumptions should be used as a guide to such improvements. The Schmidt analysis should probably be included to account for the realistic motion of the displacer in the refrigeration term (equation 15). The fact that the regenerator gap does not go precisely to zero, giving a finite dead volume and increased mass flow should also be included.

As mentioned in the introduction, efficiency is not an adequate measure of performance for low-power cryocoolers such as these. Where we require no refrigeration at all, efficiency is zero. To utilize a refrigerator which has non-zero efficiency providing some finite refrigeration power implies a waste in drive power, since no refrigeration is required by the application.

The simplest and most universal measure of performance for low-power cryocoolers is the absolute value of the power required to run them and maintain a fixed temperature with no dissipation at the cold end. All refrigeration then goes to the interception of heat losses. In principle there is no finite minimum value to this power. In fact, if all heat transfer to some cold volume is reduced to zero, it takes no power at all to maintain that temperature.

While the simple measure of drive power can be broadly applied to intercompare different low power cryocoolers, another measure of performance within particular classes of cryocoolers may prove useful. This is simply to state an efficiency based upon the ratio of the optimum drive power and the actual drive power. Thus, if it takes 50 watts to run the refrigerator shown in figure 8, then the efficiency would be (0.78/50) x 100 = 1.56% since the optimum occurs for a drive power of 0.78 watts. The inefficiency includes not only inefficiencies in the refrigeration process, but also in the compressor and drive machinery. This performance measure could be useful where other constraints are imposed. For example, suppose the application requires the use of many electrical leads to the cold space. These could be included in the analysis and the performance measure thus generated (through the optimization) would relate specifically to the application. This is more reasonable than using absolute drive power since, although no significant refrigeration power is required, the leads do require additional refrigeration along the length of the machine.

The authors would like to acknowledge the assistance of Ms. Diane Ulibarri with the computer code.

TABLE I

'Standard' Refrigerator Parameters

Stroke	= 5 mm	Frequency	= 1 Hz
Length	= 0.5 mm	Radiation Shield Diameter	= 80 mm
Cylinder Wall Thickness	= 3.0 mm	Displacer Wall	= 3.0 mm
Upper Pressure	= 5.7×10^5 Pa	Lower Pressure	= 1.6×10^5 Pa
Ambient Temperature	= 300 K		
Number of Grounded Rad. Shields	= 4	Number of Super-insulation	= 40 Layers*

Power dissipation at bottom end = 0.0 W

* Around each grounded radiation shield.

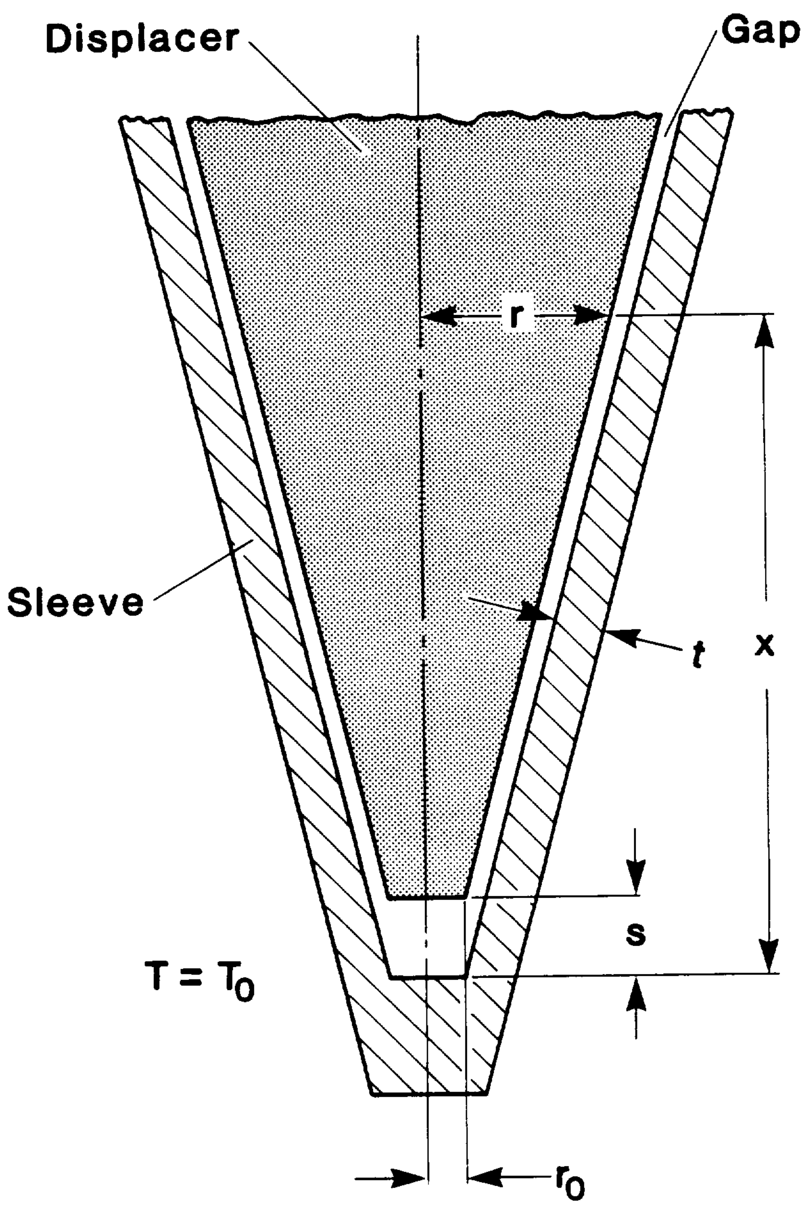

Figure 1. Schematic drawing of the tapered displacer and sleeve. The machine is shown at the top of the stroke, S. x is measured from the bottom end where the temperature is T_o and the radius of the displacer is r. The sleeve wall thickness, t, is constant. While the taper shown here is conical, r is not generally expected to be a linear function of x.

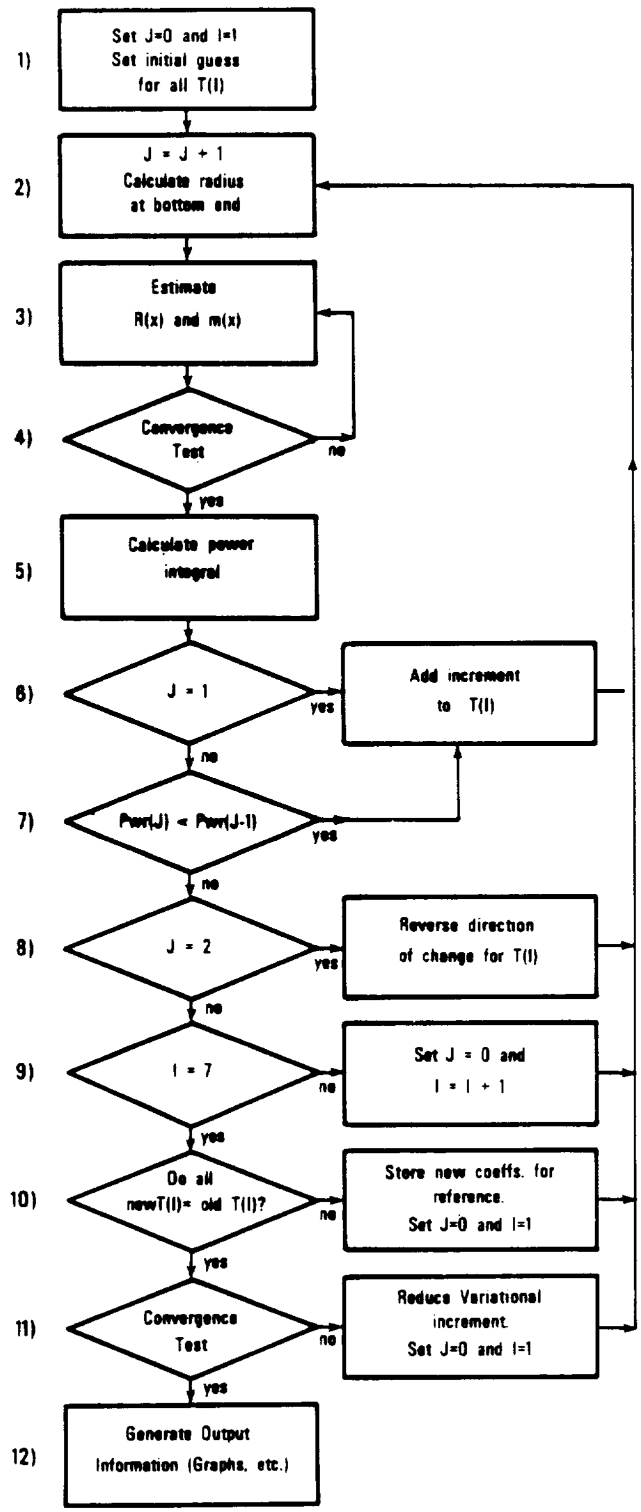

Figure 2. Flow chart for the computer program. The numbers on the left refer to descriptions of the specific steps within the text.

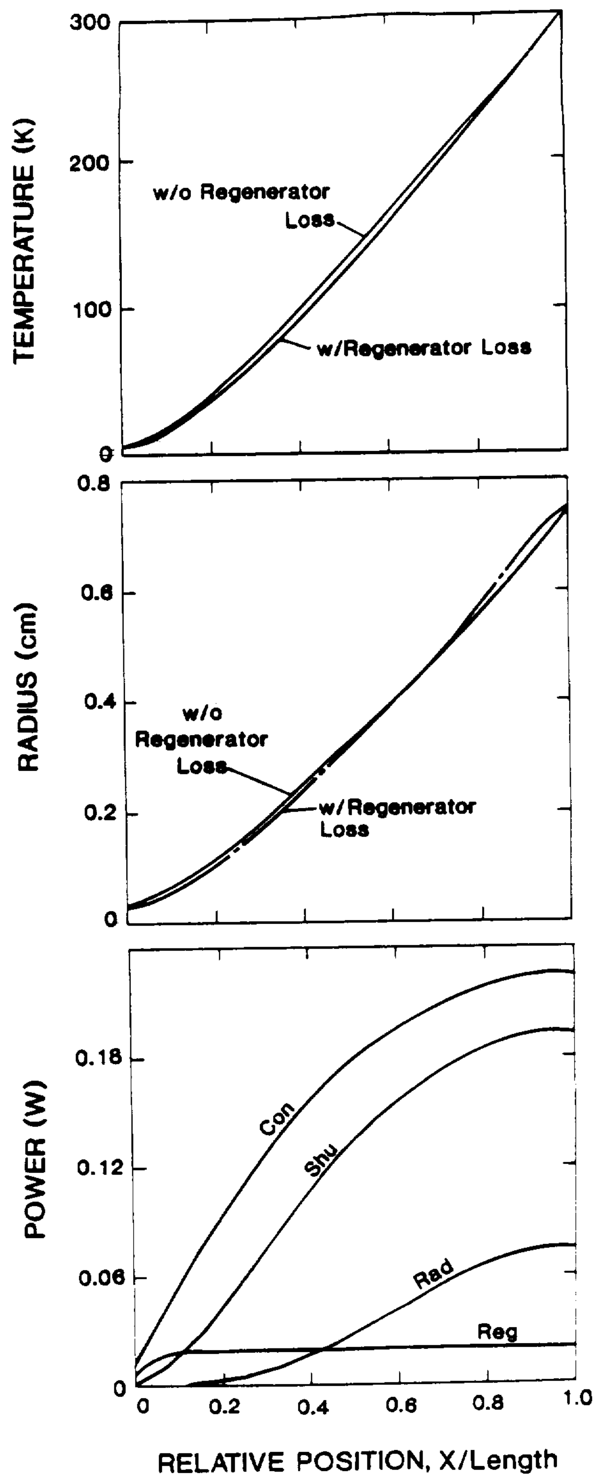

Figure 3. Results of the calculations for T_o = 5 K and the parameters shown in Table I. The curves in the top 2 boxes represent the optimum results both with and without inclusion of the regenerator inefficiency. The bottom box refers to the case including this loss term. The power is that required to take care of the particular loss term (CON = Conduction, SHU = Shuttle, REG = Regenerator, and RAD = Radiation) below the associated value of x.

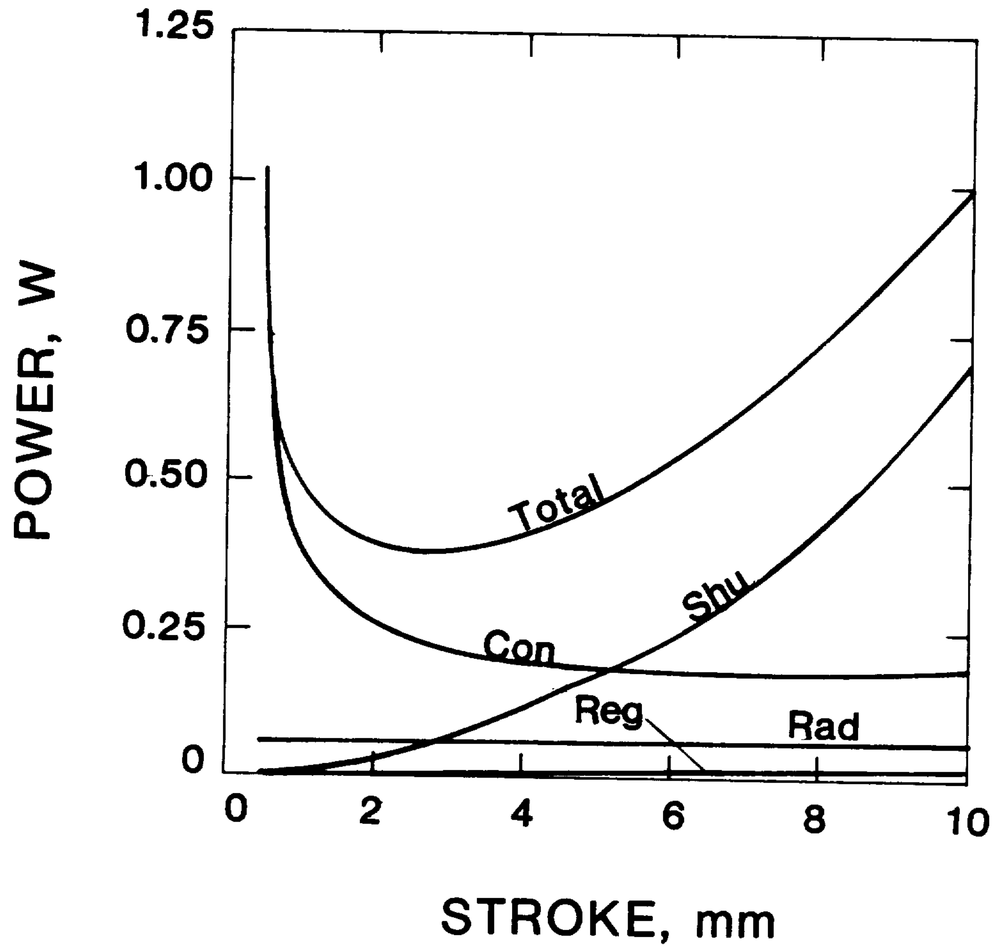

Figure 4. Bottom and top radii for the displacer and power requirement as a function of stroke for T_Q = 10 K and the parameters shown in Table I. The total power which includes the conduction, shuttle, radiation and regenerator terms shown in the figure also includes a small contribution which goes to provide refrigeration in the bottom expansion space. This contribution is negligible on this scale.

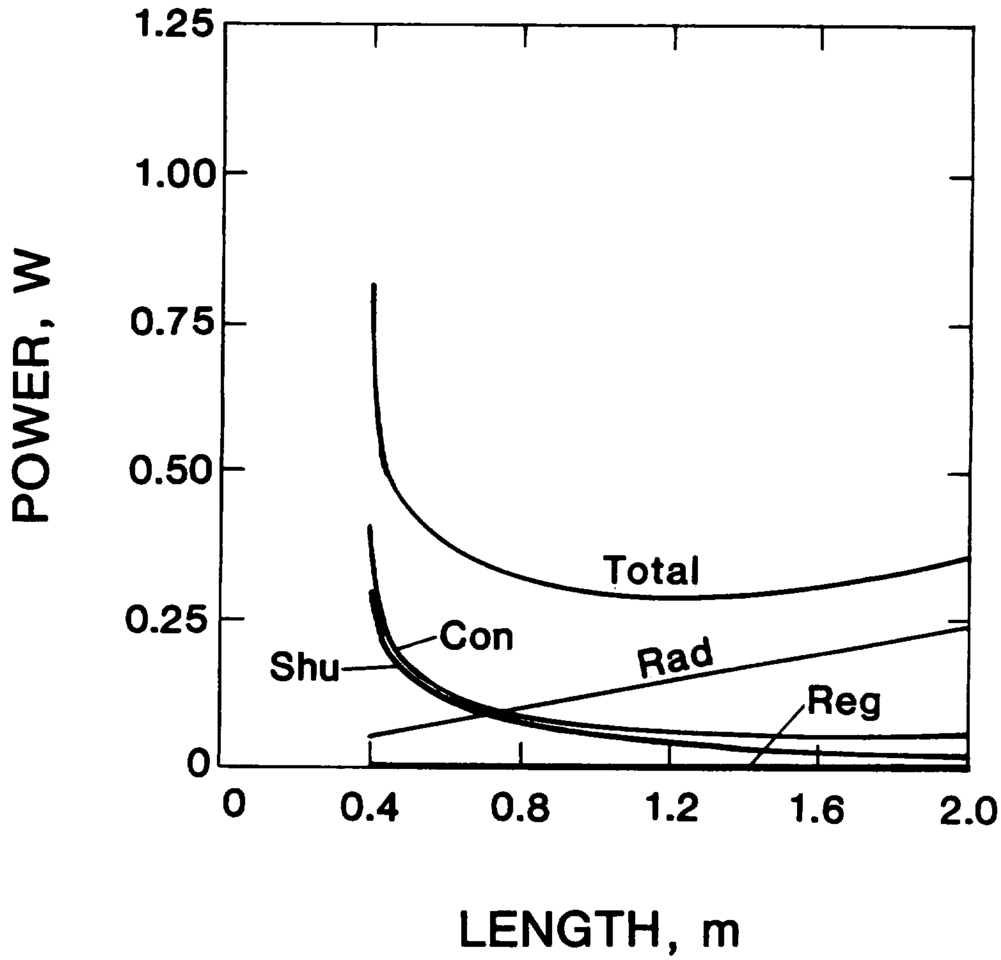

Figure 5. Bottom and top radii for the displacer and power requirement as a function of length for T_o = 10 K and the parameters shown in Table I. The total power which includes the conduction, shuttle, radiation and regenerator terms shown in the figure also includes a small contribution which goes to provide refrigeration in the bottom expansion space. This contribution is negligible on this scale.

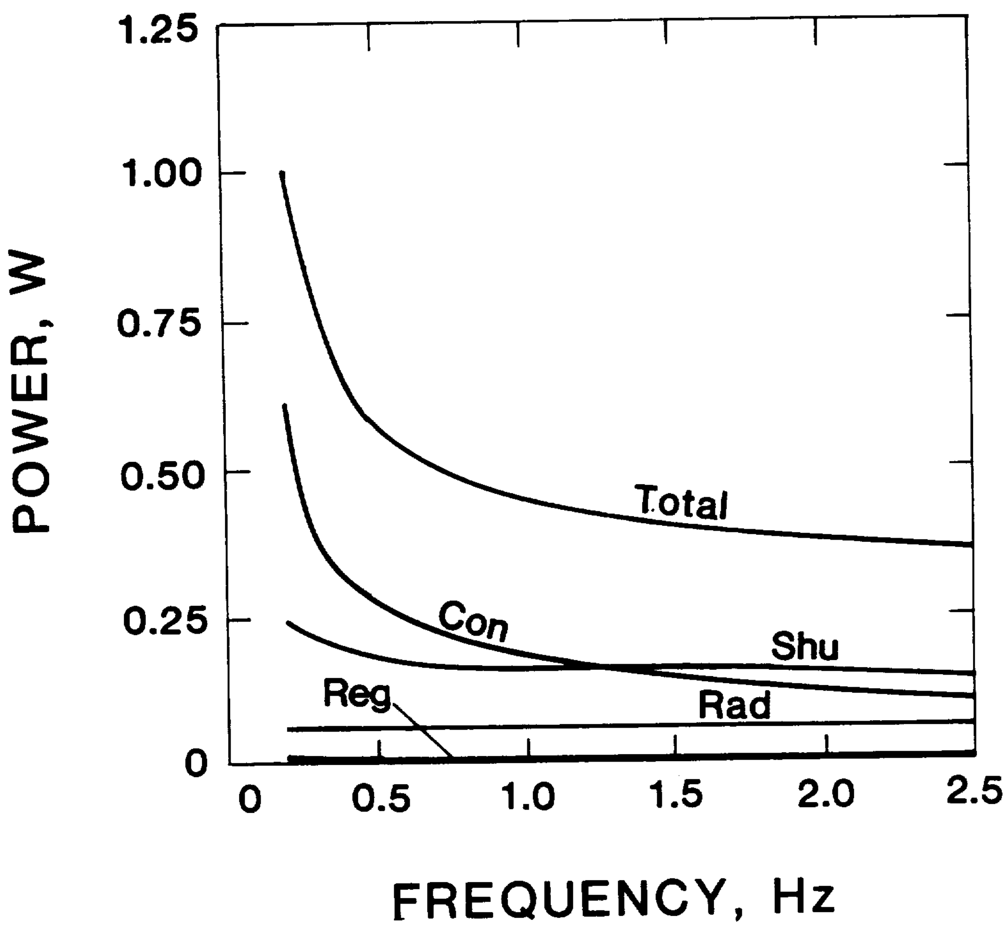

Figure 6. Bottom and top radii for the displacer and power requirement as a function of frequency for T_o = 10 K and the parameters shown in Table I. The total power which includes the conduction, shuttle, radiation and regenerator terms shown in the figure also includes a small contribution which goes to provide refrigeration in the bottom expansion space. This contribution is negligible on this scale.

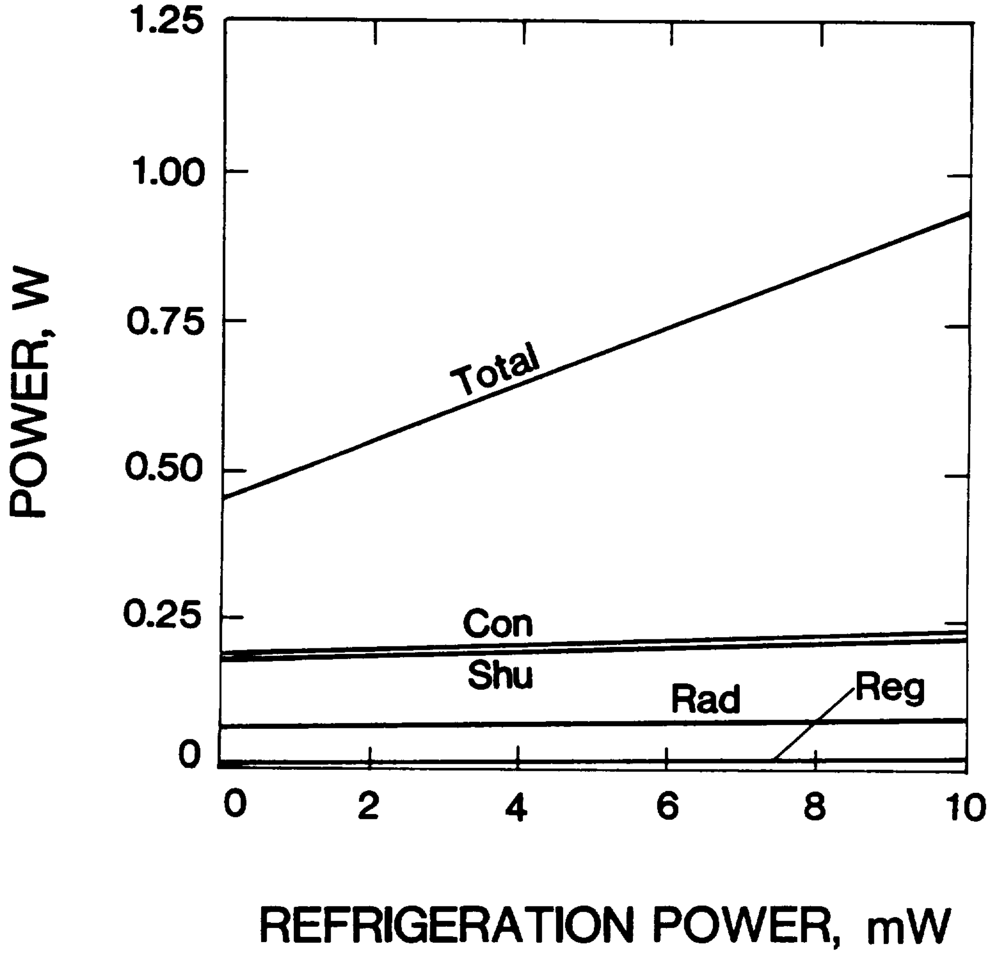

Figure 7. Bottom and top radii for the displacer and power requirement as a function of refrigeration power for T_o = 10 K and the parameters shown in Table I. The total power which includes the conduction, shuttle, radiation and regenerator terms shown in the figure also includes a small contribution which goes to provide refrigeration in the bottom expansion space. This contribution is negligible at refrigeration power = 0, but rises to 0.36 watts at 10 mW (the right intercept).

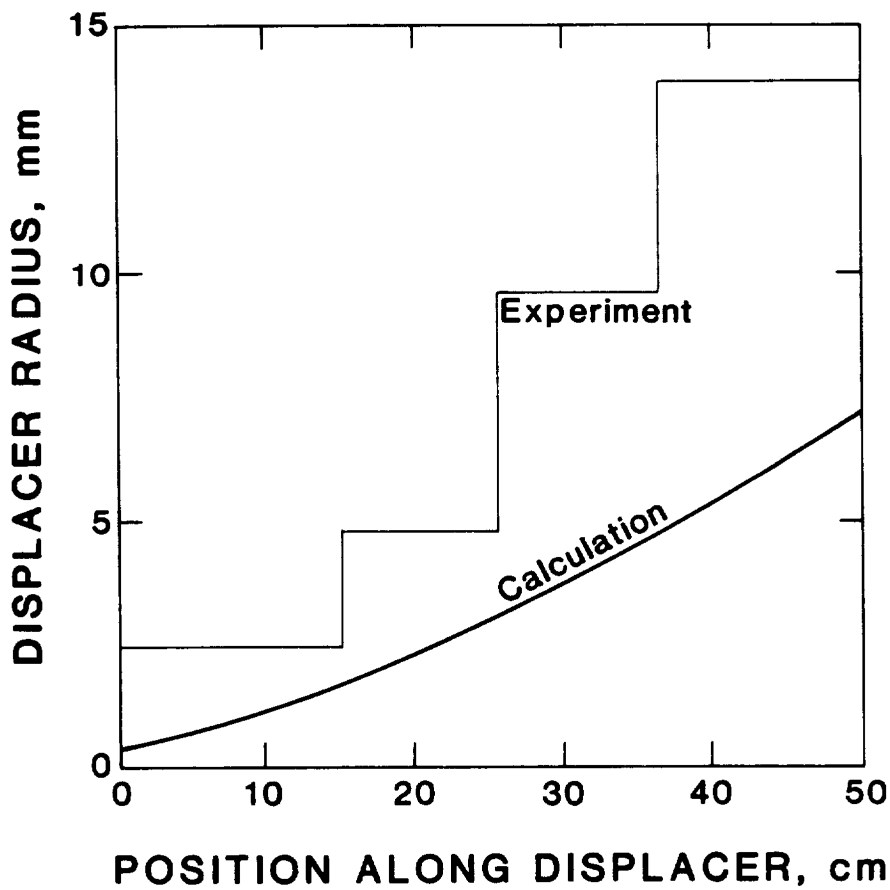

Figure 8. Comparison of a stepped displacer cryocooler with the calculation. The values of the parameters used in the calculation and which describe the refrigerator are shown in Table I.