INFLUENCE OF PARAMETER CHANGES TO STABILITY BEHAVIOR OF ROTORS

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SUMMARY

The occurence of unstable vibrations in rotating machinery requires corrective measures for improvements of the stability behavior. Before a measure will be realized, different possibilities should be investigated. In this paper a simple approximate method is represented, to find out the influence of parameter changes to the stability behavior. The method is based on an expansion of the eigenvalues in terms of systemparameters. Influence coefficients show in a very intuitive way the effect of structural modifications. The method first of all was applicated to simple nonconservative rotor models. Furthermore it was approved for an unsymmetric rotor of a test rig.

INTRODUCTION

The occurence of instability in rotating machinery may be caused by different effects, for example oil film forces in journal bearings, forces in seals, unsymmetric shafts, internal damping etc. A machine designer wants to know, whether a rotor will run stable during operation and what size the stability threshold speed will have. Furthermore he needs information about the parameters influencing the stability behavior.

Important informations about the stability of a linear rotor system can be obtained by calculations in the design stage. Because of uncertain input data for calculation, the results have to be considered critically. It is possible, that unstable vibrations may occur during operation, although the calculation was predicting a stable machine. On the other hand parameters may change during operation leading to increasing oscillations. In such cases suitable corrective measures for improvements of the stability behavior are required. Before a measure will be realized, different possibilities should be investigated finding out the simplest and most effective one. For that it would be very useful, to have approximate formulas, expressing the sensitivity of the dynamic behavior to changes of system parameters.

In linear rotor systems with nonconservative effects the stability can be valuated by means of the system eigenvalues. The real parts of the eigenvalues determine, whether the natural motion is decreasing or increasing. If the variations of eigenvalues caused by variations of systemparameters (mass-, damping-, stiffness-coefficients) are known, an estimation of system modifications to the stability behavior is possible. Such sensitivities, respectively influence coefficients, expressing the change of eigenvalues to changes of systemparameters are presented here.

Lund (ref. 1) has developed a method to calculate sensitivities of the critical speeds (eigenfrequencies) of a conservative rotor to changes in the design. Dresig (ref. 2) gives a more general development for conservative mechanical structures. The basic idea in his method is an expansion of the eigenvalues in terms of the system parameters. In Taylor's expansion derivatives of the eigenvalues to the system parameters are needed. Such derivatives were developed from Plaut and Huseyin (ref. 3) and from Fox and Kapoor (ref. 4).

Based on Taylor's expansion for complex eigenvalues in this paper an improvement of the method is presented for rotor systems with nonconservative mechanisms (oil film bearings, seals ets.) Linear, quadratic or higher order formulas are obtained, depending on the order of the derivatives taken into consideration in the expansion. With the linear formula very simple influence coefficients can be defined, pointing out the influence of special parameters to an eigenvalue. Superposition of different parameter changes is possible in this special case. Using the formulas the eigenvalues, the left-hand and right-hand eigenvectors of the original system (without parameter changes) must be known.

The application of the method is demonstrated for simple nonconservative rotor models, investigating the effects of stiffness and damping coefficients to the stability behavior. Furthermore the influence of the mass variation to the eigenvalues of an unsymmetric rotor is determined. The predicted vibration behavior caused by parameter variations could be confirmed by measurements.

NATURAL VIBRATIONS OF LINEAR ROTORS

A turbomachine consisting of a high pressure turbine, a generator and an exciter is shown in figure 1. The power of the machine is 110 MW and the operating speed 3000 rpm. Contrary to nonrotating structures the dynamic behavior of such rotating machines is influenced by additional effects. Of great importance are selfexciting and damping effects (nonconservative effects), caused by the oil film forces of journal bearings, forces in seals etc.

In linear rotor dynamics the natural vibrations can be described by linear equations of motion, usually derived by means of the finite element method.

$$\underline{M}\ddot{\underline{\tilde{q}}} + \underline{C}\dot{\underline{\tilde{q}}} + \underline{K}\underline{\tilde{q}} = \underline{0} \qquad (1)$$

$\underline{M}$ mass matrix (order NxN)

$\underline{C}$ damping matrix (order NxN)

$\underline{K}$ stiffness matrix (order NxN)

$\underline{\tilde{q}}$ displacement vector

The equations express the equilibrium of inertia, damping, and stiffness forces. Because of the nonconservative effects damping and stiffness matrix contain also skewsymmetric and nonsymmetric terms besides the symmetric ones. Furthermore some of the matrix elements depend on the operating conditions of the machine (speed, power, pressure etc.).

Investigating the natural motion (stability) of a rotor equation (1) has to be solved. The solution has the form

$$\underline{\tilde{q}}(t) = \underline{q}\, e^{\lambda t} \tag{2}$$

Substitution yields the quadratic eigenvalue problem

$$\{\lambda^2 \underline{M} + \lambda \underline{C} + \underline{K}\}\, \underline{q} = \underline{0} \tag{3}$$

with 2N eigenvalues λ_n and corresponding eigenvectors (natural modes) q_n. The eigenvalues as well as the eigenvectors mainly occur in conjugate complex pairs

$$\text{eigenvalues:} \quad \lambda_n = \alpha_n + i\omega_n \,, \quad \bar{\lambda}_n = \alpha_n - i\omega_n \tag{4}$$

$$\text{eigenvectors:} \quad \underline{q}_n = \underline{s}_n + i\underline{t}_n \,, \quad \underline{\bar{q}}_n = \underline{s}_n - i\underline{t}_n \tag{5}$$

The part of the solution, which belongs to such a conjugate complex pair, can be written as

$$\underline{\tilde{q}}_n(t) = B_n\, e^{\alpha_n t}\, \{\underline{s}_n \sin(\omega_n t + \gamma_n) + \underline{t}_n \cos(\omega_n t + \gamma_n)\} \tag{6}$$

ω_n is the circular natural frequency and α_n the damping constant (decay constant). The damping constant, respectively the real part of the eigenvalue determines, whether the solution $\underline{\tilde{q}}_n(t)$ decreases ($\alpha_n < 0$) or increases ($\alpha_n > 0$).

The four lowest natural frequencies and the corresponding damping constants of the turbomachine (figure 1) are plotted versus the running speed in figure 2. The eigenvalues are changing with speed, especially the damping constants α_1 and α_3. The diagram shows that the rotor instability onset speed is 3400 rpm determined by the zero value of α_1. The relative distance between the instability onset speed and the operating speed is very low. Therefore unstable vibrations may occur if additional destabilizing forces are acting on the rotor. If such calculated results are known in a machines design stage, corrective measures improving the stability behavior should be arranged. If instability occurs in operating machines similar problems have to be solved. Approximate formulas, respectively influence coefficients expressing the influence of parameter changes to the eigenvalues (stability behavior) may be very useful for the above mentioned requirements.

INFLUENCE OF PARAMETER CHANGES TO THE STABILITY BEHAVIOR

Eigenvalue Problem and Modal Parameters

Our aim is to find formulas for an approximate calculation of eigenvalue variations caused by system modifications (parameter changes). Besides the designed parameter changes this formulas will contain modal parameters of the initial dynamic system, that means eigenvalues λ, left-hand eigenvectors $\underline{\ell}$ and right-hand eigenvectors $\underline{q}$ of the system without parameter changes. Normally they are known from calculations (ref. 5).

Complementary to the eigenvalue problem (eq. (3)) the corresponding left-hand-eigenvalue problem with transposed matrices

$$\{\lambda^2 \underline{M}^T + \lambda \underline{C}^T + \underline{K}^T\} \; \underline{\ell} = \underline{0} \qquad (7)$$

has the same eigenvalues λ, but the mentioned left-hand eigenvectors ℓ. Working with the procedure of inverse vector iteration for complex eigenvalues (ref. 5) a desired number of the modal parameters λ, $\underline{q}$, $\underline{\ell}$, can be calculated effectively. For further derivations we suppose that the interesting modal parameters of the initial system are given.

Taylor's Expansion for the Eigenvalues

The basic idea of the approximate method is an expansion (Taylor's series) of the eigenvalues in terms of the generalized system parameters p_k $(k = 1,2, \ldots K)$

$$\lambda_n = \lambda_{no} + \sum_r \frac{1}{r!} \{ \frac{\partial}{\partial p_1}\Big|_o \Delta p_1 + \frac{\partial}{\partial p_2}\Big|_o \Delta p_2 + \ldots \frac{\partial}{\partial p_K}\Big|_o \Delta p_K \}^r \lambda(p_1, p_2, \ldots p_K) \qquad (8)$$

in which the p_k may be mass, damping, or stiffness parameters. λ_n is the changed eigenvalue after a parameter variation. λ_{no} is the corresponding eigenvalue of the initial system. The derivatives of the eigenvalues to the system parameters are expressed in operational notation. r is the order of the derivative.

Truncating equation (8) after a desired derivative, the changed eigenvalue may be calculated approximately. Easy to handle is a linear formula with first derivatives. A more exact expression can be obtained if an additional quadratic term is taken into account.

Linear Approximate Formula

Taylor's expansion (eq. (8)), truncated after the first derivatives leads to the following linear approximate equation (9)

$$\lambda_n = \lambda_{no} + \left.\frac{\partial\lambda_n}{\partial p_1}\right|_o \Delta p_1 + \left.\frac{\partial\lambda_n}{\partial p_2}\right|_o \Delta p_2 + \dots \left.\frac{\partial\lambda_n}{\partial p_K}\right|_o \Delta p_K = \lambda_{no} + \sum_k \left.\frac{\partial\lambda_n}{\partial p_k}\right|_o \Delta p_k \quad (9)$$

For determination of λ_n, respectively $\Delta\lambda_n = \lambda_n - \lambda_{no}$ the first partial derivatives $\partial\lambda_n/\partial p_k$ are required.
Using the standard subscript notation for partial derivatives

$$\lambda_{n,k} = \frac{\partial\lambda_n}{\partial p_k} \quad q_{n,k} = \frac{\partial q_n}{\partial p_k} \quad \underline{M}_{,k} = \frac{\partial \underline{M}}{\partial p_k} \quad \underline{C}_{,k} = \frac{\partial \underline{C}}{\partial p_k} \quad \underline{K}_{,k} = \frac{\partial \underline{K}}{\partial p_k} \qquad (10)$$

the differentiation of equation (3) with respect to p_k yields

$$\{\lambda_n^2 \underline{M} + \lambda_n \underline{C} + \underline{K}\} \underline{q}_{n,k} + \{2\lambda_n \lambda_{n,k} \underline{M} + \lambda_{n,k} \underline{C} + \lambda_n^2 \underline{M}_{,k} + \lambda_n \underline{C}_{,k} + \underline{K}_{,k}\} \underline{q}_n = 0 \qquad (11)$$

We premultiply equation (11) by ℓ_n^T to obtain the scalar expression

$$\underline{\ell}_n^T \{\lambda_n^2 \underline{M} + \lambda_n \underline{C} + \underline{K}\} \underline{q}_{n,k} + \underline{\ell}_n^T \{2\lambda_n \underline{M} + \underline{C}\} \underline{q}_n \lambda_{n,k}$$

$$= - \underline{\ell}_n^T \{\lambda_n^2 \underline{M}_{,k} + \lambda_n \underline{C}_{,k} + \underline{K}_{,k}\} \underline{q}_n \qquad (12)$$

The first term in this equation is zero, it represents the left-hand eigenvalue problem, multiplied with $q_{n,k}$. Furthermore it is assumed that $\underline{q}_n$ and $\underline{\ell}_n$ are normalized in a way to satisfy the relation

$$\underline{\ell}_n^T \{2\lambda_n \underline{M} + \underline{C}\} \underline{q}_n = 1 \qquad (13)$$

It follows that

$$\lambda_{n,k} = \frac{\partial\lambda_n}{\partial p_k} = - \ell_n^T \{\lambda_n^2 \underline{M}_{,k} + \lambda_n \underline{C}_{,k} + \underline{K}_{,k}\} \underline{q}_n \qquad (14)$$

Substitution of this result into equation (9) yields

$$\Delta\lambda_n = (\lambda_n - \lambda_{no}) = - \Sigma \; \underline{\ell}_n^T \{\lambda_{no}^2 \underline{M},_k + \lambda_{no} \underline{C},_k + \underline{K},_k\} \underline{q}_n \, \Delta p_k = \sum_{k=1}^{K} g_{nk} \, \Delta p_k \qquad (15)$$

with the influence coefficients g_{nk}.
Finally we introduce Δ-matrices

$$\Delta\underline{M} = \sum_{k=1}^{K} \Delta\underline{M}_k = \sum_{k=1}^{K} \underline{M},_k \Delta p_k$$

$$\Delta\underline{C} = \sum_{k=1}^{K} \Delta\underline{C}_k = \sum_{k=1}^{K} \underline{C},_k \Delta p_k \qquad (16)$$

$$\Delta\underline{K} = \sum_{k=1}^{K} \Delta\underline{K}_k = \sum_{k=1}^{K} \underline{K},_k \Delta p_k$$

expressing the change of mass, damping, and stiffness matrices and we obtain the linear expression

$$\Delta\lambda_{n\,LIN} = (\lambda_n - \lambda_{no})_{LIN} = - \underline{\ell}_n^T \{\lambda_{no}^2 \Delta\underline{M} + \lambda_{no} \Delta\underline{C} + \Delta\underline{K}\} \, \underline{q}_n \qquad (17)$$

This approximate equation is a good tool for calculation in many cases, pointing out the influence of parameter changes to the stability behavior of rotors. As above mentioned for application of the formula the modal parameters of the initial system are needed besides the parameter changes. Derivatives of the eigenvectors do not appear in equation (17).

Influence Coefficients

In equation (15) we have defined influence coefficients g_{nk}

$$g_{nk} = \left.\frac{\partial \lambda_n}{\partial p_k}\right|_o = - \underline{\ell}_n^T \{\lambda_{no}^2 \underline{M},_k + \lambda_{no} \underline{C},_k + \underline{K},_k\} \, \underline{q}_n \qquad (18)$$

This differential quotient, respectively differential sensitivity yields the influence on the eigenvalue λ_n of an infinitesimal change of a particular parameter p_k. It can be used as a meaningful approximation for finite parameter changes, when the considered modifications are relatively small.

$$g_{nk} \approx \frac{\Delta\lambda_n}{\Delta p_k} = \frac{\Delta\alpha_n}{\Delta p_k} + i \, \frac{\Delta\omega_n}{\Delta p_k} \qquad (19)$$

Influence coefficients are complex numbers. They depend on the modal parameters of the initial system.

Changing the stiffness k_k of a spring with one end fixed and the other end free (see Table 1), we obtain with $\underline{M}_{,k} = 0$ and $\underline{C}_{,k} = 0$

$$g_{nk} = - \underline{\ell}_n^T \underline{K}_{,k} \underline{q}_n \tag{20}$$

The derivation $\underline{K}_{,k}$ leads to a unity main diagonal element $K_{kk} = 1$ with all other elements zero and therefore the influence coefficient of the spring is

$$g_{nk} = - \ell_{nk} q_{nk} = - (\ell_k q_k)_n \tag{21}$$

Table 1 contains further influence coefficients for the most important elements in rotor dynamics like springs, dampers, masses, oil film bearings and seals.

Quadratic Approximate Formula

Taking into consideration the quadratic terms in Taylor's expansion too, we obtain the following expression

$$\lambda_n = \lambda_{no} + \sum_{k=1}^{K} \left.\frac{\partial \lambda_n}{\partial p_k}\right|_o \Delta p_k + \frac{1}{2} \sum_{k=1}^{K} \sum_{\ell=1}^{K} \left.\frac{\partial^2 \lambda_n}{\partial p_k \partial p_\ell}\right|_o \Delta p_k \Delta p_\ell \tag{22}$$

in this improved formula the second derivatives $\lambda_{,k\ell}$ are needed besides the first derivatives and the given parameter changes. The determination of the second derivative is shown in the appendix. The expression is more complicated than the first derivative, but all quantities can be determined from the initial system.

PARAMETER CHANGES AT SIMPLE ROTOR MODELS

We applicate the method at simple rotor models to show the influence of particular parameter changes to the stability behavior. Naturally these simple examples can be solved also exactly, on the other hand they are suitable to demonstrate the approximate formulas. A comparison of the results with exact solutions is possible.

Rigid Rotor with Flexible Supports and Clearance Excitation

The first example is an elastically supported rigid rotor with mass m (fig. 3). The supports have the stiffnesses k_{11} in horizontal direction and k_{22} in vertical direction. A clearance excitation force acts in the middle of the rotor. For instance such excitation forces appear in steam turbines. They result from the unsymmetrical fluid flow through the radial clearances at rotor and blading which appears according to the eccentricity between rotor and casing. The clearance excitation force acts rectangular to the displacement direction (fig. 3).

$$\begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = \begin{bmatrix} o & -k \\ k & o \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} \qquad (23)$$

The coefficient k depends on the power of a turbomachine, increasing with power.

Investigating the translatory natural motion we employ the coordinates q_1 and q_2 .

Equations of Motion, Eigenvalues and Eigenvectors

The equations of motion for the rigid rotor with elastic supports are

$$\begin{bmatrix} m & o \\ o & m \end{bmatrix} \begin{bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} 2k_{11} & k \\ -k & 2k_{22} \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = 0 \qquad (24)$$

or with the definitions

$$\begin{aligned} &\omega_o^2 = 2k_{11}/m \qquad \tau = \omega_o t \qquad (\ )' = d(\ )\ /\ d\tau \\ &\gamma\ = k_{22}/k_{11} \qquad \text{stiffness ratio of support} \\ &\beta\ = k/2k_{11} \qquad \text{dimensionless clearance excitation coefficient} \end{aligned} \qquad (25)$$

in dimensionless form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{q}_1'' \\ \tilde{q}_2'' \end{bmatrix} + \begin{bmatrix} 1 & \beta \\ -\beta & \gamma \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = 0 \qquad (26)$$

The corresponding eigenvalue problem is ($\mu = \lambda/\omega_o$)

$$\begin{bmatrix} 1+\mu^2 & \beta \\ -\beta & \gamma+\mu^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \qquad (27)$$

with the characteristic equation

$$\mu^4 + (1+\gamma)\,\mu^2 + \gamma+\beta^2 = 0 \qquad (28)$$

we obtain the solutions $Z = \mu^2$ from

$$Z_{1,2} = -\frac{1+\gamma}{2} \pm \sqrt{\frac{(1-\gamma)^2-4\beta^2}{4}} \,, \qquad (29)$$

respectively the four eigenvalues μ_n as a function of γ and β.

Figure 4 shows the real part and imaginary part of the essential eigenvalue with positive damping constant. The system is unstable within the range of $(1-2\beta) \leq \gamma \leq (1+2\beta)$. This range increases with increasing values of the clearance excitation coefficient β. Isotropy ($\gamma = 1$) of the supports is the most disadvantageous case. An increase of the anisotropy stabilizes the rotor system. For $(1-2\beta) \geq \gamma \geq (1+2\beta)$ the real part of the eigenvalue is zero and the natural frequencies are split.

The amplitude ratio of the right-hand eigenvector corresponding to an eigenvalue μ can be determined with equation (27), the left-hand eigenvector has the opposite sign

$$\frac{q_2}{q_1} = -\frac{1+\mu^2}{\beta} \,, \quad \frac{\ell_2}{\ell_1} = \frac{1+\mu^2}{\beta} \qquad (30)$$

Parameter Variations

To demonstrate the approximate formulas we choose an initial system with parameters $\gamma = 1.1$ and $\beta = 0.1$ (fig. 5) and investigate variations of γ, β and an additional damping ΔD. With normalized eigenvectors the particular expressions for eigenvalue changes are

-variation of γ with increment $\Delta\gamma$

$$\Delta\mu = -\,[\ell_1, \ell_2] \begin{bmatrix} o & o \\ o & \Delta\gamma \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\,\ell_2 q_2\, \Delta\gamma$$

-variation of β with increment $\Delta\beta$

$$\Delta\mu = -\,[\ell_1,\ \ell_2] \begin{bmatrix} o & \Delta\beta \\ -\Delta\beta & o \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\,(\ell_1 q_2 - \ell_2 q_1)\,\Delta\beta$$

-additional modal damping $\Delta D = c/2m\omega_o$

$$\Delta\mu = -\,\mu\,[\ell_1,\ \ell_2] \begin{bmatrix} 2\Delta D & o \\ o & 2\Delta D \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = -\,2\mu\,(\ell_1 q_1 + \ell_2 q_2)\,\Delta D$$

The linear approximate formula points out the stabilizing effect of anisotropic supports ($\Delta\gamma$ positive in fig. 5b). Figure 5a shows an increase of frequency caused by stiffening the system from γ to $\gamma + \Delta\gamma$.

For special cases the linear approximate formula has disadvantages, for instance in the isotropic case ($\gamma = 1$) with horizontal tangent line. Then better results will be obtained with the quadratic formula (fig. 5b).

The destabilizing effect of increasing clearance excitation coefficient β and the stabilizing effect of additional damping ΔD are presented in figure 5c.

Rigid Rotor in Journal Bearings

Figure 6 shows a rotor of a test rig. Compared to the bearings the stiffness of the shaft is very high, therefore it can be regarded as a rigid rotor supported in two equal bearings (cylindrical bearings B/D = 0.8), rotating with the rotational speed Ω. The rotor has a mass m = 72 kg, the distance between the bearings is 660 mm. The statical bearing load F_{stat} is equal to 353 N. Each journal has a diameter of 50 mm, the values for the radial clearance and for the oil viscosity are Δr = 210 μm, respectively 3.35 Ns/m^2.

The dynamic behavior of journal bearings can be characterized by four stiffness - and four damping coefficients k_{ik} and c_{ik}, respectively by the nondimensional quantities

$$\gamma_{ik} = k_{ik}\,\frac{\Delta r}{F_{stat}}\,,\quad \beta_{ik} = c_{ik}\,\frac{\Delta r\,\Omega}{F_{stat}} \tag{31}$$

They are functions of the Sommerfeld number. The pure translatory motion can be described by means of the coordinates q_1 and q_2.

Equations of Motion, Eigenvalues and Eigenvectors

The following equations of motion for a rigid rotor in two journal bearings describe the equilibrium of forces in the case of pure translatory motion and without external loads.

$$\begin{bmatrix} m & o \\ o & m \end{bmatrix} \begin{bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \end{bmatrix} + 2 \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} + 2 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = 0 \qquad (32)$$

Defining

$$\omega_o^2 = g/\Delta r, \quad \tau = \omega_o t, \quad w = \Omega/\omega_o, \quad (\ )' = d\ (\ )\ /\ d\tau$$

the equations of motion can be written in the nondimensional form

$$\begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix} \begin{bmatrix} \tilde{q}_1'' \\ \tilde{q}_2'' \end{bmatrix} + \frac{1}{w} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} \tilde{q}_1' \\ \tilde{q}_2' \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \qquad (33)$$

The statement for the natural motions

$$\tilde{q}_i = q_i e^{\lambda t} = q_i e^{\mu\tau}, \quad \mu = \lambda/\omega_o \qquad (34)$$

yields the eigenvalue problem

$$\begin{bmatrix} \gamma_{11} + \frac{\beta_{11}}{w}\mu + \mu^2 & \gamma_{12} + \frac{\beta_{12}}{w}\mu \\ \gamma_{21} + \frac{\beta_{21}}{w}\mu & \gamma_{22} + \frac{\beta_{22}}{w}\mu + \mu^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \qquad (35)$$

The four eigenvalues μ_n are i. g. conjugate-complex quantities. If the eigenvalues are known, the corresponding eigenvectors can be calculated from equation (35).

Figure 7 shows the eigenvalue that causes the instability in dependence of the dimensionless rotational speed w. Also the eigenvalue that belongs to a rotatory motion is plotted, but it has no meaning for the stability behavior of the rotor. The real part of the eigenvalue will come to zero at $w = 2.66$. At this rotational speed the stability threshold is reached. The Sommerfeld number here has the value $So = 0.563$. The corresponding circular natural frequency ω/ω_o is 1.435.

Influence of Parameter Changes at the Stability Threshold

We start from the stability threshold and we investigate how particular parameters effect the stability behavior. Therefore we take the linear approximate formula (17). First we ask about the influence of the single coefficients γ_{ik} and β_{ik}/w. It can be judged by the influence coefficients in Table 1. Figure 8 shows the influence coefficients in the complex plane. For example, one can see that an increase of γ_{12} is labilizing or that an increase of the damping coefficients β_{11}/w and β_{22}/w is stabilizing the rotor motion.

If we change the rotational speed, the Sommerfeld number and the stiffness and damping coefficients also will change. Figure 9 shows the variation of the eigenvalue in dependence of the rotational speed calculated by the approximate formula. The formula figures the tangent line to the eigenvalue curve at the expansion point. An increase of the rotational speed causes instability.

Finally in figure 9 the influence of the oil viscosity is shown. If we increase the oil temperature from 38°C to 44°C, the oil viscosity will decrease. The real part of the eigenvalues becomes negative and there is a stabilizing effect.

ELASTIC ROTOR WITH FLEXIBLE SUPPORTS AND SEAL FORCES

A more complicated rotor-system, an elastic turbopump rotor with flexible supports, an impeller and two plain seals is shown in figure 10. Between the bearings where the basic shaft diameter is 64 mm, the shaft carries the impeller mass of 55 kg. Two plain seals are mounted besides the impeller with a seal length of 40 mm and a radial clearance of 0.3 mm. The dynamic characteristics of the seals are given by the stiffness, damping and inertia coefficients K, k, C, c, m^* (fig. 10). The shaft is supported in two identical bearings, each one having a horizontal stiffness $k_{11} = 1.0\ 10^7$ N/m and a vertical stiffness $k_{22} = .75\ 10^7$ N/m. The distance between the bearings is equal to 1.2 m.

First of all eigenvalues and eigenvectors were calculated for the described original system in a speed range from 2000 rpm to 8000 rpm. In figure 11 two damping constants (decay constants) are plotted versus rotational speed. One of the damping constants crosses the zero axis, pointing out the instability onset speed of 5200 rpm, which is above the operating speed.

In order to improve the stability behavior, some parameter changes were investigated, starting from the initial system with a rotational speed of 5000 rpm. Figure 11 shows the results, especially the influence of seal stiffness and damping coefficients. All coefficients were changed 20 per cent. An increase of K and C stabilizes; an increase of k has a destabilizing effect.

Furthermore it is shown that a positive change of k_{22} in direction to isotropic bearings is disadvantageous.

VARIATION OF THE MASS OF A TEST RIG ROTOR

WITH UNSYMMETRIC FLEXIBLE SHAFT

The test rig rotor shown in figure 12 consists of a flexible shaft with rectangular cross section 8 x 12 mm^2. It is running in two ball bearings which are comparatively stiff. In the middle between the bearings a disk is supported, having a mass of 0.89 kg. The nonrotating shaft has two different natural frequencies 24.6 Hz and 35.1 Hz concerning to the different stiffnesses in two rectangular planes.

It is well known that in rotors with unequal moments of inertia different dynamic effects may occur, for instance vibrations caused by the rotor weight and unstable vibrations (ref. 6). Concerning the stability behavior in this case a variation of the disk mass was investigated by calculations (exact results and approximate formulas) and for control by measurements at the test rig. Theoretical results for the eigenvalues of the undamped shaft are plotted in figures 13a and 13b versus the rotational speed and the mass of the disk. The eigenvalues defined for the rotating coordinate system are either pure imaginary (fig. 13a) or pure positive real (fig. 13b). The last one occur in a rotational speed range between the two natural frequencies of the nonrotating shaft. That means that the rotor motion is unstable in this range.

In order to show the utility of the approximate formula the change of the eigenvalues (rotating system) caused by a positive 25 per cent variation of the disk mass was investigated for the three different running speeds 1200 rpm, 1800 rpm and 2400 rpm. The three figures 14a, 14b, 14c show the variation of the natural frequency, respectively the damping constant, in dependence of the mass. The mass of the original system is 0.89 kg, the changed mass is 1.1 kg. The three diagrams contain the exact, the linear and the quadratic approximate solution. Furthermore the results from measurements are plotted in figures 14a and 14c. There is a good agreement of the results. For 1200 rpm the mass change decreases the frequency (destabilizing effect), for 2400 rpm the mass change increases the frequency (stabilizing effect). A stabilizing effect is also given at the running speed 1800 rpm, where the decay constant α is lowered by the mass change.

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APPENDIX

Calculation of the Second Derivatives of the Eigenvalues

First we have to express the right- and lefthand eigenvalue problem (eq. 3), (eq. 7) in the following form:

$$(\underline{A} - \lambda_n \underline{B})\, \underline{x}_n = 0 \qquad \text{(A 1)}$$

$$\underline{y}_n^T\, (\underline{A} - \lambda_n \underline{B}) = 0 \qquad \text{(A 2)}$$

where the matrices $\underline{A}$, $\underline{B}$ and the vectors $\underline{x}_n$ and $\underline{y}_n$ are defined as

$$\underline{A} = \begin{bmatrix} \underline{M} & \underline{0} \\ \underline{0} & -\underline{K} \end{bmatrix} \quad ; \quad \underline{B} = \begin{bmatrix} \underline{0} & \underline{M} \\ \underline{M} & \underline{C} \end{bmatrix} \qquad \text{(A 3)}$$

$$\underline{x}_n = \begin{bmatrix} \lambda_n \underline{q}_n \\ \underline{q}_n \end{bmatrix} \quad ; \quad \underline{y}_n = \begin{bmatrix} \lambda_n \underline{\ell}_n \\ \underline{\ell}_n \end{bmatrix} \qquad \text{(A 4)}$$

The vectors $\underline{x}_m$ and $\underline{y}_n$ satisfy the biorthogonality relation

$$\underline{y}_n^T\, \underline{B}\, \underline{x}_m = \delta_{nm} \qquad \text{(A 5)}$$

where δ_{nm} is the Kronecker delta. The first derivative of λ_n with respect to the parameter p_j has the form

$$\lambda_{n,j}\, (\underline{y}_n^T\, \underline{B}\, \underline{x}_n) = \underline{y}_n^T\, (\underline{A}_{,j} - \lambda_n \underline{B}_{,j})\, \underline{x}_n \qquad \text{(A 6)}$$

For the calculation of the second derivatives of λ_n, we need the derivatives of the eigenvectors $\underline{x}_n$ and $\underline{y}_n$. The vectors $\underline{x}_{n,j}$ and $\underline{y}_{n,j}$ can be represented as a linear combination of $\underline{x}_n$ and $\underline{y}_n$:

$$\underline{x}_{n,j} = \sum_{i=1}^{2N} c_{nji}\, \underline{x}_i \qquad \underline{y}_{n,j} = \sum_{i=1}^{2N} d_{nji}\, \underline{y}_i \qquad \text{(A 7)}$$

Using the equations (A1), A2), (A5) and (A7), Plaut and Huseyin (ref. 3) obtain the following expressions for the coefficients c_{nji} and d_{nji}

$$c_{njn} + d_{njn} = - \underline{y}_n^T \underline{B},_j \underline{x}_n$$

$$\left.\begin{aligned} c_{nji} &= \frac{\underline{y}_i^T(\underline{A},_j - \lambda_n \underline{B},_j)\underline{x}_n}{\lambda_n - \lambda_i} \\ d_{nji} &= \frac{\underline{y}_n^T(\underline{A},_j - \lambda_n \underline{B},_j)\underline{x}_i}{\lambda_n - \lambda_i} \end{aligned}\right\} \text{for } n \neq i \qquad \text{(A 8)}$$

In order to get the second derivative of λ_n with respect to p_j and p_ℓ we must differentiate eq. (A6) with respect to p_ℓ. With eq. (A1), (A2), (A5) and (A7) we obtain for $\lambda_{n,j\ell}$ (ref. 3)

$$\begin{aligned} \lambda_{n,j\ell} = {} & \underline{y}_n^T (\underline{A},_{j\ell} - \lambda_n \underline{B},_{j\ell}) \underline{x}_n \\ & + (c_{njn} + d_{njn}) \lambda_{n,\ell} + (c_{n\ell n} + d_{n\ell n}) \lambda_{n,j} \\ & + \sum_{\substack{i=1 \\ i \neq n}}^{2N} (\lambda_n - \lambda_i) [d_{nji} c_{n\ell i} + d_{n\ell i} c_{nji}] \end{aligned} \qquad \text{(A 9)}$$

where c_{nji} and d_{nji} can be calculated by eq. (A8).

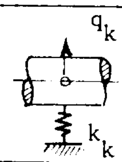
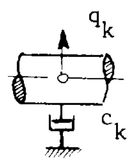
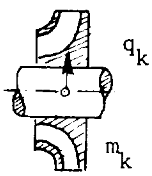
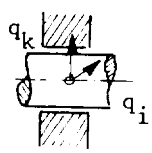
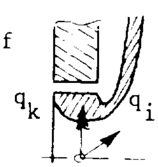
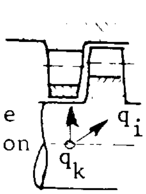
If we go back to our initial problem with the matrices $\underline{M}$, $\underline{C}$ and $\underline{K}$ and the eigenvectors $\underline{\ell}$ and $\underline{q}$, we get by substituting $\underline{A}$, $\underline{B}$, $\underline{x}$ and $\underline{y}$ for the second derivative

$$\begin{aligned} \lambda_{n,j\ell} = \frac{\partial \lambda_n}{\partial p_j \partial p_\ell} = {} & - \underline{\ell}_n^T (\lambda_n^2 \underline{M},_{j\ell} + \lambda_n \underline{C},_{j\ell} + \underline{K},_{j\ell}) \underline{q}_n \\ & - \{ \underline{\ell}_n^T (2\lambda_n \underline{M},_j + \underline{C},_j) \underline{q}_n \} \lambda_{n,\ell} \\ & - \{ \underline{\ell}_n^T (2\lambda_n \underline{M},_\ell + \underline{C},_\ell) \underline{q}_n \} \lambda_{n,j} \\ & + \sum_{\substack{i=1 \\ i \neq n}}^{2N} \frac{(\underline{\ell}_n^T \underline{G}_{nj} \underline{q}_i)(\underline{\ell}_i^T \underline{G}_{n\ell} \underline{q}_n) + (\underline{\ell}_n^T \underline{G}_{n\ell} \underline{q}_i)(\underline{\ell}_i^T \underline{G}_{nj} \underline{q}_n)}{(\lambda_n - \lambda_i)} \end{aligned}$$

where $\underline{G}_{nj}$ means (A 10)

$$\underline{G}_{nj} = \lambda_n^2 \underline{M}_{,j} + \lambda_n \underline{C}_{,j} + \underline{K}_{,j} \qquad \text{(A 11)}$$

The second derivative with respect to a single parameter p_j can be obtained by replacing the subscript ℓ in eq. (A10) by the subscript j.

Element	Force-motion-relation	Influence coefficient g_{nk}
Spring 	$\tilde{F}_k = k_k \tilde{q}_k$	$g_{nk} = \left.\frac{\partial\lambda_n}{\partial k_k}\right\vert_o = -(\ell_k q_k)_n$
Damper 	$\tilde{F}_k = c_k \dot{\tilde{q}}_k$	$g_{nk} = \left.\frac{\partial\lambda_n}{\partial c_k}\right\vert_o = -\lambda_{no}(\ell_k q_k)_n$
Mass 	$\tilde{F}_k = m_k \ddot{\tilde{q}}_k$	$g_{nk} = \left.\frac{\partial\lambda_n}{\partial m_k}\right\vert_o = -\lambda_{no}^2(\ell_k q_k)_n$
Journal bearing 	$\begin{bmatrix}\tilde{F}_i\\ \tilde{F}_k\end{bmatrix} = \begin{bmatrix}c_{ii} & c_{ik}\\ c_{ki} & c_{kk}\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}}_i\\ \dot{\tilde{q}}_k\end{bmatrix} + \begin{bmatrix}k_{ii} & k_{ik}\\ k_{ki} & k_{kk}\end{bmatrix}\begin{bmatrix}\tilde{q}_i\\ \tilde{q}_k\end{bmatrix}$	$g_{nc_{ik}} = -\lambda_{no}(\ell_i q_k)_n$ $g_{nk_{ik}} = -(\ell_i q_k)_n$
Seals of Pumps 	$\begin{bmatrix}\tilde{F}_i\\ \tilde{F}_k\end{bmatrix} = \begin{bmatrix}C & c\\ -c & C\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}}_i\\ \dot{\tilde{q}}_k\end{bmatrix} + \begin{bmatrix}K & k\\ -k & K\end{bmatrix}\begin{bmatrix}\tilde{q}_i\\ \tilde{q}_k\end{bmatrix}$	$g_{nC} - \lambda_{no}(\ell_i q_i + \ell_k q_k)_n$ $g_{nc} - \lambda_{no}(\ell_i q_k - \ell_k q_i)_n$ $g_{nK} - (\ell_i q_i + \ell_k q_K)_n$ $g_{nk} - (\ell_i q_k - \ell_k q_i)_n$
Clearance Excitation 	$\begin{bmatrix}\tilde{F}_i\\ \tilde{F}_k\end{bmatrix} = \begin{bmatrix}o & k\\ -k & o\end{bmatrix}\begin{bmatrix}\tilde{q}_i\\ \tilde{q}_k\end{bmatrix}$	$g_{nk} = -(\ell_i q_k - \ell_k q_i)_n$

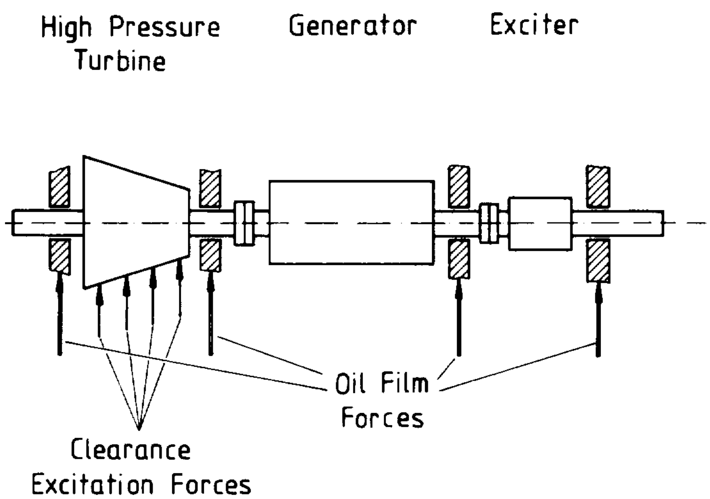


Figure 1. - Rotor of turbomachine.

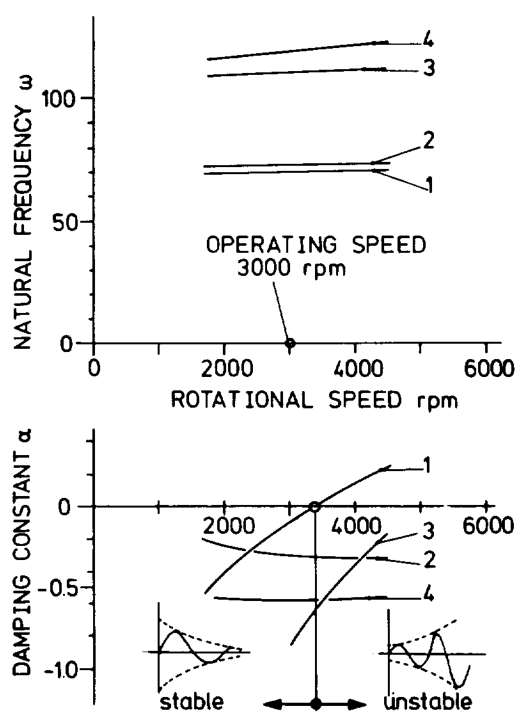


Figure 2. - Eigenvalues of turbomachine rotor.

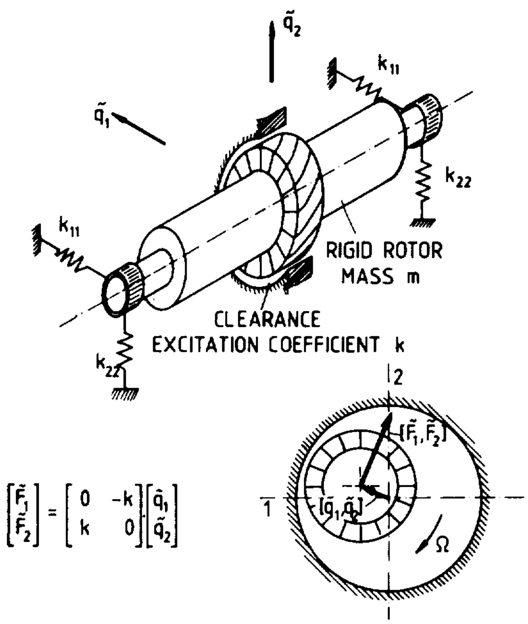


Figure 3. - Rigid rotor with flexible supports and clearance excitation.

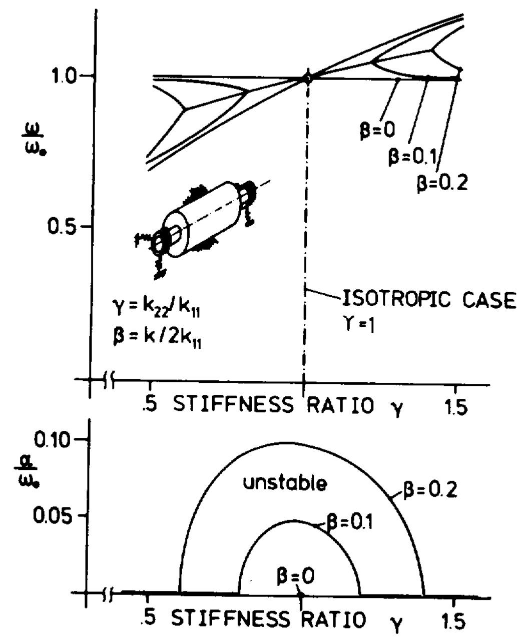


Figure 4. - Eigenvalues of rigid rotor with flexible supports and clearance excitation.

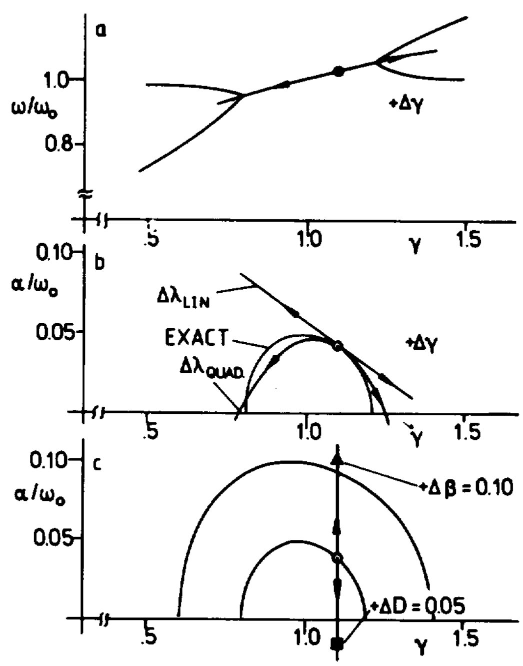


Figure 5. - Parameter variations at rigid rotor with flexible supports and clearance excitation.

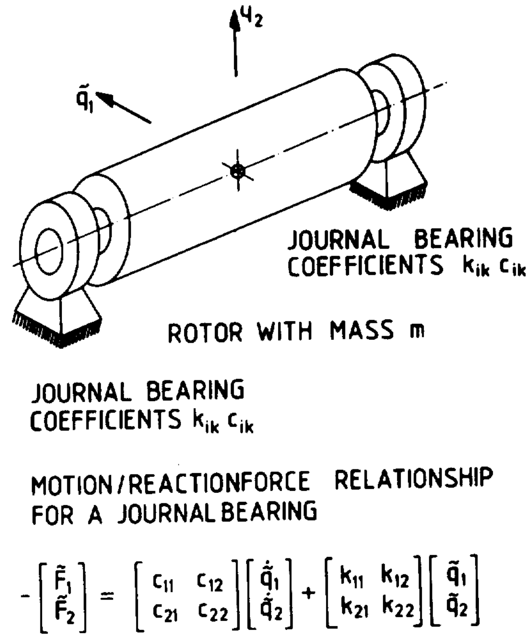


Figure 6. - Rigid rotor in journal bearings.

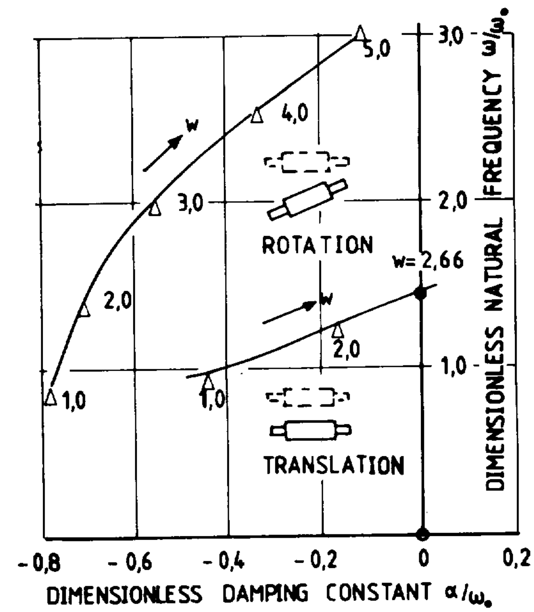


Figure 7. - Eigenvalues of rigid rotor in journal bearings.

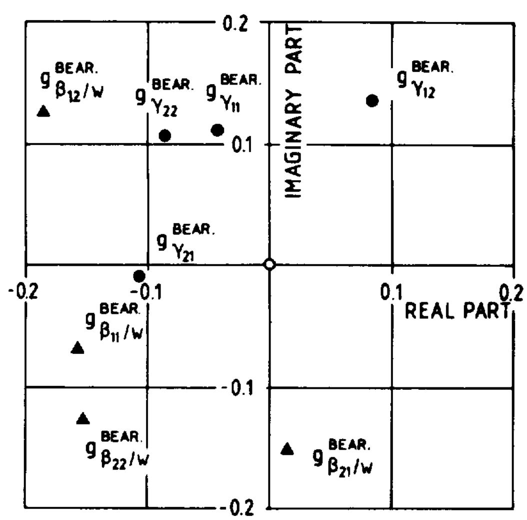


Figure 8. - Influence of bearing coefficients to stability behavior.

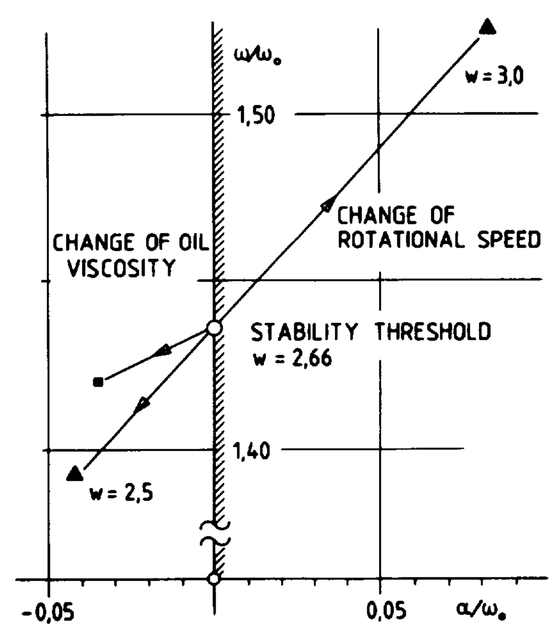
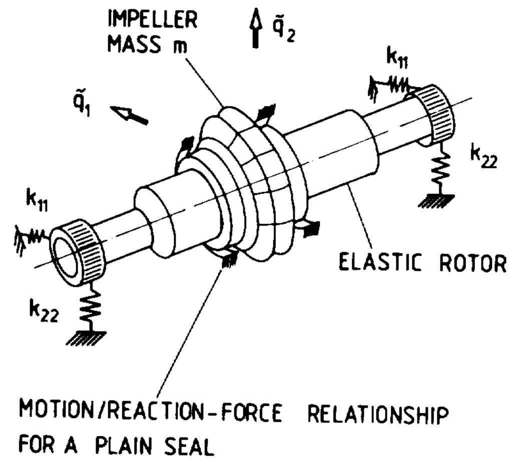


Figure 9. - Variation of rotational speed and oil viscosity.



$$-\begin{bmatrix}\tilde{F}_1\\ \tilde{F}_2\end{bmatrix} = \begin{bmatrix}m^* & \\ & m^*\end{bmatrix}\begin{bmatrix}\ddot{\tilde{q}}_1\\ \ddot{\tilde{q}}_2\end{bmatrix} + \begin{bmatrix}C & c\\ -c & C\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}}_1\\ \dot{\tilde{q}}_2\end{bmatrix} + \begin{bmatrix}K & k\\ -k & K\end{bmatrix}\begin{bmatrix}\tilde{q}_1\\ \tilde{q}_2\end{bmatrix}$$

Figure 10. - Elastic turbopump rotor.

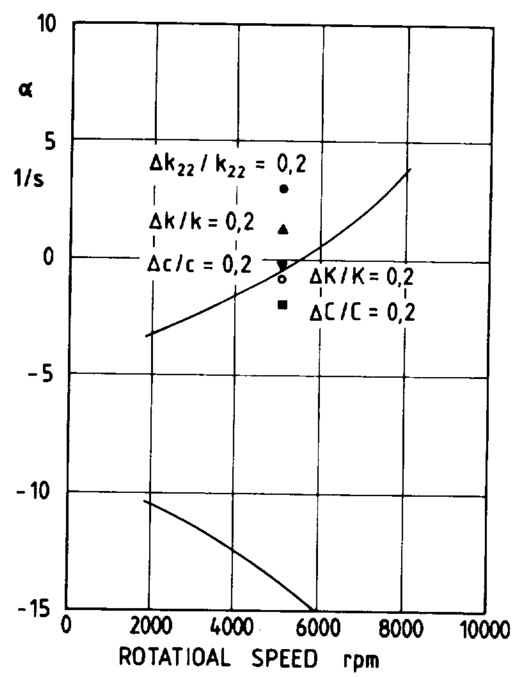


Figure 11. - Influence of seal stiffness and damping coefficients to stability behavior of turbopump rotor.

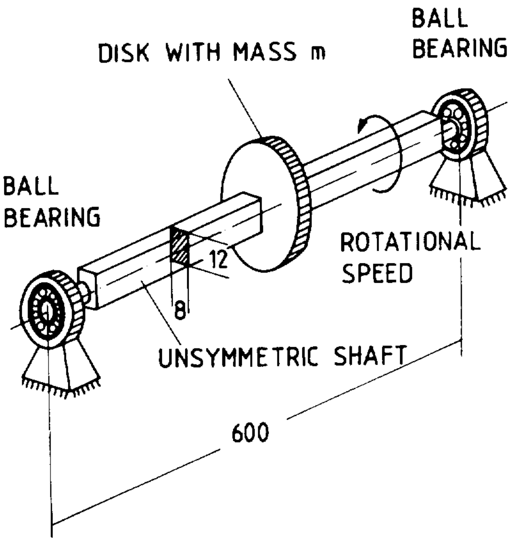


Figure 12. - Test rig rotor with unsymmetric shaft.

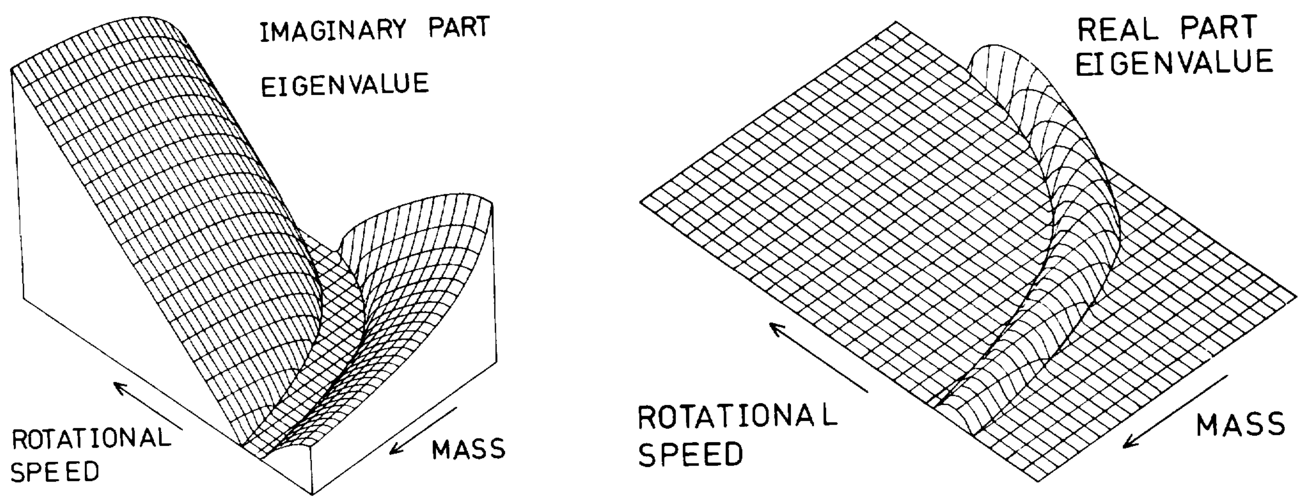


Figure 13. - Eigenvalues of rotor with unsymmetric shaft.

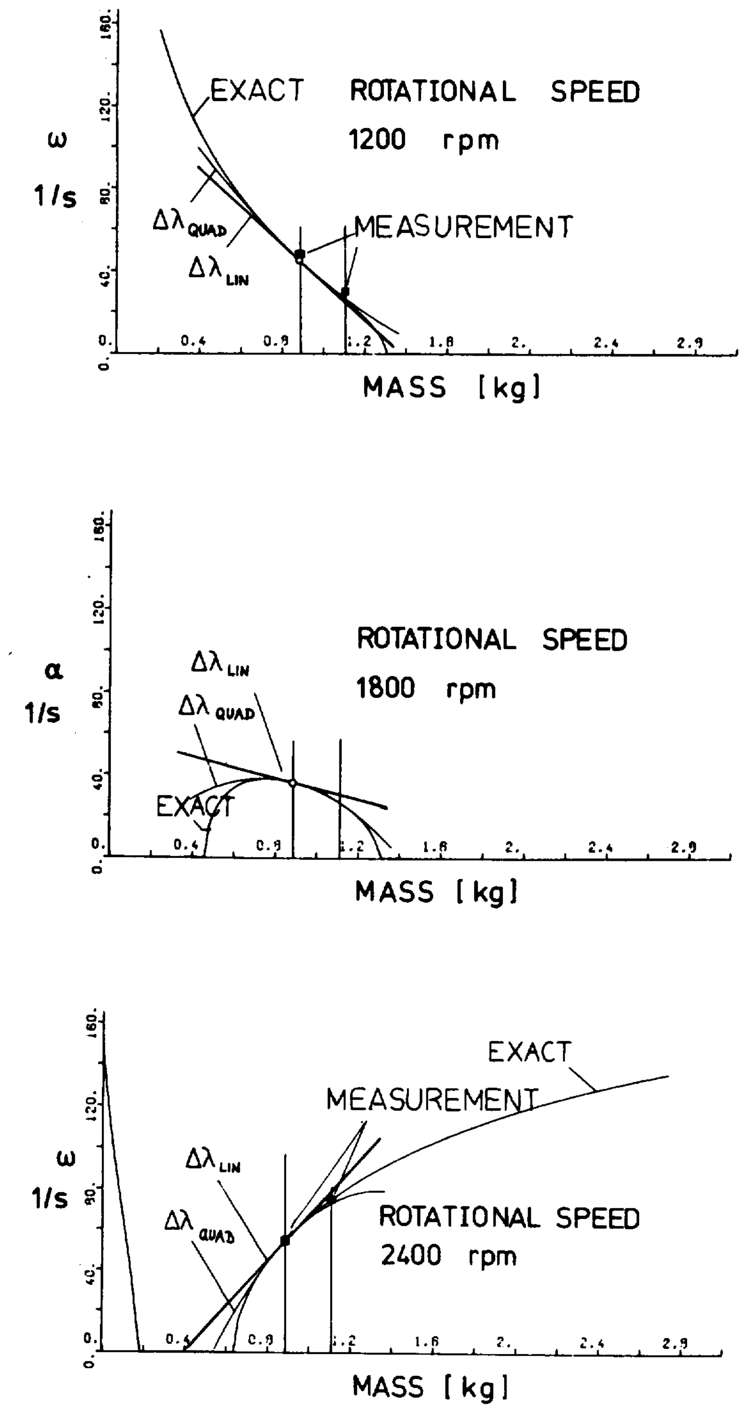


Figure 14. - Variation of eigenvalues in dependence of mass change.