# AN APPROXIMATE METHOD FOR SOLUTION TO VARIABLE MOMENT OF INERTIA PROBLEMS

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## ABSTRACT

The "weathering vaning" motion of a wind turbine with a moving rotor is an oscillatory problem with a variable moment of inertia. The analysis of such a motion requires the solution of a non-linear differential equation. In this article an approximation method is presented for reducing the problem to an equivalent constant moment of inertia problem.

The method is based on the assumption that a moving rotor is an integrator and, therefore, the problem will behave as if it has an averaged moment of inertia. It is further assumed that this will be a valid solution to the problem if the rotating speed of the wind turbine is infinite. The method consists of determing the integrated average of the moment of inertia for a single rotation. This averaged value can then be used to determine equivalent natural frequency of the system and other dynamic properties.

The method is shown to be valid by solving the non-linear differential equation for various rotating speeds. It was found that the cycle time is the equivalent cycle time if the rotating speed is 4 times greater than the systems minimum natural frequency. The ratio of equivalent to minimum cycle time is

$$\bar{t}/to = (Imax/Imin)^{\frac{1}{4}}.$$

## INTRODUCTION

The oscillatory motion of a system with a variable moment of inertia is complex. The analysis of such a motion requires the solution of a non-linear differential equation. In this article a method for reducing the problem to an equivalent constant moment of inertia problem is presented.

The application which generated interest in this problem is the "weather vaning" of a wind turbine with a moving rotor. The method established here is applicable to any problem in which the moment of inertia is a variable about the axis of rotation such as the roll of a helicopter.

The problem is illustrated in Fig. 1. It is desired to describe the motion of the system about the axis A-A while the rotor is moving. One can see from Fig. 1 that the moment of inertia about A-A varies from a minimum when the rotor is vertical (Position A) to a maximum when the rotor is horizontal (Position B). This variation is continuous and cyclic with each turn of the rotor.

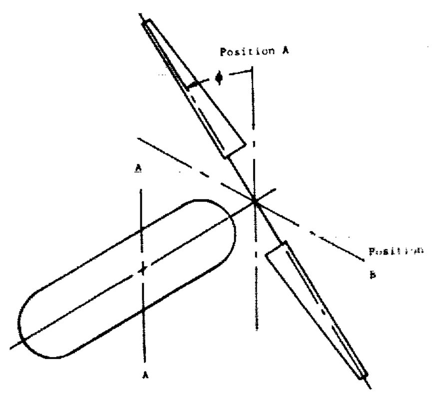


Fig. 1 - Problem Geometry

One can also see from Fig. 1 that the problem cannot be handled with a single initial condition. A different and unique motion is obtained for each initial position of the rotor. Hence, the problem has a stochastic nature.

## ANALYSIS

At any rotor position, $\phi$, the instantaneous moment of inertia of the rotor, $I_i$, about axis A-A is:

$$I_i = mr_k^2 \sin^2\phi = I_r \sin^2\phi$$

where m is the mass of the rotor. The terms $r_k$ and $I_r$ are the radius of gyration and the moment of inertia of the rotor about its axis of rotation, respectively. The moment of inertia of the rest of the system about A-A, which is constant, is:

$$I_o = I_c + mh^2$$

where $I_c$ is the moment of inertia of the center-body and h is the distance from the center of the rotor to axis A-A. The expression for the total moment of inertia I is the sum of the constant and variable portions

$$I = I_o + I_i = I_o + I_r \sin^2\phi$$

or

$$I = I_o(1 + J\sin^2\phi) \tag{1}$$

where $J = I_r/I_o$.

When the entire system is oscillating about axis A-A with a moving rotor, it is apparent that the rotor is acting as an integrator of moment of inertia. It is hypothesized that if the rotor has an infinite angular velocity the moment of inertia I behaves as a constant integrated average value. It is this hypothesis which is the basis of method presented here.

To establish the utility of the method, assuming for the moment that the hypothesis is correct, there are two questions which must be answered:

1) What averaging technique should be used?
2) How close must the speed be to infinite for the method to be useful? In other words, how fast is fast?

The hypothesis is proven by examining the simple harmonic motion of the system in Fig. 1. The proof is valid for more complex situations such as those including damping and forcing functions. In demonstrating the proof, the method of solving a variable moment of inertia problem will be established.

The equation for simple harmonic motion is:

$$\frac{d^2\alpha}{dt^2} = \frac{-K\alpha}{I} = -\frac{\omega_o^2\,\alpha}{1+J\sin^2(\omega t)} \tag{2}$$

and

$$\omega_o^2 = K/I_o \tag{3}$$

where K is the torsional constant and $\omega_o$ is defined in Eq. 3 as the reference frequency for the system. In writing Eq. 2, the moment of inertia I has been replaced by Eq. 1 and the rotor angle $\phi$ has been replaced by rotor angular velocity $\omega$ and time t.

Eq. 2 is dimensionalized by letting

$$\tau = t/t_o = \omega_o t$$

$$n = \omega/\omega_o$$

and

$$\theta = \alpha/\alpha_o$$

where $\alpha_o$ is the initial angular displacement. Substituting these expressions into Eq. 2, one has

$$\frac{d^2\theta}{d\tau^2} = -\frac{\theta}{1+J\sin^2(n\tau)} = -k\theta \tag{4}$$

The term k is defined as the natural frequency ratio

$$k = \frac{1}{1+J\sin^2(n\tau)} = \left(\frac{\omega_n}{\omega_o}\right)^2 \tag{5}$$

and can be used to obtain the natural frequency of the system at anytime.

The hypothesis states that as the rotor angular velocity approaches infinity, the natural frequency ratio k approaches a constant integrated average value. The average value is obtained by integrating Eq. 5 over a quarter cycle

$$\bar{k} = \frac{2}{\pi}\int_o^{\pi/2} k\,d\phi = \frac{2}{\pi}\int_o^{\pi/2}\frac{d\phi}{1+J\sin^2\phi} = \frac{2}{\pi}\int_o^{\pi/2}\frac{d\phi}{(J+1)\sin^2\phi+\cos^2\phi} \tag{6}$$

From a table of definite integrals*, one obtains

$$\int_o^{\pi/2}\frac{dx}{a^2\sin^2x+b^2\cos^2x} = \frac{\pi}{2ab}$$

$$\therefore\ \bar{k} = \frac{1}{\sqrt{J+1}} = \left(\frac{\bar{\omega}}{\omega_o}\right)^2 = \left(\frac{t_o}{\bar{t}}\right)^2 \tag{7}$$

Eq. 7 is the answer to the first question. Since J + 1 is $I_{max}/I_{min}$, a more convenient form for Eq. 7 is

$$\bar{t}/t_o = (I_{max}/I_{min})^{\frac{1}{4}} \tag{8}$$

The hypothesis is proven if it can be shown that the system oscillates at a cycle time defined by Eq. 7 as n increases. It should be noted in passing that the averaging method defined by Eq. 6 is the proper one. Initially, the following averaging was used which is incorrect.

$$J = \frac{1}{2\pi}\int(1+J\sin^2\phi)d\phi$$

The error the author made was that reciprocal of the average is not the average of the reciprocals.

Eq. 4 is a non-linear differential equation. It was solved numerically for a number of cases of n and J using the Continuous System Modeling Program (CSMP) which is standard IBM software.

The solution to the differential equation with time ratio $\tau$ is presented in Fig. 2 for J = 1. One can see that the frequency shifted significantly with increase in rotor speed from n = 0 to n = 1. Fig. 2 also shows a significant change in displacement with speed ratio n. In 3 reference frequency cycles, the solutions for other than n = 0 does not appear to be repeating indicating the stochastic nature of the problem.

*Handbook of Chemistry and Physics, Chemical Rubber Publishing Co.

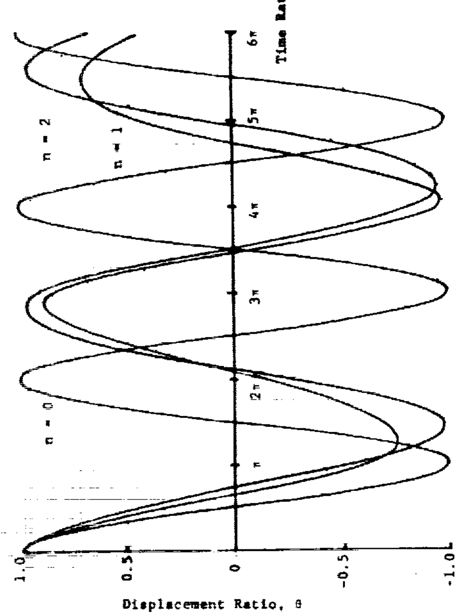


Fig. 2 - Displacement Response with Variable Rotor Speed

In Fig. 3 the time ratio $\tau$ for the first cycle is presented with increasing rotor speed ratio n. The limiting value as predicted by Eq. 7 is indicated in Fig. 3. One can see from Fig. 3 that the time ratio equals the limiting value when $n > 4$. Hence, the hypothesis is proven and the second question is answered.

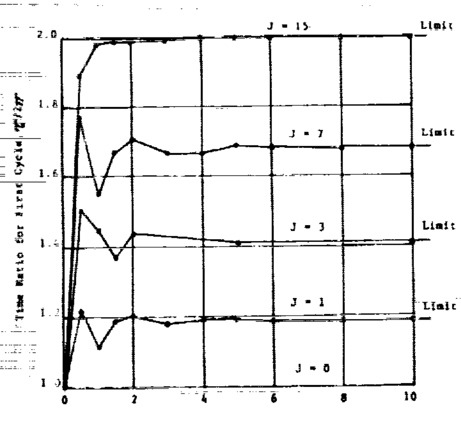


Fig. 3 - First Cycle Time Response

The curves in Fig. 3 are smooth even though the Fig. 3 does not indicate it. The curves all start at $\tau = 2\pi$ because the problem was always started with the system at minimum moment of inertia or with rotor at Position A. If the problem were started at maximum moment of inertia, the solution would have started at $\tau = 2\pi\sqrt{J+1}$.

At any other position the solution would start anywhere in between. It is the variation in moment of inertia with rotor position that causes the oscillation of the first cycle time ratio in Fig. 3 before it "damps" down to the limiting value.

It appears from Fig. 3 that as J increases the first cycle time ratio approaches the limiting value more quickly. The reverse is also true. The solution for $J = 0$ is a constant horizontal line as indicated in Fig. 3. The speed ratio for utilizing the limit solution may maximumize with J.

## CONCLUDING REMARKS

In conclusion, it can be stated that:

1) The hypothesis is true.

2) A variable moment of inertia problem can be analyzed as a constant moment of inertia problem if the rotor speed is 4 times the reference frequency, which is based on minimum moment of inertia.

3) The method of solution is to multiply the reference frequency by the fourth root of the minimum to maximum moment of inertia ratio to obtain the system's natural frequency.

4) If $n > 4$ the initial starting point is not important and the problem is deterministic.

## QUESTIONS AND ANSWERS

E.W. Beans

From: T. Base

Q: When you set up your initial equations, why didn't you write them in form

$$\frac{d}{dt}(I\dot{\phi}) = \text{Applied Torques (including damping)}$$

so that: $I_M \frac{d\dot{\phi}}{dt} + \dot{\phi}\frac{dI_m}{dt} = \text{Applied Torques}$

then the rate of change of $I_M$ with time could be used directly in the equations.

A: *The equation you wrote is not the one I wanted to solve. Your equation appears to have a velocity dependent term. Since dI/dt varies cyclic I would try to solve it using the averaging technique.*

From: Art Smith

Q: Do you plan to check your results with a more exact method?

A: *No. My results appear to satisfy application for which it was developed.*

From: G. Beaulieu

Q: You have considered rigid blade for your analysis; could the harmonic deformation of a vibrating blade significantly change the moment of inertia?

A: *Yes, if the mass center is displaced.*

From: Dan Schiff

Q: Does your solution account for dynamic effects, e.g., gyroscopic effects--or only static?

A: *No, but the method should be applicable to the solution of any equation with cyclic coefficients.*