REVIEW OF ANALYSIS METHODS FOR ROTATING SYSTEMS WITH PERIODIC COEFFICIENTS

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ABSTRACT

The present article reviews two of the more common procedures for analyzing the stability and forced response of equations with periodic coefficients, namely, the use of Floquet methods, and the use of multiblade coordinate and harmonic balance methods. The analysis procedures of these periodic coefficient systems are compared with those of the more familiar constant coefficient systems.

INTRODUCTION

In dynamic analyses of rotating wind turbine systems, one frequently encounters equations of motion with periodic coefficients. Unlike systems with constant coefficients whose analysis techniques are well known and familiar, the analysis of these periodic coefficient equations are somewhat less familiar. The present paper reviews two of the more common procedures for analyzing the stability and response of these periodic coefficient equations, namely, the use of Floquet methods and the use of multiblade coordinate and harmonic balance methods. To put things in proper perspective and to make comparisons, the paper will briefly review the constant coefficient systems first. The paper is essentially based on Appendices A, B, C, and D of a recent report by the authors, (ref. 1).

CONSTANT COEFFICIENT SYSTEMS

Given a system of N linear differential equations with constant coefficients,

$$\underline{M}\,\ddot{\underline{q}} + \underline{B}\,\dot{\underline{q}} + \underline{K}\,\underline{q} = \underline{F}\,(t) \qquad (1)$$

where $\underline{M}$, $\underline{B}$, and $\underline{K}$ are square matrices of order NxN, while $\underline{q}$ and $\underline{F}$ (t) are column matrices of order Nx1. These can be rearranged as,

$$\begin{bmatrix} \underline{M} & 0 \\ 0 & \underline{M} \end{bmatrix} \begin{Bmatrix} \dot{\underline{q}} \\ \ddot{\underline{q}} \end{Bmatrix} - \begin{bmatrix} 0 & \underline{M} \\ -\underline{K} & -\underline{B} \end{bmatrix} \begin{Bmatrix} \underline{q} \\ \dot{\underline{q}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \underline{F} \end{Bmatrix} \qquad (2)$$

Then, multiplying through by the inverse of the mass matrix gives 2N first order equations,

$$\dot{\underline{y}} - \underline{A}\,\underline{y} = \underline{G} \qquad (3)$$

where $\underline{A}$ is a square matrix of order 2Nx2N, while $\underline{y}$ and $\underline{G}$ are column matrices of order 2Nx1 given by

$$\underline{A} = \begin{bmatrix} 0 & \underline{1} \\ -\underline{M}^{-1}\underline{K} & -\underline{M}^{-1}\underline{B} \end{bmatrix}, \quad \underline{y} = \begin{Bmatrix} \underline{q} \\ \dot{\underline{q}} \end{Bmatrix}, \quad \underline{G} = \begin{Bmatrix} 0 \\ \underline{M}^{-1}\underline{F} \end{Bmatrix} \qquad (4)$$

The above rearrangement, Eq. (3), is valid providing the mass $\underline{M}$ is not singular, which is usually the case with physical systems.

(a) Stability

To investigate stability, one sets $\underline{F} = 0$ (which gives $\underline{G} = 0$) in Eq. (3), to obtain a set of homogeneous equations. Then one seeks exponential solutions of the form $\underline{y} = \underline{y}\,e^{pt}$. Placing these into Eq. (3) leads to the standard eigenvalue problem,

$$\underline{A}\;\underline{y} = p\;\underline{y} \qquad (5)$$

Eigenvalues p_k of the matrix $\underline{A}$ can be obtained by standard numerical eigenvalue routines. If any eigenvalue p_k is positive real or has a positive real part, the system represented by Eq. (3) or equivalently by Eq. (1) is unstable.

(b) Forced Response

Under steady-state conditions, the forces $\underline{F}$ (t) on a rotating system tend to occur periodically in multiples of the rotation frequency Ω. One can then express the force for a particular frequency $\omega_m = m\Omega$, in the form,

$$\underline{F}(t) = \mathrm{Re}(\underline{F}\;e^{i\omega_m t}) = \underline{F}_R \cos\omega_m t - \underline{F}_I \sin\omega_m t \qquad (6)$$

The response $\underline{q}(t)$ is similarly of the form,

$$\underline{q}(t) = \mathrm{Re}(\underline{q}\;e^{i\omega_m t}) = \underline{q}_R \cos\omega_m t - \underline{q}_I \sin\omega_m t \qquad (7)$$

Placing Eqs. (6) and (7) into the basic Eq. (1) and matching sine and cosine terms gives a set of 2Nx2N real equations,

$$\begin{bmatrix} \underline{G} & \underline{H} \\ -\underline{H} & \underline{G} \end{bmatrix} \begin{Bmatrix} \underline{q}_R \\ \underline{q}_I \end{Bmatrix} = \begin{Bmatrix} \underline{F}_R \\ \underline{F}_I \end{Bmatrix} \qquad (8)$$

where one has the matrix elements,

$$\underline{G} = \underline{K} - \omega_m^2\,\underline{M}\,, \qquad \underline{H} = \omega_m \underline{B} \qquad (9)$$

Given the amount of the m^{th} harmonic force present $\underline{F}_R^{(m)}$ and $\underline{F}_I^{(m)}$, Eq. (8) can be solved by simple inversion to find the response $\underline{q}_R^{(m)}$ and $\underline{q}_I^{(m)}$ for each harmonic. Then, one may sum up all the harmonics to give the total periodic response as,

$$\underline{q}(t) = \sum_{m=0}^{N} \underline{q}_R^{(m)} \cos\omega_m t - \sum_{m=0}^{N} \underline{q}_I^{(m)} \sin\omega_m t \qquad (10)$$

Finding the response $\underline{q}(t)$ this way rather than by direct numerical integration, allows one to assess the effects of a particular harmonic on the resulting response of the system.

FLOQUET METHODS

Assume the coefficients $\underline{M}$, $\underline{B}$, $\underline{K}$ in Eq. (1) or equivalently the coefficients $\underline{A}$ in Eq. (3) vary periodically in time, rather than being constants. For illustrating Floquet methods, it will be convenient to use the first order representation, namely 2N equations of the form,

$$\dot{\underline{y}} - \underline{A}(t)\underline{y} = \underline{G}(t) \quad (11)$$

where $\underline{A}(t)$ and $\underline{G}(t)$ are periodic over an interval T.

(a) Stability

The Floquet stability analysis described here follows that given by Peters and Hohenemser (ref. 2). To investigate stability, one sets $\underline{G}$=0 in Eq. (11) to obtain homogeneous equations. The Floquet theorem states the solution of Eq. (11) with $\underline{G}$=0 is of the form

$$\underline{y}(t) = \underline{B}(t) \left\{C_k e^{p_k t}\right\} \quad (12)$$

where $\underline{y}(t)$ and $\{C_k e^{p_k t}\}$ are $2N \times 1$ column matrices, and $\underline{B}(t)$ is a $2N \times 2N$ square matrix periodic over period T, that is, $\underline{B}(T)=\underline{B}(0)$. From the above, one can express

$$\underline{y}(0) = \underline{B}(0)\ \{C_k\} \quad (13)$$

$$\underline{y}(T) = \underline{B}(T)\left\{C_k e^{p_k T}\right\} = \underline{B}(0)\left\{C_k e^{p_k T}\right\} \quad (14)$$

Also, one can express $\underline{y}(T)$ as,

$$\underline{y}(T) = \underbrace{[\underline{y}^{(1)}\ \underline{y}^{(2)} \ldots]}_{[Q]} \begin{Bmatrix} y_1(0) \\ y_2(0) \\ \vdots \end{Bmatrix} \quad (15)$$

where $\underline{y}^{(1)}$ is the solution at t=T of Eq. (11) with $\underline{G}$=0, for the initial conditions y_1=1 and all remaining $y_i(0)=0$, $\underline{y}^{(2)}$ is the solution for $y_2(0)=1$ and all remaining $y_i(0)=0$, etc. The square matrix [Q] is called the "Transition Matrix." Equating Eq. (15) to (14) and introducing Eq. (13) gives,

$$[Q]\left[\left\{B(0)\right\}_1 C_1 + \left\{B(0)\right\}_2 C_2 + \ldots\right] =$$

$$= \left\{B(0)_1\right\} C_1 e^{p_1 T} + \ldots \quad (16)$$

Since C_k are independent, one must have

$$[Q]\ \{B(0)\}_k = \lambda_k \{B(0)\}_k \quad (17)$$

where $\lambda_k = e^{p_k t}$ are the eigenvalues of the [Q] matrix. One then has the relation

$$p_k = \frac{1}{T}\,\ell n\,\lambda_k = \alpha_k + i\,\omega_k \quad (18)$$

from which the real and imaginary parts of the stability exponent p_k are given as

$$\alpha_k = \frac{1}{T}\,\ell n|\lambda_k| = \frac{1}{2T}\,\ell n[(\lambda_k)_R^2 + (\lambda_k)_I^2] \quad (19)$$

$$\omega_k = \frac{1}{T}\tan^{-1}[(\lambda_k)_I/(\lambda_k)_R] \quad (20)$$

The real part α_k is a measure of the growth or decay of the response, as can be seen from Eq.(12). Values of $\alpha_k>0$ (or equivalently $|\lambda_k|>1$) indicate instability. The imaginary part ω_k represents the frequency. However, because $\tan^{-1}$ is multivalued, one can only obtain ω_k to within a multiple of 2π. To obtain the actual frequency and motion corresponding to the k^{th} root, p_k, one sets C_k=1 and all other remaining C_i=0 in Eqs. (12) and (13). Then, using the k^{th} eigenvector $\{B(0)\}_k$ from Eq. (17) as an initial condition, one would solve Eq. (11) with $\underline{G}$=0 by numerical techniques for the resultant motion.

Summarizing: To check for stability of a system of linear equations with periodic coefficients, obtain the eigenvalues λ_k of the "Transition Matrix" [Q] . If $|\lambda_k|>1$, one has instability. The traditional stability exponent p_k is related to λ_k through Eqs. (18) to (20). Two remarks on the above procedure should be noted. (1) The "Transition Matrix" [Q] can be formed by solving either the first order equations, Eqs. (11) with $\underline{G}$=0, or the second order equations, Eqs. (1) with $\underline{F}$=0 and periodic coefficients, whichever is more convenient for the integration scheme. (2) The above procedure will still apply even if the equations have constant coefficients. However, for such cases it is usually easier to form the matrix $\underline{A}$ given by Eq. (4) and obtain its eigenvalues p_k rather than to form the "Transition Matrix" [Q] and obtain its eigenvalues λ_k.

(b) Forced Response

Solutions of Eq. (11), or equivalently Eq. (1) with periodic coefficients, can be obtained by direct numerical integration using some convenient integration scheme. By proper choice of the initial conditions, one can eliminate all transients from the response and obtain the desired steady-state dynamic response by integrating through only one period T, instead of the very large number usually required to reach steady-state for lightly damped systems. A procedure for finding the proper initial conditions is given below.

Solutions of Eq. (11) are of the general form,

$$\underline{y}(t) = \underline{y}_H(t) + \underline{y}_P(t) \quad (21)$$

where $\underline{y}_H(t)$ is the homogeneous solution and $\underline{y}_P(t)$ is the particular solution. One can obtain a complete solution of Eq. (11) numerically for any given set of initial conditions. Call this solution $\underline{y}_E(t)$. One can add any number of additional homogeneous solutions $\Delta\underline{y}_H(t)$ having different initial conditions, to this solution. This would give a new solution to Eq. (11),

$$\underline{y}(t) = \underline{y}_E(t) + \Delta\underline{y}_H(t) \quad (22)$$

which would have different initial conditions than those for $\underline{y}_E(t)$.

One can obtain all the homogeneous solutions of Eq. (11) by solving Eq. (11) with $\underline{G}=0$ a total of 2N times, subject to the initial conditions $y_1=1$ and all remaining $y_i=0$, then $y_2=1$ and all remaining $y_i=0$, etc. In fact, this was done earlier to investigate stability and resulted in the 2N homogeneous solutions $\underline{y}^{(1)}(t)$, $\underline{y}^{(2)}(t)$, etc., respectively.
Thus one may write

$$\Delta \underline{y}_H(t) = \underbrace{[\underline{y}^{(1)}(t)\ \underline{y}^{(2)}(t)\ \ldots]}_{[Q(t)]} \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \end{Bmatrix} \tag{23}$$

where [Q(t)] is the transition matrix at any instant of time, and C_1, C_2, ... are 2N arbitrary constants. The new solution Eq. (22) can be rewritten as

$$\underline{y}(t) = \underline{y}_E(t) + [Q(t)]\underline{C} \tag{24}$$

For a periodic solution over period $T=2\pi/\Omega$, one must have $\underline{y}(T)=\underline{y}(0)$. Placing Eq. (24) into this condition and solving for the arbitrary constants $\underline{C}$ gives,

$$\underline{y}_E(T) + [Q(T)]\underline{C} = \underline{y}_E(0) + [Q(0)]\underline{C}$$

$$\underline{C} = \left[\underline{1} - [Q]\right]^{-1} \left\{\underline{y}_E(T) - \underline{y}_E(0)\right\} \tag{25}$$

where it was noted that $[Q(0)]=\underline{1}$, and $[Q(T)]=[Q]$ is the "Transition Matrix" found earlier for the stability investigation. Placing these values of $\underline{C}$ back into Eq. (24), the initial conditions for insuring a periodic solution become

$$\underline{y}(0) = \underline{y}_E(0) + [\underline{1} - \underline{Q}]^{-1} \left\{\underline{y}_E(T) - \underline{y}_E(0)\right\} \tag{26}$$

One can then solve the basic Eq. (11) numerically with these initial conditions to obtain a periodic solution over one period. It should be noted that if one had chosen the initial conditions for $\underline{y}_E(t)$ as $\underline{y}_E(0)=0$, one would obtain simply

$$\underline{y}(0) = [\underline{1} - \underline{Q}]^{-1}\ \underline{y}_E(T) \tag{27}$$

This is a particularly convenient form for finding the initial conditions for periodic solutions.

An alternative form for determining the proper initial conditions for periodic solutions has been proposed by Friedmann and his coworkers (refs. 3 and 4) in their work on wind turbines, namely,

$$\underline{y}(0) = [\underline{1} - \underline{Q}]^{-1}\ \underline{Q} \int_0^T [Q(t)]^{-1}\underline{F}(t)dt \tag{28}$$

This is similar to Eq. (27), but does not use $\underline{y}_E$. It seems easier to obtain $\underline{y}_E(T)$ with initial conditions $\underline{y}_E(0)=0$ and use Eq. (27), rather than obtaining [Q(t)] at every point and performing the indicated operations required by Eq.(28).

The general procedure described by Eqs. (21) to (27) may be extended to deal also with nonlinear equations,

$$\dot{\underline{y}} - \underline{A}(t)\underline{y} = \underline{F}(t, \underline{y}, \dot{\underline{y}}) \tag{29}$$

where the right hand side now contains nonlinear functions of the coordinates. An iterative variation of the previous linear procedure to obtain the initial conditions for periodic solutions of nonlinear equations is used by the MOSTAS Code (ref. 5). The procedure is as follows. First, a numerical solution $\underline{y}_E(t)$ is obtained to the nonlinear Eq. (29) for some estimate of the initial conditions $\underline{y}_E(0)$. Then each of the 2N elements of $\underline{y}_E(0)$ is perturbed a small amount ε_i and the resulting 2N solutions are obtained. This involves solving the nonlinear Eq. (29) subject to the initial conditions,

$$\underline{y}_E(0) + \begin{Bmatrix} \varepsilon_1 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}, \quad \underline{y}_E(0) + \begin{Bmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \vdots \end{Bmatrix}, \text{ etc.} \tag{30}$$

and will result in 2N responses of the form

$$\underline{y}^{(i)}(t) = \underline{y}_E(t) + \Delta\underline{y}_E^{(i)}(t) \tag{31}$$

where $\Delta\underline{y}_E^{(i)}(t)$ represents the effect of each perturbation ε_i, and is found by <u>subtracting</u> $\underline{y}_E(t)$ from each of the 2N resulting responses $\underline{y}^{(i)}(t)$. One can then express the total solution approximately as,

$$\underline{y}(t) = \underline{y}_E(t) + \underbrace{\left[\frac{1}{\varepsilon_1}\Delta\underline{y}_E^{(1)}, \frac{1}{\varepsilon_2}\Delta y_E^{(2)}, \ldots\right]}_{[Q]} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \end{Bmatrix} \tag{32}$$

which is in the same form as Eq. (23). Then, again requiring the periodicity condition $\underline{y}(T)=\underline{y}(0)$ and following through as before, will result in the same relation Eq. (26) found previously. Because of the nonlinearities now present, the elements of [Q] as found from Eqs.(32), (31), (30) may vary with the amplitude of the initial condition used, $\underline{y}_E(0)+\underline{\varepsilon}_i$. This is in contrast to the linear case where [Q] remains always constant. Hence, an iterative application of Eq.(26) with a new corrected $\underline{q}_E(0)$ should be done. If the nonlinearities are not too great, convergence to the required $\underline{y}_E(0)$ should be rapid.

It should be remarked that the numerical procedure for forced response described in this section, can also be used for the constant coefficient linear case, although it is probably easier there to obtain the solution by using Harmonic response methods given by Eqs.(6) to (10). However, for cases where there is some nonlinearity, the present iterative approach becomes attractive.

MULTIBLADE COORDINATES AND HARMONIC BALANCE

Given a rotor with N blades rotating with rotation speed Ω, attached to a flexible tower. Because the tower motions x_i are described in a fixed reference frame while the blade motions β_i are described relative to a rotating frame, the resulting equations may have mass, damping, or stiffness coefficients which are functions of the azimuthal position of the k^{th} blade ψ_k. A typical such set of equations is given, for example, in refs. 6 and 7 as,

$$M\ddot{x} + C_x\dot{x} + k_x x + S \frac{d^2}{dt^2} \sum^{k} \beta^{(k)} \cos \psi_k = F_x(t) \tag{33}$$

$$S\ddot{x} \cos \psi_k + I\ddot{\beta}^{(k)} + C_\beta \dot{\beta}^{(k)} + k_\beta \beta^{(k)} = F_\beta{}^{(k)}(t)$$

$$(k = 1, 2, \ldots N)$$

where the azimuthal position ψ_k is,

$$\psi_k = \Omega t + (k - 1)\, 2\pi/N \tag{34}$$

The first equation above represents force equilibrium for the tower motion x, while the remaining N equations represent force equilibrium for the motion of each of the N blades $\beta^{(k)}$. The above equations are readily generalized to more tower motions x_i, and more blade coordinates for each blade $\beta_i{}^{(k)}$.

(a) Stability

To examine Eqs. (33) for stability, one sets $F_x=0$ and $F_\beta{}^{(k)}=0$ to obtain homogeneous equations.

For rotors with 3 or more blades $N \geq 3$, one may eliminate the periodic coefficients in these equations by introducing new multiblade coordinates $b_o(t)$, $b_s(t)$, $b_c(t)$ such that

$$\beta^{(k)} = b_o(t) + b_s(t) \sin \psi_k + b_c(t) \cos \psi_k \tag{35}$$

Substituting these into Eqs. (33), then multiplying the last N equations by $\sin \psi_k$, $\cos \psi_k$, and 1 respectively, then summing these last N equations and noting that

$$\left.\begin{aligned} &\sum_{k=1}^{N} \sin \psi_k = \sum_{k=1}^{N} \cos \psi_k = 0 \\ &\sum_{k=1}^{N} \sin^2 \psi_k = \sum_{k=1}^{N} \cos^2 \psi_k = N/2 \\ &\sum_{k=1}^{N} \sin \psi_k \cos \psi_k = 0 \end{aligned}\right\} \text{For } N \geq 3 \tag{36}$$

results in a new set of differential equations in the variables x, b_s, b_c, b_o which now all have constant coefficients, namely,

$$M\ddot{x} + C_x\dot{x} + k_x x + \frac{N}{2} S\, \ddot{b}_c = 0$$

$$\frac{N}{2}\left[I\, \ddot{b}_s + C_\beta \dot{b}_s + (k_\beta - I\Omega^2) b_s - 2\Omega I\, \dot{b}_c - \Omega C_\beta\, b_c\right] = 0$$

$$\frac{N}{2}\left[S\ddot{x} + 2\Omega I \dot{b}_s + \Omega c_\beta b_s + I\ddot{b}_c + c_\beta \dot{b}_c + (k_\beta - I\Omega^2) b_c\right] = 0 \tag{37}$$

$$N\left[I\ddot{b}_o + C_\beta \dot{b}_o + k_\beta b_o\right] = 0$$

These equations may then be investigated for stability using the standard constant coefficient techniques described earlier. For additional details and applications of multiblade coordinates, see Hohenemser and Yin (ref. 8). Multiblade coordinates were originally introduced by Coleman and Feingold (ref. 9) in their studies of helicopter ground resonance.

For rotors with 2 blades, N=2, the analysis is more difficult because the rotor disk no longer has polar symmetry. If the same multiblade coordinates given by Eq.(35) are used in the basic Eqs.(33), the periodic coefficients would not be entirely eliminated since now,

$$\begin{aligned} &\sum_{k=1}^{2} \sin^2 \psi_k = 1 - \cos 2\psi_1 \\ &\sum_{k=1}^{2} \cos^2 \psi_k = 1 + \cos 2\psi_1 \\ &\sum_{k=1}^{2} \sin \psi_k \cos \psi_k = \sin 2\psi_1 \end{aligned} \tag{38}$$

instead of the convenient constant terms given by Eqs.(36). A rough estimate of the stability and response can be obtained by simply time-averaging the resulting $\cos 2\psi_1$ and $\sin 2\psi_1$ variations to zero and using only the constant coefficient terms. This is equivalent to setting N=2 in the multiblade transformed Eqs.(37).

For more accurate estimates for these 2-bladed rotors, one may use harmonic balance methods. This consists of first introducing new coordinates $b_T(t)$ and $b_A(t)$ for these two blades such that,

$$\beta^{(1)} = \beta_T + \beta_A\,, \quad \beta^{(2)} = \beta_T - \beta_A \tag{39}$$

then summing and subtracting the last two equations of Eqs.(33) while noting that $\sin \psi_2 = -\sin \psi_1$ and $\cos \psi_2 = -\cos\psi_1$, then expanding each of the coordinates in a harmonic series,

$$x = x_o + x_{1S} \sin \Omega t + x_{1C} \cos \Omega t + x_{2S} \sin 2\Omega t + \ldots$$

$$b_T = b_{T0} + b_{T1S} \sin \Omega t + b_{T1C} \cos \Omega t + \ldots$$

$$b_A = b_{A0} + b_{A1S} \sin \Omega t + b_{A1C} \cos \Omega t + \ldots \tag{40}$$

where x_0, x_{1S}, b_{T0}, b_{T1S}, ... are all functions of time. Placing these into Eqs.(33) and balancing out each harmonic term in each equation will yield a truncated series of constant coefficient differential equations. These equations may again be examined for stability using the standard constant coefficient techniques described earlier.

Often, depending on the form of Eqs.(33), the resulting constant coefficient differential equations will uncouple into several smaller coupled systems of equations which may be examined independently of one another. For example, for the case of Eqs.(33), one smaller coupled system would involve the variables x_0, x_{2S}, x_{2C}, b_{A1C}, b_{A1S}, ... while another would involve x_{1C}, x_{1S}, b_{A0}, b_{A2C}, b_{A2S}, ... For such systems, one could use an alternate extended form of the multiblade coordinate transformation Eq.(35) namely,

$$x = x_0 + x_{2S} \sin 2\Omega t + x_{2C} \cos 2\Omega t + \ldots$$
$$\beta^{(k)} = b_{1S} \sin\psi_k + b_{1C} \cos\psi_k + \ldots \quad (41)$$

together with the harmonic balance method to solve the problem. This works here, since the form given by Eq.(41) exactly duplicates the motion of the two blades given by the general case Eqs.(39) and (40), since $\sin\psi_2 = -\sin\psi_1$, $\cos\psi_2 = -\cos\psi_1$, and only x_0, x_{2S}, x_{2C}, b_{A1C}, b_{A1S}, ... would be present. However, in more general cases (for example, if the first equation of Eqs.(33) had an additional term $M_1\ddot{x}\cos\psi_1$ or $k_1 x \cos\psi_1$ present), the resulting equations would not split into two smaller groups, and the general harmonic balance method Eqs.(39) and (40) would have to be used.

Indeed, for the more general case mentioned above, one would also investigate the system for direct Mathieu equation type instabilities of half integer order $\Omega/2$. $3\Omega/2$, ... by introducing additional harmonic terms $\sin m\Omega t$ and $\cos m\Omega t$ where $m=1/2, 3/2, 5/2, \ldots$ into Eqs.(40), and harmonically balancing as before. These terms would not couple in with the previous equations and can be solved independently of them. The primary instability region would result from the $\Omega/2$ terms. See Bolotin (ref. 10) for further details of the general harmonic balance method. Also see Sheu (ref. 7) for an application of the alternate extended form of the multiblade transformation Eq.(41), to a simple two bladed rotor in ground resonance.

(b) Forced Response

For rotors with 3 or more blades, $N \geq 3$, one uses the multiblade coordinate transformation Eq.(35) to eliminate the periodic coefficients in the basic equations of motion Eqs.(33), as described in the preceding section. The equations then reduce to the constant coefficient equations given by Eqs.(37), only now the right-hand-sides are

$$\text{R.H.S.} = \begin{Bmatrix} F_x(t) \\ \sum_{k=1}^{N} F_\beta^{(k)}(t) \sin\psi_k \\ \sum_{k=1}^{N} F_\beta^{(k)}(t) \cos\psi_k \\ \sum_{k=1}^{N} F_\beta^{(k)}(t) \end{Bmatrix} \quad (42)$$

instead of the previous value of zero. Under steady-state conditions, the tower and blade forces generally occur periodically in multiples of the rotation frequency Ω, and can generally be expressed as,

$$F_x(t) = F_{x0} + F_{x1S}\sin\psi_1 + F_{x1C}\cos\psi_1 + F_{x2S}\sin 2\psi_1 + \ldots \quad (43)$$
$$F_\beta^{(k)}(t) = F_{\beta 0} + F_{\beta 1S}\sin\psi_1 + F_{\beta 1C}\cos\psi_1 + \ldots$$

where $\psi_k = \Omega t + (k-1)2\pi/N$. Placing the above forces into Eqs.(42) and using the trigonometric identities and summations,

$$\sin m\psi_k \sin\psi_k = \frac{1}{2}\cos(m-1)\psi_k - \frac{1}{2}\cos(m+1)\psi_k$$

$$\cos m\psi_k \sin\psi_k = \text{etc.} \quad (44)$$

$$\sum_{k=1}^{N} \sin m\psi_k = \begin{cases} N \sin m\psi_1 & \leftarrow m = N, 2N, \ldots \\ 0 & \leftarrow m \neq N, 2N, \ldots \end{cases}$$

$$\sum_{k=1}^{N} \cos m\psi_k = \begin{cases} N \cos m\psi_1 & \leftarrow m = N, 2N, \ldots \\ 0 & \leftarrow m \neq N, 2N, \ldots \end{cases}$$

one can obtain the right-hand-sides of Eqs.(37) in terms of either constants or harmonic functions of $m\Omega t$. The forced responses $x(t)$, $b_s(t)$, $b_c(t)$, $b_0(t)$ can then be found using the standard techniques for constant coefficient systems discussed previously. It should be noted that because of the multiblade transformation Eq. (35), the resulting responses for the tower motion and blade motions corresponding to the m^{th} harmonic $\omega_m = m\Omega$, would be of the form,

$$x = x_R \cos\omega_m t - x_I \sin\omega_m t$$
$$\beta^{(k)} = b_{0R}\cos\omega_m t - b_{0I}\sin\omega_m t + (b_{SR}\cos\omega_m t - b_{SI}\sin\omega_m t)\sin\psi_k + (b_{CR}\cos\omega_m t - b_{CI}\sin\omega_m t)\cos\psi_k \quad (45)$$

The tower thus oscillates at frequency ω_m in the fixed frame whereas the blades may oscillate at frequencies ω_m, $\omega_m + \Omega$, $\omega_m - \Omega$ relative to the rotating frame.

For rotors with 2 blades, N = 2, the multiblade coordinate transformation Eq. (35) does not eliminate the periodic coefficients, but rather changes the $\cos\psi_k$ variations to $\cos 2\psi_k$ variations. A rough estimate of the response can be obtained by simply time-averaging the resulting $\sin 2\psi_k$ and $\cos 2\psi_k$ variations to zero, and then proceeding with the remaining constant coefficient terms, as was done for the $N\geq 3$ case. The results are likely to be somewhat off for the second harmonic, $\sin 2\psi_k$ and $\cos 2\psi_k$ responses.

For more accurate estimates for these 2-bladed rotors, one can use the harmonic balance methods of the previous section. The steady-state periodic tower and blade forces given by Eqs. (43) are substituted into the basic equations of motions Eqs. (33). One then introduces the new coordinates given by Eqs. (39), then sums and subtracts the last two blade equations, then expands the tower and blade motions as given by Eq. (40), only now the coordinates x_0, x_{IS}, b_{T0}, b_{TIS}, b_{A0},...etc. are taken to be constants rather than functions of time. Harmonically balancing the various terms in each equation results in a truncated set of algebraic equations which can be solved to obtain the coordinates x_0, x_{1S}, b_{T0},... etc., corresponding to the given forcing excitations F_{x0}, F_{x1s}, $F_{\beta 0}$, $F_{\beta 1s}$,...etc. The resulting tower and blade motions are then given directly by Eqs. (40) and (39). The resulting set of algebraic equations will often uncouple into smaller coupled sets of equations which can be examined independently of one another. This procedure is similar to that for the constant coefficient forced response case Eq. (8), except now, the periodic coefficients couple the different harmonics together. Thus, the solution will consist of many harmonics $m\Omega$ even if only one forcing harmonic $F_{\beta 1s}$ were present alone.

ROTATING COORDINATES

As an addendum to the previous multiblade coordinates and harmonic balance methods, it should be mentioned that for some problems, the use of rotating coordinates is also convenient. For example, in the case of a 2-bladed rotor on isotropic tower supports (same tower mass, damping, and stiffness in two directions, x_1 and x_2), Eqs. (33) would read,

$$M\ddot{x}_1 + C_x\dot{x}_1 + k_x x_1 + S\frac{d^2}{dt^2}\sum^{k}\beta^{(k)}\cos\psi_k = F_{x1}(t) \tag{46}$$

$$M\ddot{x}_2 + C_x\dot{x}_2 + k_x x_2 - S\frac{d^2}{dt^2}\sum^{k}\beta^{(k)}\sin\psi_k = F_{x2}(t)$$

$$S\ddot{x}_1\cos\psi_k - S\ddot{x}_2\sin\psi_k + I\ddot{\beta}^{(k)} + C_\beta\dot{\beta}^{(k)} + k_\beta\beta^{(k)} = F_\beta^{(k)}(t) \qquad (k=1,2)$$

One can then express the tower motions x_1 and x_2 in terms of rotating coordinates ξ_1 and ξ_2 which rotate with the rotor, as

$$x_1 = \xi_1\cos\Omega t + \xi_2\sin\Omega t \tag{47}$$

$$x_2 = -\xi_1\sin\Omega t + \xi_2\cos\Omega t$$

where the rotation $\psi_1=\Omega t$ is taken from the x_2 axis towards the x_1 axis. Placing these equations into Eqs. (46), then multiplying the first two equations by $\cos\psi_1$ and $\sin\psi_1$ respectively and subtracting, then multiplying the first two equations by $\sin\psi_1$ and $\cos\psi_1$ and adding, then subtracting the third and fourth equations, then adding the third and fourth equations will result in a new set of differential equations in the variables ξ_1, ξ_2, b_A, b_T which now all have constant coefficients, namely,

$$M(\ddot{\xi}_1 + 2\Omega\dot{\xi}_2 - \Omega^2\xi_1) + C_x(\dot{\xi}_1 + \Omega\xi_2) + k_x\xi_1 + 2S(\ddot{\beta}_A - \Omega^2\beta_A) = F_{x1}\cos\Omega t - F_{x2}\sin\Omega t$$

$$M(\ddot{\xi}_2 - 2\Omega\dot{\xi}_1 - \Omega^2\xi_2) + C_x(\dot{\xi}_2 - \Omega\xi_1) + k_x\xi_2 - 4S\Omega\dot{\beta}_A = F_{x1}\sin\Omega t + F_{x2}\cos\Omega t$$

$$2S(\ddot{\xi}_1 + 2\Omega\dot{\xi}_2 - \Omega^2\xi_1) + 2I\ddot{\beta}_A + 2C_\beta\dot{\beta}_A + 2k_\beta\beta_A = F_\beta^{(1)} - F_\beta^{(2)}$$

$$2I\ddot{\beta}_T + 2C_\beta\dot{\beta}_T + 2k_\beta\beta_T = F_\beta^{(1)} + F_\beta^{(2)} \tag{48}$$

In the above, $\beta_T=(\beta^{(1)}+\beta^{(2)})/2$ and $\beta_A=(\beta^{(1)}-\beta^{(2)})/2$ are the same coordinates introduced earlier in Eqs. (33). These differential equations may then be investigated for stability and forced response using the standard constant coefficient techniques described earlier. Such analyses of a 2-bladed rotor on isotropic tower supports were also performed by Coleman and Feingold (Ref. 9) in their studies of helicopter ground resonance.

Rotating coordinates are often used in rotating machinery shaft critical speed problems, and are useful for dealing with problems of rotors with unsymmetrical mass, unsymmetrical damping, or unsymmetrical shaft stiffness supported on isotropic bearings. See for example, Bolotin (Ref. 11). For such problems, one can readily set up the equations of motion in the rotating frame directions, and the fixed supports will introduce no periodic terms because of their isotropic nature. For vertical axis wind turbines, such rotating coordinates for the blades are useful since the tower supports are generally isotropic due to the symmetrically arranged guy wires. For horizontal axis wind turbines, the tower supports are generally not isotropic, hence periodic coefficients will remain in the equations when using rotating coordinates. If the support anisotropy is not too large, one can again additionally introduce harmonic balance methods to eliminate the periodic coefficients, as was done in the previous section.

CONCLUDING REMARKS

The present article has reviewed two of the more common procedures for analyzing the stability and forced response of rotating systems with periodic coefficients, namely, Floquet methods and multiblade coordinate, harmonic balance methods. Also, the use of rotating coordinates was discussed.

The Floquet methods are based on a convenient numerical integration scheme and involves the computation of the "Transition Matrix," [Q], from which stability and the initial conditions for steady-state response solutions can be obtained. These methods seem attractive for large systems and can be modified to include nonlinearities in the equations.

The multiblade and harmonic balance methods involve first the introduction of multiblade coordinates in order to take out the periodic coefficients from the blades [Eqs.(35) for N≥3], or to obtain a better ordered system of equations [Eqs.(39) for N=2]. Then, harmonic balance methods Eqs.(40) are used to deal with any remaining periodic coefficients. These methods seem attractive for smaller systems and can give considerable insight into the origin and nature of instabilities and the various harmonics present in the forced response.

Rotating coordinates can also be used to effectively eliminate the periodic coefficients in problems involving unsymmetrical rotors on isotropic tower supports. These can often be used in rotating shaft critical speed problems and for vertical axis wind turbines. If the support anisotropy is not too large, harmonic balance methods may additionally be used to deal with any remaining periodic coefficients.

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QUESTIONS AND ANSWERS

J. Dugundji

From: P.R. Barnes

Q: The Singularity Expansion Method (SEM) introduced by Dr. Carl Baum of Kirkland AFB, Albuquerque, NM is another, perhaps better, approach to solving these problems. Do you know about SEM?

A: *No, I do not. I have just dealt here with two of the more common methods for dealing with these problems.*

From: W.E. Holley

Q: Are you aware of any treatments of stochastic problems with periodic coefficients?

A: *I have not dealt with that aspect of the problem, so I am not aware of them. I believe though that there is considerable literature on that subject.*