An Analysis of GDOP in Global Positioning System Navigation

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ABSTRACT

The accuracy of user navigation fix based on the NAVSTAR Global Positioning System is described by a 4x4 position-time error covariance matrix. The "trace" of this matrix serves as a convenient navigation performance index and the square-root of the trace is called Geometric Dilution of Precision (GDOP). In this paper, certain theoretical results concerning the general properties of the navigation performance are derived. An efficient algorithm for the computation of GDOP is given. Applications of the results are illustrated by numerical examples.

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INTRODUCTION

The NAVSTAR Global Positioning System (GPS), when fully operational in the early 1990's, will provide world-wide navigation through synchronized transmissions from a constellation of eighteen 12-hour period satellites in three 55^{o}-inclination orbital planes. An accurate user navigation fix (position and time) will be obtainable by receiving transmissions from four satellites and decoding the signal transit times.

One may relate the measurements, referred to as the pseudo-ranges, to the navigation state as follows

$$C T_j = \sqrt{(X_1 - x_1)^2 + (X_2 - x_2)^2 + (X_3 - x_3)^2} + X_4 + n_j \quad (1)$$

where
- C = velocity of light
- T_j = Signal transit time from GPS satellite "j" to user, not corrected for user clock offset, Δt
- X_1, X_2, X_3, X_4 = user naviagation state, the first three represent a set of convenient Cartesian user coordinates, $X_4 = C\Delta t$ is a range bias equivalent of user clock offset
- x_1, x_2, x_3 = corresponding Cartesian coordinates of GPS satellite "j"
- n_j = random measurement noise

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From a set of four measurements, a user navigation fix may be determined. The accuracy of the fix is characterized by the following 4x4 position-time navigation error covariance matrix

$$P = (H^T W H)^{-1} \tag{2}$$

where H = measurement parital derivative matrix

$$= \left[\begin{array}{c|c|c} a^T & & \\ \hline b^T & & \\ \hline c^T & & \\ \hline d^T & & \end{array}\right] \tag{3}$$

a, b, c and d = line-of-sight unit vectors from a set of four GPS satellites to the user, W = 4x4 covariance matrix of random measurement noise, superscript "T" = transpose of matrix.

The measurement error covariance matrix W is generally taken to be diagonal, which is strictly true for uncorrelated measurements only. In practice, assignments of quantitative values to the elements of W also takes into consideration such factors as the elevation and health status of individual GPS satellites. Thus W may be more appropriately be referred to as the weighting matrix. For uniform weighting, P is proportional to $(H^T H)^{-1}$, which depends only on the relative geometry of the user and the four GPS satellites, as is evident from Equation (3). The square-root of the "trace" of $(H^T H)^{-1}$ is referred to as Geometric Dilution of Precision (GDOP), a self-explanatory name. Whatever the weighting strategy, the "trace" of the navigation error covariance matrix serves as a covenient and natural performance index characterizing the accuracy of the naviagation fix. For a diagonal weighting matrix W,

TRACE "P" = sum of diagonal terms of $(H^T H)^{-1}$ weighted by the inverses of the corresponding elements of W

Thus the evaluation of the GPS naviagation performance is essentially equivalent to the computation of the diagonal terms of $(H^T H)^{-1}$, which may be called the GDOP matrix for convenience.

The navigation performance index, Trace "P", also serves as a criterion for the selection of a set of four best GPS satellites among those visible, which may be as many as ten for users which are satellites themselves. If, for optimum performance, each of the different combinations of four has to be evaluated, the computational burden can be considerable. In the following, certain theoretical results concerning the general properties of the GDOP matrix are derived. An efficient algorithm for the computation of GDOP matrix and the navigation performance index is given. Applications of the results are illustrated by numerical examples.

ANALYTICAL RESULTS

To solve for a navigation fix from four measurements, the partial derivative matrix H must be non-singular. Since

$$\text{determinant } H = \begin{vmatrix} a^T-d^T & 0 \\ b^T-d^T & 0 \\ c^T-d^T & 0 \\ d^T & 1 \end{vmatrix} = \begin{vmatrix} a & -d \\ b & -d \\ c & -d \end{vmatrix}$$

a navigation fix can be determined from four GPS satellites with line-of-sight directions a, b, c, d, if and only if the three vectors (a-d), (B-d), and (c-d) are linearly independent, i.e., non-coplanar. This shall be assumed to be the case in the following development.

Since Trace $(H^TH)^{-1}$ = Trace $(HH^T)^{-1}$, by making use of Equation (3) and the fact that a, b, c, d, are unit vectors, one obtains,

$$\text{Trace } (H^TH)^{-1} = \text{Trace } (HH^T)^{-1}$$

$$= \text{Trace} \begin{bmatrix} 2 & a^Tb+1 & a^Tc+1 & a^Td+1 \\ & 2 & b^Tc+1 & b^Td+1 \\ & & 2 & c^Td+1 \\ (\text{symmetric}) & & & 2 \end{bmatrix}^{-1} \qquad (4)$$

The advantages of dealing with HH^T instead of H^TH will become obvious below.

The following may be observed from Equation (4):

1. The matrix HH^T in Equation (4) is non-negative, symmetric, and with identical diagonal terms which are greater than the off-diagonal terms. (Expressions such as a^Tb are scalar product of unit vectors and are less than unity). These properties give rise to good behaviour in numerical operations.

2. Since the Trace of a matrix is equal to the sum of its eigenvalues and the eigenvalues of the matrix inverse are inverses of the eigenvalues of the matrix itself, one has the following results:

a. Trace $(HH^T)^{-1} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}$, where the λ's are eigenvalues of (HH^T) with

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 8.$$

b. From "a" above and the fact that the λ's are non-negative, one may conclude that

$$\text{Trace } (HH^T)^{-1} \geq 2 \qquad (5).$$

c. Let us order the eigenvalues of HH^T as

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 < 8.$$

One has the obvious inequality

$$\text{Trace } (HH^T)^{-1} = \frac{1}{\lambda_1} + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right)$$

$$> \frac{1}{\lambda_1} + \underset{\lambda_2+\lambda_3+\lambda_4<8}{\text{lower bound}} \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}\right),$$

or,

$$\text{Trace } (HH^T)^{-1} > \frac{1}{\lambda_1} + \frac{9}{8} \qquad (6)$$

Thus knowledge of the smallest eigenvalue of HH^T provides another lower bound for the navigation performence index. Sometimes this lower bound also serves as a good estimate.

d. The 2x2 principal submatrix of HH^T, e.g. $\begin{bmatrix} 2 & a^Tb+1 \\ a^Tb+1 & 2 \end{bmatrix}$ has eigenvalues $3+a^Tb$ and $1-a^Tb$. From the Theorem of Root Separation for Symmetric Matrices[1] one obtains the following bounds on the eigenvalues of HH^T

$$\lambda_1 \le 1-a^Tb \le \lambda_3 \tag{7}$$

$$\lambda_2 \le 3 + a^Tb \le \lambda_4 \tag{8}$$

These inequalities have no preferences for the labeling of the unit vectors. That is, a, b may be replaced by c, d, etc., to obtain sharper bounds. In particular, one must have $\lambda_1 < 2$ and $\lambda_4 > 2$. Therefore, the eigenvalues of HH^T cannot be all identical and the equality sign in (5) may be deleted. Physically, this follow from the fact that the four unit vectors in three-dimensional space cannot play identical roles in the four-dimensional position-time space. Combining inequalities (6) and (7), one obtains another inequality.

$$\text{Trace } (HH^T)^{-1} > \frac{9}{8} + \frac{1}{1-\cos\theta} \tag{9}$$

where θ = smallest angle subtended by two line-of-sight vectors.

This inequality, although not sharper than Inequality (8), is easier to calculate, and expresses the intuitive rule of thumb that an accurate navigation should not rely on a GPS constellation that is clustered together. We shall see later that with good geometry, navigation performance index of magnitude less than 3 may be obtained. On the other hand, as indicated by Inequality (9), a navigation performance index in excess of 8.5 would result if any two line-of-sight vectors to GPS satellites are separated by 30° or less.

An upper bound for the navigation performance index may be obtained as

$$\text{Trace}\,(HH^T)^{-1} < \frac{3}{\lambda_1} + \frac{1}{2} \tag{10}$$

by noting $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $\lambda_4 > 2$.

It may also be pointed out that because the determinant of a matrix product is equal to the product of the individual determinants, and that the determinant of a matrix is equal to the product of its eigenvalues, one has the relation

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 = (\text{determinant } H)^2 = \begin{vmatrix} a-d \\ b-d \\ c-d \end{vmatrix}^2 .$$

The maximization of $\begin{vmatrix} a-d \\ b-d \\ c-d \end{vmatrix}$ has been suggested as a convenient GPS selection criterion[3]. It is seen from the above equation that this criterion is equivalent to a maximization of the denominator of our performance index,

$$\text{Trace}\,(HH^T)^{-1} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4}$$

$$= \frac{\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_4}{\lambda_1\lambda_2\lambda_3\lambda_4}.$$

ALGORITHM

An efficient algorithm for the computation of the GDOP matrix may be obtained by noting the following decomposition of the measurement partial derivative matrix:

$$H = \left[\begin{array}{c|c} a^T & 1 \\ \hline b^T & 1 \\ \hline c^T & 1 \\ \hline d^T & 1 \end{array}\right] = \left[\begin{array}{c|c} a^T & 1 \\ b^T & 1 \\ c^T & 1 \\ \hline 0 & 1 \end{array}\right] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \left[\begin{array}{c|c} d^T & 0 \end{array}\right]$$

$$\triangleq A + BD^T$$

From this decomposition, the Sherman-Morrison Formula[2] gives us

$$H^{-1} = A^{-1} - (A^{-1}B)(D^TA^{-1}) \big/ (1 + D^TA^{-1}B) \qquad (11).$$

Let $\quad (f \,\vdots\, g \,\vdots\, h) \triangleq \left([a \,\vdots\, b \,\vdots\, c]^{-1}\right)^T \qquad (12).$

Then one has, by straight-forward simple algebra,

$$A^{-1} = \left[\begin{array}{c|c|c|c} f & g & h & -g \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

and

$$H^{-1} = \frac{1}{\alpha}\left[\begin{array}{c|c|c|c} \alpha f + (d^T f) g & \alpha g + (d^T g) g & \alpha h + (d^T h) g & -g \\ \hline -d^T f & -d^T g & -d^T h & 1 \end{array}\right] \qquad (13)$$

where $q \triangleq f + g + h$

$\alpha \triangleq 1 - d^T q.$

When H^{-1} is obtained, one may obtain the GDOP matrix as $(HH^T)^{-1} = (H^{-1})^T (H^{-1})$. In particular,

Trace $(HH^T)^{-1}$ = sum of the squares of the elements of H^{-1}

$$= f^T f + g^T g + h^T h + \frac{2}{\alpha}\left\{ (d^T f) g^T f + (d^T g) g^T g + (d^T h) g^T h \right\}$$

$$+ \frac{g^T g + 1}{\alpha^2}\left\{ 1 + (d^T f)^2 + (d^T g)^2 + (d^T h)^2 \right\} \qquad (14)$$

Equations (11), (13) and (14) constitute the algorithm. It reduces the inversion of the 4x4 matrix HH^T to the inversion of a 3x3 matrix (a ¦ b ¦ c) plus the scalar products of several 3x1 vectors. Notice that Eq. (13) may also be obtained from inverting H by partitioning[2]. But the Sherman-Morrison Formula provides additional flexibility as will be discussed below.

An important advantage of this algorithm is that very little recomputation is required when the fourth GPS satellite is switched. In selecting the best set of four GPS satellites from the many possible combinations, a simple combinatorial test logic may be advantageous. For this purpose, one may need the flexibility of changing any one of the rows of H. Although Eq. (14) remains valid provided one interprets the vectors f, g, and h accordingly, this does mean these vectors have to be recomputed. In that case it is preferable to use Eq. (11) directly instead of Eq. (14). To illustrate let us assume that for a particular GPS configuration, $H^{-1}=G$ is already obtained. If the nth (n = 1, 2, 3, 4) GPS Satellite with line of sight vector r is to be replaced by another satellite with line-of-sight vector p, the new measurement partial derivative matrix may be written as

$$H' = H + \begin{bmatrix} \delta_{1n} \\ \delta_{2n} \\ \delta_{3n} \\ \delta_{4n} \end{bmatrix} \left[(p-r)^T \,\vdots\, 0 \right]$$

where δ_{in} is the Kronecker delta ($\delta_{in}=0$ for $i \neq n$ $\delta_{in}=1$ for $i = n$).

From the above decomposition the Sherman-Morrison Formula gives us

$$H'^{-1} = G - \frac{\begin{bmatrix} G_{1n} \\ G_{2n} \\ G_{3n} \\ G_{4n} \end{bmatrix} \left([p-r]^T \begin{bmatrix} G_{11} & G_{12} & G_{13} & G_{14} \\ G_{21} & G_{22} & G_{23} & G_{24} \\ G_{31} & G_{32} & G_{33} & G_{34} \end{bmatrix} \right)}{1 + [p-r]^T \begin{bmatrix} G_{1n} \\ G_{2n} \\ G_{3n} \end{bmatrix}} \qquad (15)$$

The computational economy provided by this equation is obvious.

Intuitively, orthogonal line-of-sight user-to-GPS satellite configurations are favorable. In three-dimensional space, it is, of course, impossible to have a set of four mutually orthogonal unit vectors. An alternative has three of the line-of-sight vectors a, b, c orthogonal. For this case the vectors f, g and h become the same orthogonal unit vectors as a, b, c and Eq. (14) simplifies to

$$\begin{aligned}\text{Trace } (HH^T)^{-1} &= 3 + 2(d_1+d_2+d_3)/\alpha + 4(1+d_1^2+d_2^2+d_3^2)/\alpha^2 \\ &= 1 + \{8 + 2(1-d_1-d_2-d_3)\}/(1-d_1-d_2-d_3)^2\end{aligned}$$

where d_1, d_2, d_3 with $d_1^2+d_2^2+d_3^2 = 1$ are components of the line-of-sight unit vector d along the orthogonal a, b, c directions. It is of interest to note that for this case the Navigation Peformance Index depends only on $(d_1+d_2+d_3)$, the simplest symmetric function of the components of the vector d. The best performance index of 2.80 is achieved for $d^T = (-1,-1,-1)/\sqrt{3}$. This is the situation that the line of sight to GPS satellite "d" shows no preference to, but is directed away from the other GPS satellites, an artificial but not improbable configuration for an user satellite. For $d^T = (1,1,1)/\sqrt{3}$, i.e., d having the same general direction as the other three lines-of-sight, the performance index degrades to 13.20. This degradation reminds us of the statement made earlier about avoiding closely-grouped GPS satellites. For d = -a, i.e. for an user located between two GPS satellites, the performance index has the value 4.00.

There is reason to think that a GPS constellation with a-d, b-d, c-d orthogonal may give good navigation performance.

This may be realized with the set of line-of-sight vectors $a^T = (-1, 1, 1)\sqrt{3}$, $b^T = (1, -1, 1)/\sqrt{3}$, $c^T = (1, 1, -1)/\sqrt{3}$ and $d^T = (1, 1, 1)/\sqrt{3}$. However, for this configuration, the angle between the vector d and any other vector is $\cos^{-1}(2/3)$, which is comparatively small, and may be undesirable from the consideration of the preceeding section. Indeed, it follows immediately from Inequality (10) that the navigation performance index must be in excess of $9/8 + 1/(1 - 2/3) = 4\frac{1}{8}$, a lower bound which may be compared with the exact index of 5.5 obtainable from straightforward simple computation. On the other hand, by reversing the direction of the vector d given above, one has the completely symmetrical configuration that the line-of-sight vectors are all separated by the same angle $\cos^{-1}(-1/3)$. For this configuration one may compute the eigenvalues of HH^T as $\lambda_1 = \lambda_2 = \lambda_3 = 4/3$ and $\lambda_4 = 4$, giving rise to a navigation performance index

$$\mathrm{Trace}\,(HH^T)^{-1} = \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2.5$$

Notice that for this configuration,

1. The upper bound for λ_1 given in Inequality (7) is achieved.
2. Any perturbation of the configuration will result in a decrease in the minimum angle between two line-of-sight vectors, and therefore a decrease in λ_1.

Thus this configuration maximizes the smallest eigenvalue of HH^T, or equivalently, minimizes the largest eigenvalue of $(HH^T)^{-1}$. Whether this also happens to be the best configuration remains to be investigated.

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