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STOCHASTIC MODELING OF THE TIME-AVERAGED EQUATIONS FOR CLIMATE DYNAMICS

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Long Term Objectives

The grant study deals overall with long term, climatologically significant statistical properties of the atmospheric component of the climate system. Understanding should lead to an ability to correctly and efficiently incorporate this component into larger models for climate that include oceanic and cryospheric influences and deal realistically with climate replication. An immediately usefully and critically important consequence of successful completion of the study would be the ability to assess the magnitude of the unpredictable (noise) component of long term atmospheric behavior, and hence the potential for viable climatological forecasting.

Immediate Program Content And Objectives

The Stanford program in climate is new and in process of development. Both because of the relative novelty of our approach to the problem and because our students have minimal backgrounds in meteorology and atmospheric sciences, we have elected to start at a simple but fundamental level of analysis. Thus, the initial effort is concentrating on the use of Lorenz's "minimum hydrodynamic equations" as a prototype for describing at least some of the non-linear characteristics of long term atmospheric behavior that we believe are important for climate. We have two main thoughts in mind:

a) that techniques for closure of the moment equations that arise from treating time averaged climatic variables can be tested with the minimum equations. These do indeed exhibit the counter-gradient momentum transfer and vacillatory behaviors that are known as essential features on the non-linear behavior of the atmosphere;

and b) in view of the fact that temporal response characteristics need to be known for developing atmospheric climate models, it is hoped that study of such properties of the minimum equations (hopefully with only minor modifications) will lead to a rationale for simplified atmospheric climate models via a subdivision into more easily treated sub-components. For example, we might consider long term, large scale changes to be describable in terms of a barotropic model that is driven baroclinically, perhaps by stochastic forcing via quasi-geostrophic turbulence, as well as by deterministic terms. The minimum equations would represent the simplest possible description for the barotropic component of such a model.

Accomplishments

We are advocating the introduction of time averaging for the treatment of climate questions. The approximations that then become possible (and, we suggest, necessary for practical viability of answering climate problems) will vary with the averaging time selected, which in term will vary with the particular climate question being addressed. In every case, however, the averaging process, when applied to the (non-linear) differential equations governing atmospheric motions, introduces new, statistically defined variables (moments and correlations) which are formally governed by an infinite set of equations. For large scale motions conventional turbulence theory approaches for closing these equations do not apply, and for the climatological case we must resort to other techniques, the nature of which we expect to discover only from better understanding of the atmospheric circulation system itself. The simplest model that incorporates the important counter-gradient transport of momentum, necessary for the maintenance of the atmospheric energy cycle, is represented in Lorenz's introduction of the "minimum hydrodynamic" system of equations. These represent the first three Fourier modes of a standing wave system for a rectangular domain, with β effect neglected, and their solution (available analytically) describes the temporal non-linear oscillations of the standing waves. The equations governing the time-variation of the amplitudes of the three wave components, A_1, A_2, A_3, are

$$\frac{dA_1}{dt} = C_1 A_1 A_2 \; ; \quad \frac{dA_2}{dt} = C_2 A_1 A_3 \; ; \quad \frac{dA_3}{dt} = C_3 A_2 A_1 \; ; \qquad (1)$$

where $C_1 = -[\alpha(\alpha^3 + 1)]^{-1}$; $C_2 = \alpha^2(\alpha^2 + 1)^{-1}$, $C_3 = -(\alpha^2 - 1)/2\alpha$, and $\alpha = k/\ell$, the ratio of zonal to meridional wave number. Jacobian elliptic functions constitute the analytic solution to (1):

$$\begin{aligned} A_1 &= A_1^* \;\; dn(\overline{ht - t^*}, h_o^2) \\ A_2 &= A_2^* \;\; sn(\overline{ht - t^*}, h_o^2) \\ A_3 &= A_3^* \;\; cn(\overline{ht - t^*}, h_o^2) \end{aligned} \qquad (2)$$

for $\alpha > 1$ and $2k^2E/V < 1$, where E and V are the enstrophy and energy of the system. The parameters A_1^* , A_2^* , A_3^* , t^*, h and k_o are defined in terms of E and V and a phase angle.

The description and analysis can be simplified by a scaling transformation:

$$\begin{aligned} &X_1 = A_1/A_1^* \; , \; X_2 = A_2/A_2^* \; , \; X_3 = A_3/A_3^* \; , \\ &t' = ht = A_1^* \; \frac{\alpha}{2}\left(\frac{\alpha^2 - 1}{\alpha^2 + 1}\right)^{\frac{1}{2}} t \; . \end{aligned} \qquad (3)$$

to give

$$\begin{aligned} \dot{X}_1 &= -k_o^2 X_2 X_3 \qquad & X_1 &= dn(t' - t'^* , k_o^2) \\ \dot{X}_2 &= \quad X_3 X_1 \qquad & X_2 &= sn(t' - t'^* , k_o^2) \\ \dot{X}_3 &= \; - X_1 X_2 \qquad & X_3 &= cn(t^1 - t'^* , k_o^2) \end{aligned} \qquad (4)$$

Provided we modify the conventional definition for the moment of a time-averaged variable from

$$M_{ij\cdots\ell}(t) = \frac{1}{T} \int_{t-T}^{T} [X_i(t') - \mu_i(t')]$$
$$[X_j(t') - \mu_i(t')]\cdots[X_\ell(t') - \mu_\ell(t')]dt'$$

to

$$M_{ij\cdots\ell}(t) = \frac{1}{T} \int_{t-T}^{t} [X_i(t') - \mu_i(t)] [X_j(t') - \mu_j(t)] \ldots [X_\ell(t') - \mu_\ell(t)] dt' \qquad (5)$$

moment equations for the time averaging interval T can be readily derived. It is this (infinite) system of equations that we need to solve for the climate case, and since the solutions (4) can be used to calculate the moments (5) exactly for the minimum equations, we are in position to evaluate the accuracy of any proposed closure scheme, at least as far as the properties of the minimum equations are concerned. Exact moments, for arbitrary values of averaging time T can in fact be derived in analytic, though rather cumbersome form, and this has been done up to the third order moments τ_{ijk}.

A possible approach for achieving approximate closure that has worked well in the case of stochastic dynamics weather forecasting, is linearisation of the moment equations. In our case, the linearisation would be made about an assumed known climatic state, which, for problems of the first kind would be based on observed records over the period $0 > t > -T$, and, for prediction of the second kind, would correspond to climatological mean estimates of statistically stationary states. Note that in the second case we need to treat ensemble as well as temporal means, as situation which leads to complications in the development of the moment equations which we are starting to look into. Tests on the efficacy of this concept of a linearised approximation are now being carried out by numerical solution of truncated forms of the (linearised) moment equations.

However, for the minimum equations we have discovered an unexpected property that indeed seems to obviate the need for devises such as linearisation for closing the moment equations. It turns out that for small values of the parameter

$$k_o^2 = \frac{1}{\alpha^4} \frac{A_2^{*2}}{A_1^{*2}} \qquad (6)$$

moments of the solutions (2) for the minimum equations tend to decrease with increasing order of the moment. In fact power series of the form

$$
\begin{aligned}
\dot{\mu}_i &= \sum_{n=0}^{\infty} k_o^{2n}\,\mu_i^{(n)} \qquad [\mu_1^{(0)} = 1] \\
\dot{\sigma}_{ij} &= k_o^{2(\delta_{1i} + \delta_{1j})} \sum_{n=0}^{\infty} k_o^{2n}\,\sigma_{ij}^{(n)} \qquad (7) \\
\dot{\tau}_{ijk} &= k_o^{2(\delta_{1i} + \delta_{ij} + \delta_{ik})} \sum_{n=0}^{\infty} k_o^{2n}\,\tau_{ijk}^{(n)}
\end{aligned}
$$

when $\delta_{ab} = 0$ $a \neq b$, $\delta_{ab} = 1$, $a = b$, yield series of of differential equations for the moments which are <u>automatically</u> <u>closed.</u> For example, the lowest order equations in k_o^2 turn out to be

$$
\begin{aligned}
\dot{\mu}_1 &= 0 \\
\dot{\mu}_2 &= \mu_3 \\
\dot{\mu}_3 &= -\mu_2 \\
\dot{\sigma}_{22} &= 2\sigma_{32} \\
\dot{\sigma}_{33} &= -2\sigma_{32} \\
\dot{\sigma}_{32} &= \sigma_{33} - \sigma_{22}
\end{aligned}
$$

which can readily be shown to have solutions identically equal to those obtained from the exact solution (4), when the exact moments are expanded for $k_o^2 \ll 1$. Small values of k_o in fact correspond to mild vacillations of the zonal flow and, as discussed by Lorenz, represent the predominant mode of atmospheric vacillation phenomena. For k_o^2 close to 1 (which is the upper bound for k_o^2) , corresponding to large oscillations in the east-west wind, this approach does not appear to be applicable.

The fact that closure of the time-averaged moment equations for the minimum equations is possible for physically reasonable values of parameters immediately brings to the fore the question of the applicability of moment equation closure procedures such as (7) to more realistic equations for atmospheric motions. It is our intent to enter at depth into this question, both by judging which properties of the minimum equations can be expected to hold more generally and by modifying the minimum equations to allow for clearly omitted features (such as β , or adding secular and stochastic forcing terms).

One of the important results looked for in the statistical approach we are trying to develop is the determination of the principal time-scaled components into which the atmospheric system can be subdivided. These separate, but coupled components are distinguished by their disparate time scales, and their content would surely vary with climatic average of concern; the aim is to obtain the simplest possible mode of description of the atmospheric portion of the climate system. Whether, or under what circumstances such a form of representation is possible is still at issue. Our investigations currently seek to evaluate the 'response time' for the minimum equations, so as to better understand how both long term forcing (e.g. seasonal variation) and rapid fluctuations (e.g. from two-dimensional turbulence) should be coupled to the equations. We are in process of determining how best to proceed with this problem, and we envisage that the coupling has at least in part to be stochastic, leading to variability in climate evolution as prescribed by the equations of currently unknown magnitude.

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