A FINITE ELEMENT-ANALYTICAL METHOD FOR MODELING A STRUCTURE IN AN INFINITE FLUID

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SUMMARY

A method is described from which the interaction of an elastic structure with an infinite acoustic fluid can be determined. The displacements of the structure and the pressure field of the immediate surrounding fluid are modeled by finite elements, and the remaining pressure field of the infinite fluid region is given by an analytical expression. This method yields a frequency dependent boundary condition for the outer fluid boundary when applied to the frequency response of an elastic beam in contact with an acoustic fluid. The frequency response of the beam is determined using NASTRAN, and compares favorably (1-2% error) to the exact solution which is also presented. The effect of the fluid on the response of the structure at low and high frequencies is due to added mass and damping characteristics, respectively.

INTRODUCTION

The interaction of an acoustic fluid with an elastic solid has received considerable attention in the literature. Some areas of investigation in the frequency domain include underwater vibrations, vibrations of liquids in elastic containers, and the evaluation of the near and far pressure field of an acoustical fluid surrounding a sinusoidally excited elastic structure. A finite element modeling of the combined problem was formulated by Zienkiewicz and Newton (ref. 1). Their finite element modeling of the displacements of a structure and the pressure field of a finite acoustical fluid leads to a system of unsymmetric linear equations to be solved.

Problems involving a finite domain can at least conceptually (and usually practically) be modeled using finite elements (see ref. 2, for example), but those problems involving an infinite fluid domain must necessarily be modeled with only a finite portion of the fluid if the finite element method is to be used. The appropriate boundary condition at the truncated fluid boundary is often in doubt. Zienkiewicz and Newton (ref. 1) suggest a system of dashpots at this outer fluid boundary, but it will be shown that this is the proper boundary condition only in the high frequency limit. This paper formulates the the boundary condition that should be applied at this outer boundary, and shows how this condition is incorporated into the finite element method. To this end, the fluid is divided into a region immediately surrounding the structure (which is to be modeled by finite elements) and an infinite region. Within the infinite region a series expansion is chosen for the pressure, the coefficients of which are unknowns of the problem. Hunt et al (ref. 3) have a similar model, except the pressure field in the infinite region is given by the surface Helmholtz integral equation. In any case, the expression for the pressure field in the infinite region identically satisfies the governing wave equation and the proper boundary conditions at infinity.

A variational principle, presented specifically for a beam with one face in an acoustical fluid, suggests the proper coupling not only between the structure and the fluid but also between the finite and infinite fluid regions. If a coordinate surface is chosen as the outer fluid boundary, orthogonality relationships of the series expansion may be used to satisfy continuity of the pressure field at this boundary. This orthogonality allows the coefficients of the series expansion to be eliminated as unknowns from the problem, and results in an additional stiffness matrix for the nodal pressures on the outer boundary. This matrix is full, symmetric, and frequency-dependent, and is implemented in NASTRAN by direct matrix input.

This method is applied specifically to the frequency response of a simply supported beam with one face in contact with an infinite acoustical fluid (2-D problem). The exact solution for the frequency response of the beam is presented, and the finite element results compare favorably with the exact solution. It is also shown that at low frequencies the effect of the fluid on the structure is an added mass, while at high frequencies it is a damping. Moreover, the far field pressure in the infinite region can be determined from the series expansion once the nodal pressures at the outer fluid boundary are known.

While this method is applied for a 2-D frequency response, it can be generalized to the response of a 3-D elastic structure in an infinite acoustic fluid. The outer surface of the fluid must be a coordinate surface of a space in which the wave equation is separable since the orthogonality of the series expansion on this surface is used. Once again, the structure and the portion of the fluid between the structure and this coordinate surface are modeled by finite elements. Unfortunately, the additional stiffness matrix couples all the pressure nodes at the outer boundary, and, in general, is frequency dependent. If the frequency is specified, the additional stiffness matrix is known, although in general it could increase the bandwidth of the problem. On the other hand, for determining the submerged natural frequencies of structures an iterative procedure is necessary since the natural frequency is unknown.

A VARIATIONAL PRINCIPLE

It is convenient in applying the finite element method to have a variational principle on which the discretized finite element model can be based. Such principles involving the displacements of an elastic structure can be found in references 4 and 5; similar principles for fluid mechanics problems are presented by Olson in reference 6. Gladwell (refs. 7, 8, 9) presents variational theorems for the acoustic fluid for both pressure and displacement formulations. For the coupled structural-fluid problem, a suitable variational formulation can be found by properly combining those for an elastic structure and an acoustic fluid. Such a principle is a reliable basis and guide for numerically solving a fluid-structure problem using finite elements. Moreover, with the fluid divided into a finite region (modeled by finite elements) and an infinite region (fluid described by an analytical expression), the variational formulation will necessarily point to the proper coupling of each.

Finite Fluid Region

A simply supported beam is shown in figure 1 which has one side in contact with a finite acoustic fluid and subjected to a sinusoidal load per unit length of $w(x)e^{i\Omega t}$. The deflection of the beam in the y-direction, $u(x)e^{i\Omega t}$, satisfies the differential equation

$$EI\frac{d^4u}{dx^4} - m\Omega^2 u = -p(x,0)h + w(x) \tag{1}$$

where E is the modulus of elasticity of the beam, I is the moment of inertia, m is the mass per unit of length of the beam, and h is the depth of the beam in the z-direction. The pressure $p(x,y)e^{i\Omega t}$ of the fluid region A satisfies the wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\frac{\Omega^2}{c^2}p \tag{2}$$

where c is the speed of sound in the fluid. It is also assumed that

$$\frac{\partial p}{\partial n} = 0 \quad \text{on } S \tag{3}$$

where S, shown in figure 1, is the boundary of A excluding the beam's surface. On the surface of the beam, it is also necessary to enforce (see refs. 1, 2) the condition, which comes from conservation of momentum, that

$$\frac{\partial p}{\partial y} = -\rho\ddot{u} = \rho\Omega^2 u \quad \text{on } y = 0 \tag{4}$$

where ρ is the density of the fluid.

It is possible to formulate a mixed variational principle that will incorporate both equations (1) and (2) and the appropriate boundary conditions for each. Consider the functional F(u,p) given by

$$F(u,p) = \frac{1}{2}\int_0^{\ell} EI\left(\frac{d^2u}{dx^2}\right)^2 dx - \frac{1}{2}m\Omega^2\int_0^{\ell} u^2 dx + \int_0^{\ell} p(x,0)hu\,dx$$
$$- \int_0^{\ell} w(x)u\,dx + \frac{1}{2\rho\Omega^2}\int_A\left[\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2\right]hdA - \frac{1}{2\rho c^2}\int_A p^2 hdA \tag{5}$$

The first two terms are the strain energy and kinetic energy of the beam, respectively. The next two terms are minus the work done by the pressure and forcing function on the beam, and the last two terms represent the kinetic and potential energies of the fluid, respectively. The functional F is a function of both the structural displacements u and the fluid pressure p. If independent variations of F are taken with respect to u and p, it follows that

$$\delta F(u,p) = \int_0^{\ell}\left[EI\frac{d^4u}{dx^4} - m\Omega^2 u + ph - w(x)\right]\delta u\,dx$$
$$- \int_A\left[\frac{1}{\rho\Omega^2}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + \frac{p}{\rho c^2}\right]\delta p\,hdA + EI\frac{d^2u}{dx^2}\delta u'\Bigg|_0^{\ell} \tag{6}$$
$$- EI\frac{d^3u}{dx^3}\delta u\Bigg|_0^{\ell} + \frac{1}{\rho\Omega^2}\int_S\frac{\partial p}{\partial n}\delta p\,hds + \int_0^{\ell}\left[-\frac{1}{\rho\Omega^2}\frac{\partial p}{\partial y} + u\right]\delta p\,h\,dx$$

If u and p are found such that

$$\delta F(u,p) = 0 \tag{7}$$

with any trial function u satisfying

$$u(0) = u(\ell) = 0 \tag{8}$$

then it can be seen from equation (6) that u and p necessarily satisfy equations (1) and (2) and the boundary conditions given by equations (3) and (4).

Coupling of the Infinite Fluid

If the region of the fluid is infinite, as shown in figure 2, the fluid is subdivided: the finite element description of the pressure in the fluid is used in a finite region A_1 surrounding the structure, and an analytical expression (which identically satisfies the wave equation) is used in the remaining infinite region A_2. In order to properly couple the two solutions, the pressure field must be continuous and consistent with the variational principle. The functional F(u,p) in equation (5) now contains two additional terms which are the same as the last two terms but integrated over the remaining infinite region. The analytical expression for p in this region satisfies the proper boundary conditions at infinity (Somerfeld radiation condition). Variations of

F taken with respect to u and p in both regions give

$$\delta F(u,p) = \int_0^{\ell} \left[EI \frac{d^2u}{dx^4} - m\Omega^2 u + ph - w(x) \right] \delta u \, dx + \frac{1}{\rho\Omega^2} \int_S \frac{\partial p}{\partial n} \delta p \, h \, ds$$

$$+ EI \frac{d^2u}{dx^2} \delta u' \Big|_0^{\ell} - EI \frac{d^3u}{dx^3} \delta u \Big|_0^{\ell} + \int_0^{\ell} \left[\frac{1}{\rho\Omega^2} \frac{\partial p}{\partial n} + u \right] \delta p \, dx \tag{9}$$

$$- \int_{A_1} \frac{1}{\rho\Omega^2} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{p}{\rho c^2} \right) \delta p \, h dA_1 + \frac{1}{\rho\Omega^2} \int_s \frac{\partial p}{\partial n_1} \delta p \, h \, ds$$

$$- \int_{A_2} \frac{1}{\rho\Omega^2} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{p}{\rho c^2} \right) \delta p \, h dA_2 + \frac{1}{\rho\Omega^2} \int_s \frac{\partial p}{\partial n_2} \delta p \, h \, ds$$

where s is the boundary between the finite and infinite fluid regions. With the analytical function p identically satisfying the wave equation, the coefficient of δp in the next to last term is identically zero. Hence, only the term

$$I = \frac{1}{\rho\Omega^2} \int_s \frac{\partial p}{\partial n_s} \delta p \, h \, ds \tag{10}$$

which is the loading of the infinite region on the finite must be included. In the integral I, p is given by an analytical expression which must match the finite element nodal approximation on the fluid-fluid interface. Assume the analytical expression for p is given by an expansion

$$p = \sum_{n=1}^{N} A_n f_n(x,y) \tag{11}$$

where the A_n's are undetermined coefficients and the functions $f_n(x,y)$ identically satisfy the wave equation. Equation (11), together with the finite element description of the pressure at the interface and the continuity of the pressure field, will permit the integral in equation (10) to be evaluated. The continuity of the pressure field can be easily obtained by choosing an outer boundary on which the orthogonality of the functions $f_n(x,y)$ can be used. The evaluation of equation (11) will be carried out specifically for the frequency response of a beam in an infinite fluid.

FINITE ELEMENT FORMULATION

Beam and Neighboring Fluid

The finite element method approximates the displacements u of the beam by

$$u = \sum_i N_i^S u_i \tag{12}$$

where u_i is a generalized nodal displacement, and N_i^S is a shape function for the displacements of the beam. Similarly, the pressure field in the fluid is approximated by

$$p = \sum_i N_i^F p_i \tag{13}$$

where p_i is a nodal pressure. Substituting these approximations into equation (6) and interpreting equation (7) to mean that partial derivatives with respect to nodal displacements and pressures should equal zero, the following set of equations is determined:

$$\begin{bmatrix} K & L \\ 0 & \frac{1}{\rho\Omega^2}H \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} - \Omega^2 \begin{bmatrix} M & 0 \\ -\frac{1}{\Omega^2}L^T & \frac{1}{\rho c^2 \Omega^2}Q \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{14}$$

where

$$K_{ij} = \int_0^\ell EI \frac{d^2 N_i^S}{dx^2} \frac{d^2 N_j^S}{dx^2} dx \tag{15}$$

$$M_{ij} = m \int_0^\ell N_i^S N_j^S dx \tag{16}$$

$$f_i = \int_0^\ell w(x) N_i^S dx \tag{17}$$

$$L_{ij} = \int_0^\ell N_i^S N_j \, hdx \tag{18}$$

$$H_{ij} = \int_A \left[\frac{\partial N_i^F}{\partial x} \frac{\partial N_j^F}{\partial x} + \frac{\partial N_i^F}{\partial y} \frac{\partial N_j^F}{\partial y} \right] hdA \tag{19}$$

$$Q_{ij} = \int_A N_i^F N_j^F \, hdA \tag{20}$$

Multiplying the second set of equations by $(\rho c \Omega)^2$ gives

$$\begin{bmatrix} K & L \\ 0 & \rho c^2 H \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} - \Omega^2 \begin{bmatrix} M & 0 \\ -(\rho c)^2 L^T & \rho Q \end{bmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{21}$$

This form is the same as that derived by Zienkiewicz and Newton (ref. 1), but equations (21) are based on a variational principle. The set of equations (11) can be generated by NASTRAN as outlined in reference 2, although matrices L and L^T are inputted directly by DMIG cards.

For more complicated geometries and structures, the form of equation (21) is unchanged. While the variational principle and the finite element formulation were given specifically for the elastic beam-acoustic fluid problem, they can be easily generalized to account for an elastic structure bounded by an acoustic fluid.

Infinite Fluid Coupling Matrix

The loading of the infinite fluid on the finite portion is found by computing the integral

$$I = \frac{1}{\rho\Omega^2} \int_s \frac{\partial p}{\partial n_2} \delta p \, h \, ds \tag{22}$$

over the outer fluid face (see eq. (10)). This integral is to be discretized and then added to the set of equations (14). At the fluid-fluid interface the pressure is given by

$$p = \sum_{i=1}^{M} N_i p_i \tag{23}$$

where N_i is the shape function for the pressure in the fluid evaluated at the fluid-fluid interface, p_i is a nodal pressure on the face, and M is the number of pressure nodes at the fluid-fluid interface. The δp in equation (22) is equal to the partial of p with respect to p_i (which is equal to N_i from equation (23)). Then the term

$$\frac{1}{\rho\Omega^2} \int_s \frac{\partial p}{\partial n_2} N_i \, h \, ds \tag{24}$$

is added to the p_i equation of equations (14).

Consider the frequency response of a beam with one side immersed in an infinite acoustic fluid, as shown in figure 3. Both the displacements of the beam and the pressure field of the neighboring acoustic fluid are modeled by finite elements. The pressure field in the fluid for $y>b$ must be bounded and satisfy

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\left(\frac{\Omega}{c}\right)^2 p \tag{25}$$

with the boundary condition that

$$p = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = \ell \tag{26}$$

and the condition that only waves outgoing from the structure are allowed.

Separation of variables leads to the following expression for p:

$$p = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\alpha_n y} \tag{27}$$

with

$$\alpha_n^2 = \left(\frac{n\pi}{\ell}\right)^2 - \left(\frac{\Omega}{c}\right)^2 \tag{28}$$

The N arbitrary coefficients A_n are yet to be determined, and α_n may be either real or imaginary depending on n and Ω.

The integral in equation (22) can now be evaluated. From equation (27),

$$\frac{\partial p}{\partial n_2} = -\left.\frac{\partial p}{\partial y}\right|_{y=b} = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{\ell}\right) \alpha_n e^{-\alpha_n b} \tag{29}$$

Substituting equation (29) into the expression (24), the following matrix expression is added to the left-hand side of equations (14):

$$\frac{1}{\rho\Omega^2}[G]\{A\} \tag{30}$$

where {A} is a vector of the N coefficients A_j and [G] is an MxN matrix given by

$$G_{ij} = \int_0^\ell N_i \sin\left(\frac{j\pi x}{\ell}\right) \alpha_j e^{-\alpha_j b} dx \tag{31}$$

The number of unknowns in equation (14) has been increased by N, the number of A_j coefficients. An additional set of equations to make the set complete is found by requiring the pressure to be continuous; that is, equation (23) must match equation (27) evaluated at the interface y=b:

$$\sum_{i=1}^{M} N_i p_i = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\alpha_n b} \tag{32}$$

Multiplying both sides by

$$\sin\left(\frac{k\pi x}{\ell}\right)$$

and integrating from 0 to ℓ with the orthogonality condition that

$$\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{k\pi x}{\ell}\right) dx = \begin{cases} 0 & n \neq k \\ \frac{\ell}{2} & n = k \end{cases} \tag{33}$$

gives

$$\{A\} = [R]\{p\} \tag{34}$$

where [R] is an NxM matrix and is given by

$$R_{ij} = \frac{2\,e^{\alpha_i b}}{\ell} \int_0^\ell N_j \sin\left(\frac{i\pi x}{\ell}\right) dx \tag{35}$$

The additional N equations from equation (34) form a complete set with equations (14) and (30). Alternatively, equation (34) can be used to eliminate the series coefficients A_j from the expression (30) in favor of the nodal pressures of the fluid-fluid interface. Thus,

$$\frac{1}{\rho\Omega^2} [G]\{A\} = \frac{1}{\rho\Omega^2} [G][R]\{p\}$$

Define

$$[H'] = [G][R] \tag{36}$$

with matrices [G] and [R] given by equations (31) and (35). The effect of the infinite fluid on the finite is to add to the fluid stiffness matrix [H] of equations (14) the matrix [H']. [H'] is an MxM symmetric matrix which may be complex. It is full, frequency-dependent and couples just the nodes at the outer fluid boundary.

If $\Omega \to 0$ or equivalently $c \to \infty$, the effect of the infinite fluid on the finite fluid is one of stiffness. From equation (28),

$$\alpha_j \to \frac{j\pi}{\ell} \quad \text{as} \quad \Omega \to 0 \tag{37}$$

and

$$G_{ij} \to a_j e^{-\alpha_j b} F_{ij} \tag{38}$$

where

$$F_{ij} = \int_0^\ell N_i \sin\left(\frac{j\pi x}{\ell}\right) dx \tag{39}$$

The matrix [F] does not depend on the frequency Ω. Similarly,

$$R_{ij} \to \frac{2\,e^{\alpha_i b}}{\ell} F_{ij} \quad \text{as} \quad \Omega \to 0 \tag{40}$$

Then matrix [H'] = [G][R] is given by

$$H'_{ij} = \sum_{k=1}^{N} G_{ik} R_{kj} \tag{41}$$

Substituting equations (37), (38), and (40) into equation (41) gives

$$H'_{ij} = \frac{2\pi}{\ell^2} \sum_{k=1}^{N} k\, F_{ik} F_{jk} \tag{42}$$

where [H'] is a constant matrix (independent of frequency) and is added directly to the stiffness matrix [H] of the fluid. If the outer finite element boundary is chosen to be the surface of the beam, then the matrices [H] and [Q] of equation (14) are zero. Then the second set of equations (14) can be written, with the matrix H' now included, as

$$\frac{1}{\rho\Omega^2}[H']\{p\} + [L^T]\{u\} = 0$$

This gives

$$\{p\} = -\rho\Omega^2 [H']^{-1}[L^T]\{u\}$$

Substitution of this equation into the first set of equations (14) gives the following added mass matrix:

$$[M'] = \rho[L][H']^{-1}[L^T] \tag{43}$$

Matrix [M'] is symmetric and full and shows that the effect of the fluid on the structure is added mass.

When $\Omega \to \infty$ or $c \to 0$, the effect of the infinite fluid on the finite is a pure damping. For, from equation (28),

$$\alpha_j \to \frac{\Omega}{c} i$$

where i is $\sqrt{-1}$. Then

$$H'_{rs} = \frac{2}{\ell}\frac{\Omega}{c} i \sum_{k=1}^{N} F_{rk} F_{sk} \tag{44}$$

[H'] is a pure imaginary matrix, linear in Ω, which is to be added to the stiffness matrix [H] in equation (14). Then rewriting equation (14) in the form of equation (21) gives

$$\begin{bmatrix} K & L \\ 0 & \rho c^2 H \end{bmatrix}\begin{Bmatrix} u \\ p \end{Bmatrix} + i\Omega \begin{bmatrix} 0 & 0 \\ 0 & \rho c B \end{bmatrix}\begin{Bmatrix} u \\ p \end{Bmatrix} - \Omega^2 \begin{bmatrix} M & 0 \\ -(\rho c)^2 L^T & \rho Q \end{bmatrix}\begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix} \tag{45}$$

where [B] is an MxM damping matrix and an element of matrix [B] is given by

$$B_{rs} = \frac{2}{\ell} \sum_{k=1}^{N} F_{rk} F_{rs} \tag{46}$$

The coefficient F_{ij}, defined in equation (39), is equal to ($\ell/2$) times the j^{th} Fourier sine coefficient of the shape function N_i. Hence the sum of terms on the right-hand side of equation (46) is the dot product of the Fourier coefficients of the function N_r with N_s. It can be shown that this dot product is equal to ($2/\ell$) times the inner product of the shape function N_r with N_s over the length of the beam. Thus,

$$B_{rs} = \frac{2}{\ell} \sum_{k=1}^{N} F_{rk} F_{sk} = \int_0^{\ell} N_r N_s \, dx \tag{47}$$

This term is identical to the one suggested by Zienkiewicz and Newton (ref. 1), which is a boundary condition derived by assuming that the pressure in the fluid takes the form of a plane wave. The boundary condition is to be applied at a boundary which has been placed "far enough" from the structure and is the proper boundary condition only in the high frequency limit.

If the outer fluid boundary is reduced to that of the beam, then both matrices [H] and [Q] in equation (45) are zero. Solving the second set of equations (45) for {p} in terms of {u} gives

$$\{p\} = \rho c(\Omega i)[B]^{-1}[L^T]\{u\} \tag{48}$$

Substituting this equation into the first set gives

$$[K]\{u\} + (\Omega i)\rho cL[B]^{-1}[L^T]\{u\} - \Omega^2[M]\{u\} = \{f\} \tag{49}$$

Then the matrix

$$\rho c[L][B]^{-1}[L^T]$$

is a damping matrix, which means that the effect of the fluid on the structure at high frequencies (or small c) is damping.

EXACT SOLUTION - FREQUENCY RESPONSE OF BEAM

The differential equation of motion for the elastic beam shown in figure 3 subjected to a uniform load varying sinusoidally in time is

$$EI \frac{\partial^4 u}{\partial x^4} + m \frac{\partial^2 u}{\partial t^2} + p(x,0,t)h = w_0 e^{i\Omega t} \tag{50}$$

where p(x,y,t) is the acoustical pressure which must satisfy the wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \tag{51}$$

A solution for u of the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{\ell}) e^{i\Omega t} \tag{52}$$

is chosen, where the A_n's are undetermined coefficients of $\sin \frac{n\pi x}{\ell}$, which are the out-of-fluid eigenvectors for the simply supported beam. Equation (51) is solved using separation of variables. The pressure field is bounded and the boundary condition at x=0 and x=ℓ is that p=0. Allowing only outgoing waves from the beam leads to the following equation for p:

$$p = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi x}{\ell}) e^{-\alpha_n y} e^{i\Omega t} \tag{53}$$

where

$$\alpha_n^2 = (\frac{n\pi}{\ell})^2 - (\frac{\Omega}{c})^2$$

The C_n's are undetermined coefficients to be found by properly coupling the fluid and the structure. At the fluid-structure interface, one requires (see, for example, ref. 2)

$$\frac{\partial p}{\partial n} = - \rho \ddot{u}_n \quad \text{at} \quad y = 0 \tag{54}$$

Since the fluid and structural modes are uncoupled for this problem, equation (54) yields

$$C_n = - \frac{\rho \Omega^2}{\alpha_n} A_n \tag{55}$$

Substituting equation (55) into equation (53), and the expressions for u and p into equation (50), gives

$$\sum_{n=1}^{\infty} \left\{ -(m + \frac{\rho h}{\alpha_n}) \Omega^2 + EI (\frac{n\pi}{\ell})^4 \right\} A_n \sin(\frac{n\pi x}{\ell}) e^{i\Omega t} = w_0 e^{i\Omega t} \tag{56}$$

with

$$\alpha_n^2 = (\frac{n\pi}{\ell})^2 - (\frac{\Omega}{c})^2$$

When both sides are multiplied by $\sin (\frac{k\pi x}{\ell})$ and integrated from 0 to ℓ to take advantage of orthogonality relationships, the solution is

$$u(x,t) = \sum_{n=1,3,...}^{\infty} A_n \sin \frac{n\pi x}{\ell} e^{i\Omega t} \tag{57}$$

$$p(x,y,t) = \sum_{n=1,3,\ldots}^{\infty} -\frac{\rho\Omega^2}{\alpha_n} A_n \sin\frac{n\pi x}{\ell} e^{-\alpha_n y} e^{i\Omega t} \tag{58}$$

with

$$A_n = \frac{4w_0}{n\pi\{-\Omega^2(m+\frac{\rho h}{\alpha_n}) + EI(\frac{n\pi}{\ell})^4} \tag{59}$$

and

$$\alpha_n^2 = (\frac{n\pi}{\ell})^2 - (\frac{\Omega}{c})^2 \tag{60}$$

In general α_n may be real or imaginary depending on the driving frequency Ω. If

$$\Omega \leq \frac{\pi c}{\ell}$$

then for all n, α_n is real. In this case the pressure and displacement are real and in phase and no radiation occurs. If $\Omega > (\pi c/\ell)$, then for some n, α_n becomes imaginary. In this case, both u and p are complex and out of phase and hence radiation may occur.

Peaks in the frequency response will occur at the in-fluid natural frequencies of the beam. Although the in-fluid and in-air mode shape of the beam are unchanged in this particular problem, the natural frequency of the beam does change. The in-fluid natural frequencies are found by setting the term in brackets in equation (56) equal to zero and solving for Ω. The solution always gives $\Omega_n < (n\pi c/\ell)$, which means that the in-fluid modal shapes of the beam do not radiate.

RESULTS

Computations were carried out by NASTRAN using the finite element-analytical method previously described. A typical grid is that shown in figure 3, where CBAR elements were used to model the beam and 2-D isoparametric elements (with quadratic approximation for the pressure) were chosen to model the fluid. The usual double numbering of grid points at the fluid-structure interface is necessary with this formulation (this procedure is outlined in ref. 2), and the nodal pressures and displacements of the interface are coupled through matrix [L] of equation (14). This matrix is entered into NASTRAN by DMIG cards. The frequency dependent matrix [H'] defined in equation (36), which models the effect of the infinite fluid on the finite, is also inputted into NASTRAN by DMIG cards. The results shown in figures 4 through 8 are for the following values: h = 2.54 cm (1 inch), ℓ = 50.8 cm (20 in.), c = 1.460 km/sec (5.748×10^4 in/sec), E = 206.8 GPa (3×10^7 psi), I = 3.468 cm^4 (.08333 in^4), m = 7.827 g/cm^3 (7.324×10^{-4} $lb\text{-}sec^2/in^4$), ρ = 1.029 g/cm^3 (9.633×10^{-5} $lb\text{-}sec^2/in^4$), w_0 = 1.751 N/cm (1 lb/in). The solid line in each of these figures is the exact solution given by equations (57) through (60). The finite element solution is shown at specific plotted points.

Figure 4 is a plot of the magnitude of the pressure at the center of the beam versus the driving frequency Ω. The pressure peaks at approximately 1062 rad/sec and 11350 rad/sec, which are the in-fluid natural frequencies of the beam for modes n = 1 and n = 3 (these values can be determined by solving equation (56) for Ω with $w_0 = 0$). There is a discontinuity in the slope of the curve for $\Omega \simeq 9029$ radians/sec, which is the frequency at which radiation occurs ($\Omega = \pi c/\ell = 9029$ rad/sec). For frequencies greater than $\pi c/\ell$, energy is being carried away by the outgoing pressure waves and the beam is said to radiate. In this case net work is done by the forcing function.

Figures 5 and 6 are plots of the magnitude of the beam's displacement at its center as a function of the driving frequency Ω. In figure 6 the displacement shows the discontinuity in slope that the pressure exhibits when the beam begins to radiate. As Ω goes through $\pi c/\ell$, the displacement of the beam increases corresponding to the reduction in pressure.

The variations of the phase angles of the pressure and displacement with frequency are shown in figures 7 and 8, respectively. For $\Omega \leq \pi c/\ell$ (9029 rad/sec), the displacement is in phase (or 180° out of phase) with the driving force and no work is done. For $\Omega > \pi c/\ell$, the displacement is out of phase with the driving frequency and radiation occurs. The only exception to this condition occurs when Ω approaches a natural frequency. The mode shape for that frequency dominates, and, since the in-fluid mode shapes of the beam do not radiate, the phase angle of the displacement is in phase (or 180° out of phase) with the driving force.

Figures 4-8 show that the finite element solution obtained through NASTRAN was reliable in modeling the elastic beam in the infinite acoustical fluid. The errors of the results shown were 1-2% for the grid shown in figure 3. The same accuracy was also obtained at a few frequencies in which the outer fluid boundary was chosen to be that of the beam (that is, b=0). In these cases, matrix [H'] (defined in eq. (36)) corresponds to the nodal pressures at the fluid-structure interface.

For the limiting case of $\Omega \to 0$, the effect of the fluid on the structure is an added mass; this effect is approximated within the finite element method by modeling the structure with NASTRAN and adding to the mass matrix generated by NASTRAN the additional mass matrix [M'] given by equation (43). The natural frequencies and mode shapes of this computation agreed favorably (less than 1% error) with those from the exact solution. The exact solution is determined from equation (56) by solving for Ω with $w_0 = 0$ and $c \to \infty$.

CONCLUSIONS

The boundary condition at the truncated fluid boundary of an infinite acoustical fluid is, in general, frequency dependent. For a finite element formulation this condition leads to a stiffness matrix [H'] which is added to the stiffness matrix of the fluid. [H'] is a full, symmetric, complex, frequency dependent matrix which couples the infinite region to the finite region and involves only the outer boundary nodes. If the driving frequency is specified (in the case of frequency response), the coupling matrix [H'] can be inputted into NASTRAN by DMIG cards. The computation of eigenvalues and eigenvectors, on the other hand, would necessarily involve an iteration scheme since the frequency of the mode shape is not known.

Although only the portion of the fluid immediately surrounding the structure is modeled by finite elements, the infinite fluid region is effectively modeled through the coupling matrix [H']. Moreover, the far field pressure can be determined once the outer boundary pressures are computed. This pressure is given by equation (27) with the series coefficients {A} determined from equation (34). If a finite portion of the fluid is modeled without including the boundary condition matrix [H'], then the fluid region is actually a finite domain. Not only is it impossible to determine the far field pressure but also some of the eigenvalues and eigenvectors found can be shown to be associated with the finite problem. These additional modal values do not appear if [H'] is included.

This type of finite element-analytical solution, presented here for a two-dimensional problem, can be readily generalized to a three-dimensional problem of modeling an elastic structure in an infinite acoustic medium. In this case the outer fluid boundary would be a sphere, and the pressure field would be given by an expansion of spherical harmonics. The frequency dependent matrix could be generated in NASTRAN by program modifications and would be accessed through DMAP alters. Unfortunately, because of the full coupling of all the outer boundary nodes, the increase in the bandwidth for a three-dimensional problem might make the computer cost prohibitive.

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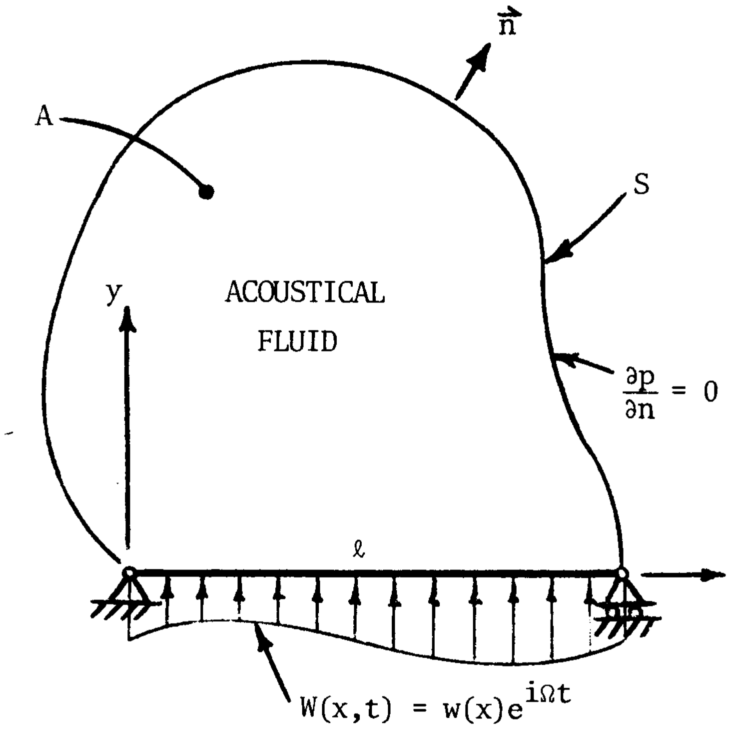

Figure 1. Sinusoidally Loaded Beam with One Side Immersed in a Finite Acoustical Fluid.

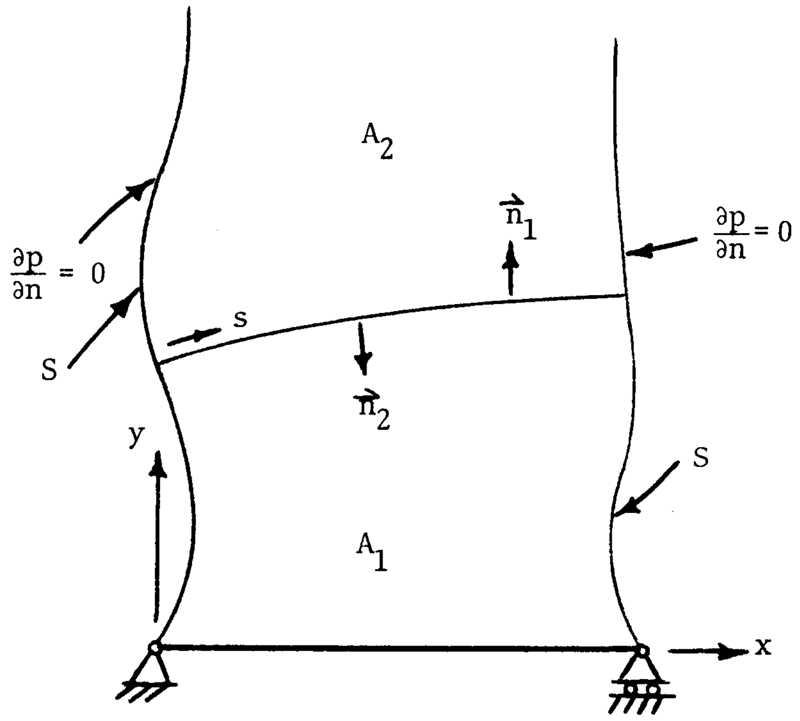

Figure 2. Fluid Region Divided into Finite Region A_1 and Infinite Region A_2.

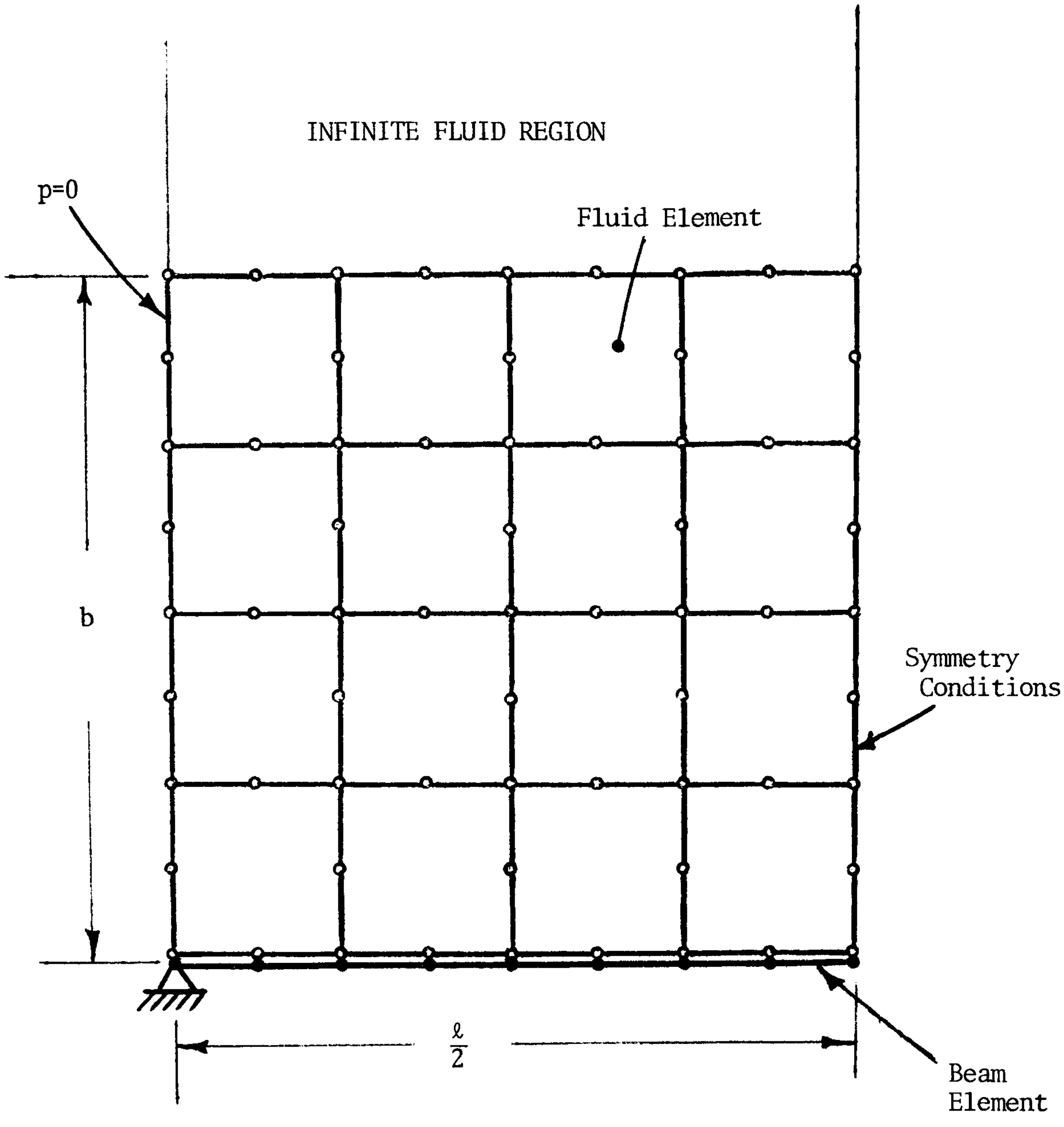

Figure 3. Finite Element Grid Showing Both the Fluid and Beam Elements. Nodal Pressures of the Fluid and Nodal Displacements of the Structure are the Unknowns.

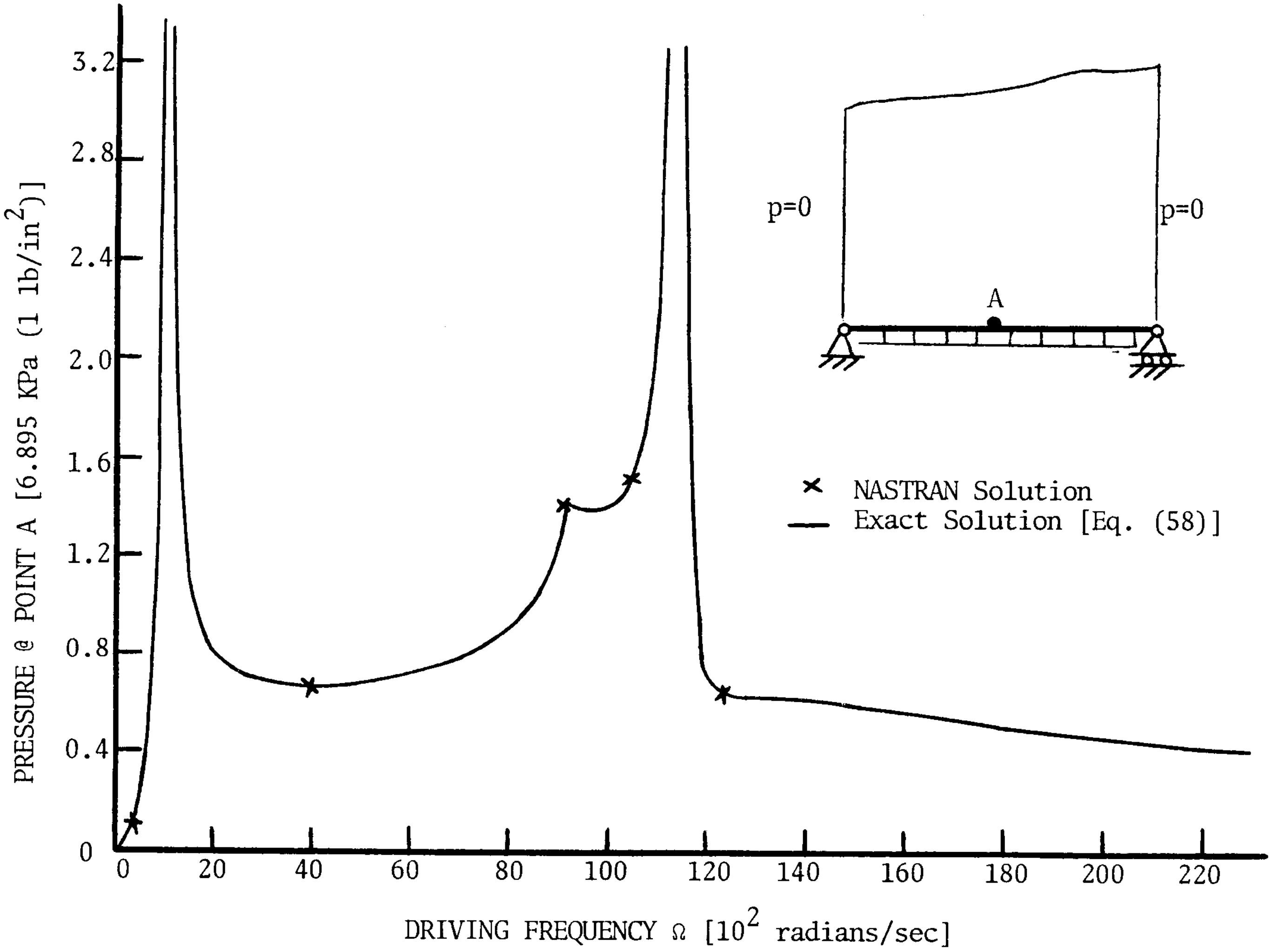

Figure 4. Magnitude of the Acoustical Pressure at Point A (Center of Beam) as a Function of the Driving Frequency. Peaks Occur at the In-Fluid Natural Frequencies Given by Equation (56).

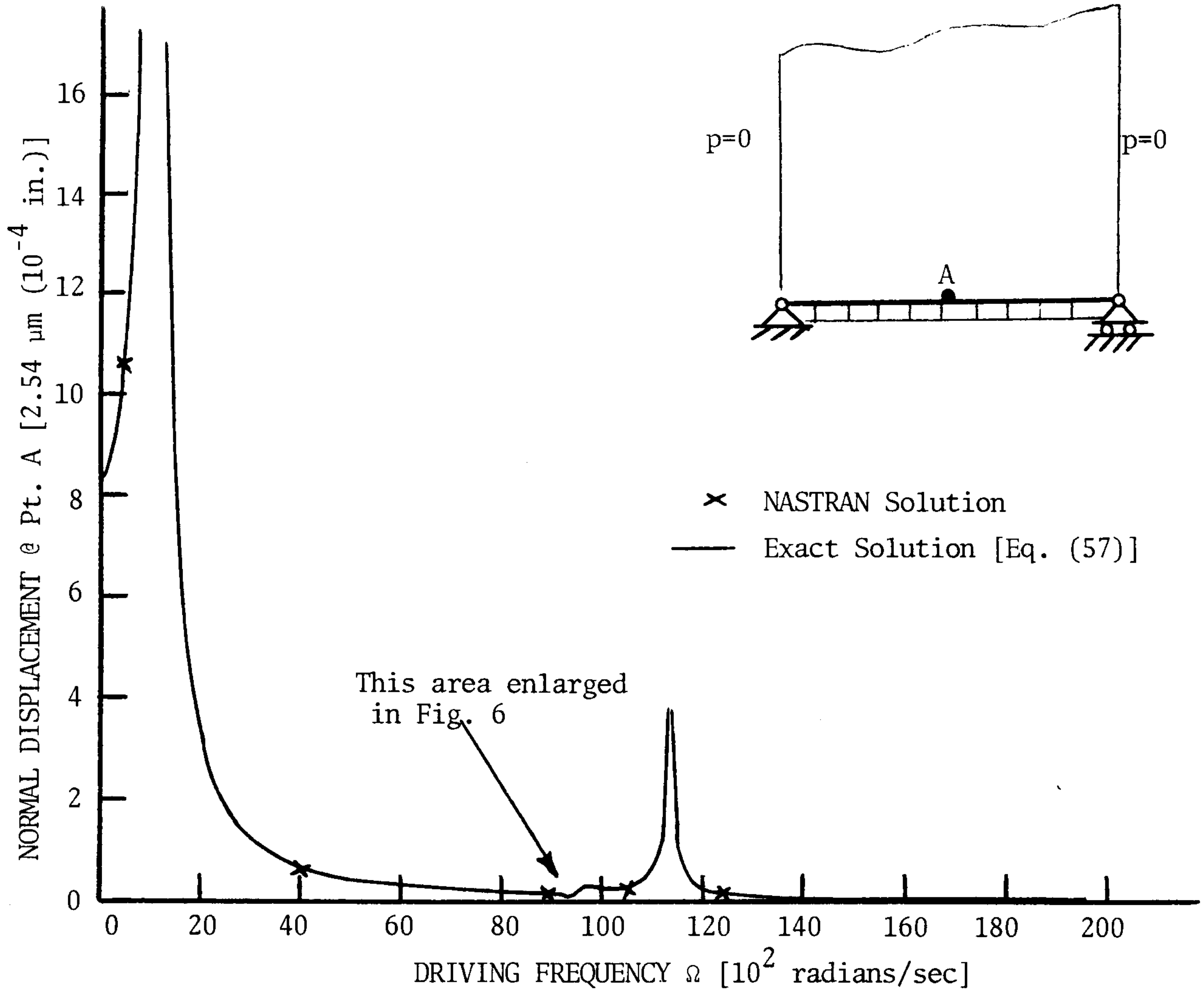

Figure 5. Magnitude of the Normal Displacement of the Center Beam Point (Point A) as a Function of the Driving Frequency. Peaks Occur at the Natural Frequencies Given by Equation (56).

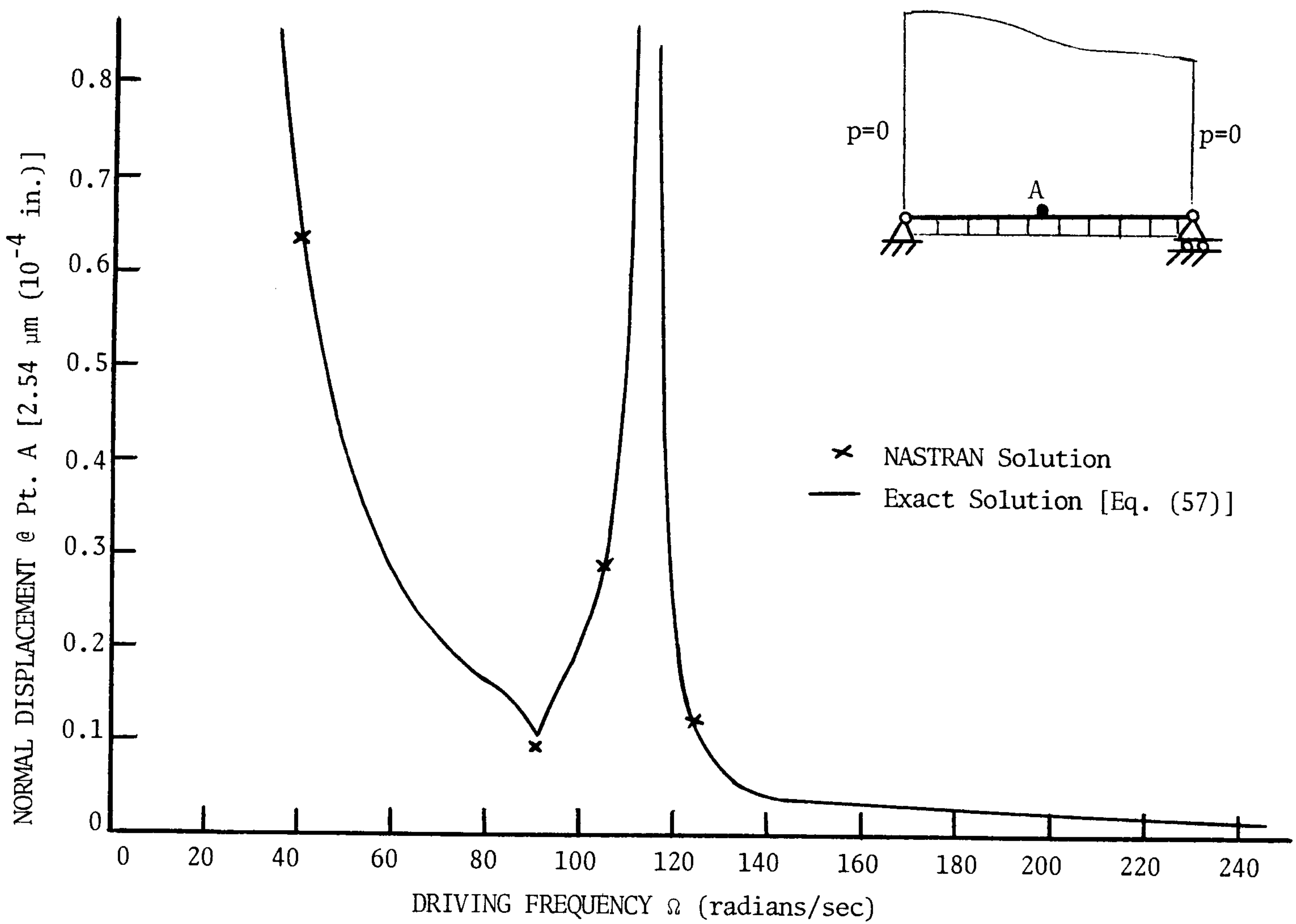

Figure 6. Magnitude of the Normal Displacement of the Center Beam Point (Pt. A) as a Function of the Driving Frequency. Peaks Occur at the Natural Frequencies Given by Equation (56).

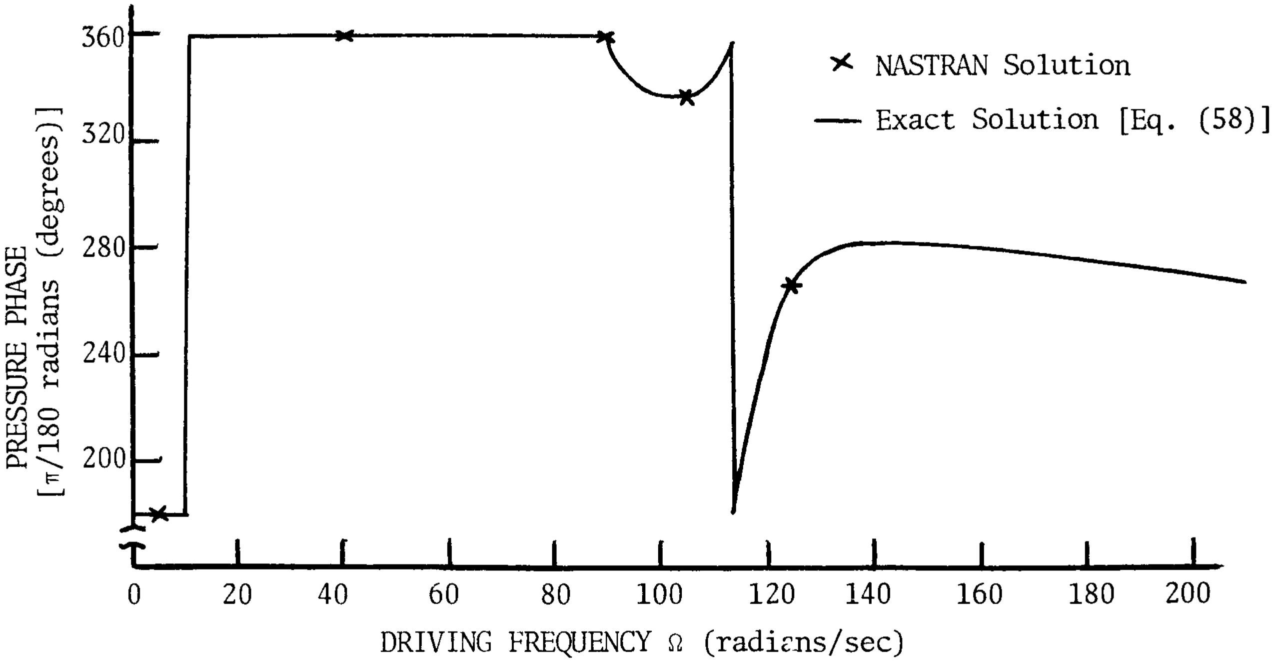

Figure 7. Phase Angle of the Acoustical Pressure at the Center of the Beam as a Function of the Driving Frequency Ω.

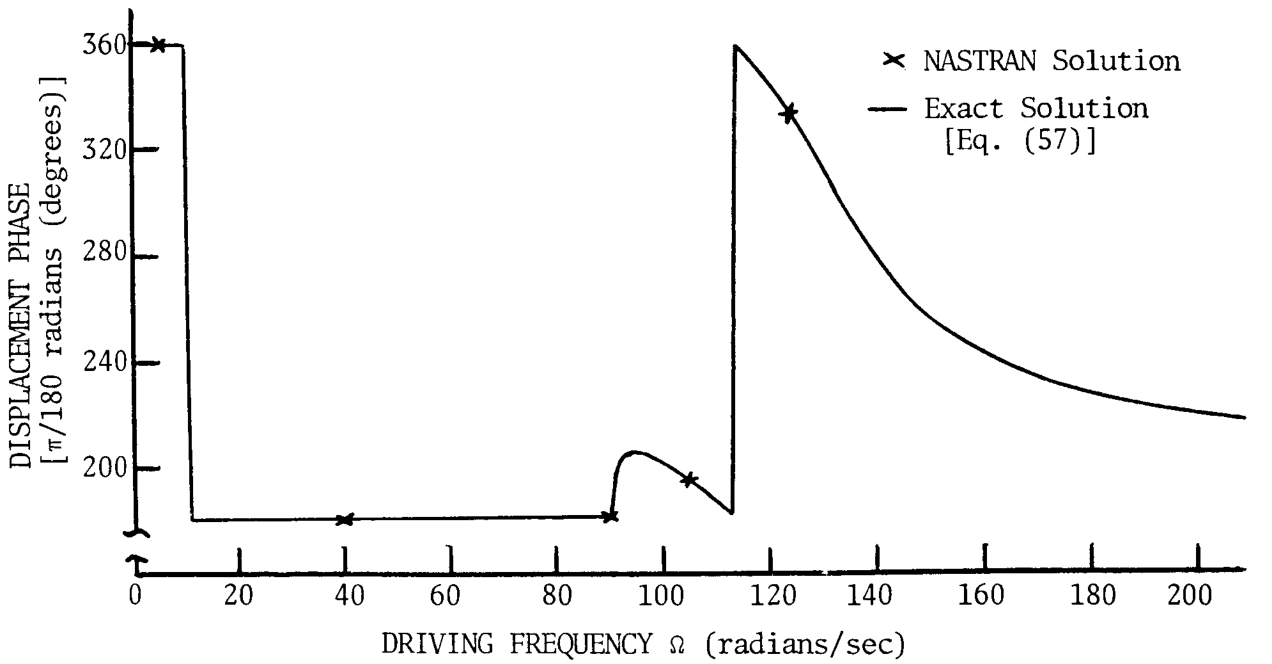

Figure 8. Phase Angle of the Normal Displacement of the Center Beam Point as a Function of the Driving Frequency Ω.