THREE-DIMENSIONAL FINITE ELEMENT ANALYSIS OF ACOUSTIC INSTABILITY OF SOLID PROPELLANT ROCKET MOTORS*

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SUMMARY

A three-dimensional finite element solution of the acoustic vibration problem in a solid propellant rocket motor is presented. The solution yields the natural circular frequencies of vibration and the corresponding acoustic pressure mode shapes, considering the coupled response of the propellant grain to the acoustic oscillations occurring in the motor cavity. The near incompressibility of the solid propellant is taken into account in the formulation. A relatively simple example problem is solved in order to illustrate the applicability of the analysis and the developed computer code.

INTRODUCTION

Present solid propellant rocket combustion instability state-of-the-art technology employs a linear analysis for predicting unsteady motions in the combustion chamber of the rocket based on the hypothesis that the influences of combustion and flow can be represented as perturbations of an acoustic problem in a closed chamber. In order to perform an accurate combustion instability analysis, one must have adequate knowledge of the natural circular frequency and corresponding pressure mode shape for the different modes of acoustic oscillation occurring in the motor cavity in the absence of combustion and flow.

The cavities in functional rocket motors generally have complex shapes that may include a number of symmetrically located slots (or fins) cut into the propellant, the purpose of which is to increase the area of the burning surface (fig. 1). During the past several years, the need for experimental acoustic modeling of such irregular cavity geometries has been greatly reduced through the use of finite element modeling techniques (ref. 1). In the finite element method, a continuum is modeled as an assemblage of elements, connected at discrete locations (nodes). Natural frequencies and mode shapes are then obtained from an eigensolution of the resulting equations of equilibrium of the modeled dynamic system. Applications of the finite element method to more generalized acoustic analysis problems are found in references 2 and 3.

The presence of the solid propellant grain can significantly shift the acoustic system frequency from that of the gas phase alone, a portion of the acoustic energy being dissipated by the deformable solid material. This effect can be one of the more significant sinks for acoustic energy in both large and small rocket motors, the amount of damping depending on the grain geometry, propellant mechanical properties, and the acoustic mode shape and frequency. It

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is thus important to have the capability of finite element modeling of both the acoustic cavity and the propellant grain structure and, therefore, of a combination and coupling of fluid and solid elements. Analyses of coupled plate-acoustic systems are found in references 4 and 5, and a three-dimensional finite element coupled acousto-mechanical analysis is formulated in reference 6.

In the case at hand, the problem involves the coupling of the irrotational motions of an inviscid compressible fluid with the deformations of an incompressible (viscoelastic) solid material enclosing the fluid. Since the propellant grain is only accurately modeled as a nearly incompressible material, the well-known minimum potential energy finite element formulation, based upon Navier's equations of equilibrium and the Ritz procedure, is invalid. Formulations for cases of incompressible and nearly incompressible materials are found in references 7, 8, 9, 10, and 11. The solid finite element formulation utilized in this development is a linear displacement-linear mean pressure tetrahedron (ref. 10), similar to the Herrmann variational formulation (ref. 7) which employs a linear displacement function and a constant mean pressure function.

The eigensolution technique employed in the developed program is that of condensation, which is discussed in reference 12. The technique utilizes the master and slave degree-of-freedom (d.o.f.) concept. In this application, the fluid pressure d.o.f. are designated master and the solid displacement and mean pressure d.o.f. are designated slave.

SYMBOLS

Values are given in both SI and U.S. Customary Units. The calculations were made in U.S. Customary Units.

a,b,c,d	exponential factors
B	bulk modulus
[D]	degree-of-freedom to strain transformation matrix
$[\bar{E}]$	constitutive coefficient matrix
[f]	fluid element potential energy coefficient matrix
[F]	fluid system potential energy coefficient matrix
G	shear modulus
H	mean pressure parameter
[I]	identity matrix
[k]	solid element stiffness matrix
[K]	solid system stiffness matrix
L	natural or volume tetrahedral coordinate
[m]	solid element consistent mass matrix
[M]	solid system consistent mass matrix
p	pressure
P	point (defined in fig. 2)
S	surface area
[t]	fluid element kinetic energy coefficient matrix
[T]	fluid system kinetic energy coefficient matrix
u,v,w	displacement components
[U]	acoustic-solid coupling matrix
V	element volume
x,y,z	rectangular Cartesian coordinates
β	direction cosine
$\{\Delta\}$	displacement vector
ε	strain

ν	Poisson's ratio
$[\Phi]$	transformation matrix defined by equation (14)
ρ	mass density
ω	natural circular frequency

Subscripts:

C	coupled
i,j,ℓ,m	element node identifiers
p,q	number of degrees-of-freedom
r,s	node identifiers (eq.(4))
S	solid

Superscripts:

e	element
T	transpose (of a matrix)

EQUATIONS OF EQUILIBRIUM

Acoustic Cavity

In applying the finite element method, the region of concern is divided into a number of subregions, or elements. Applications of the method in modeling the behavior of a fluid continuum involve several pecularities not encountered in the usual structural applications. In the discretization of a fluid continuum, the finite elements represent spatial rather than material subregions of the continuum; i.e., instead of representing finite elements of the fluid material, the elements represent subregions in the space through which the fluid moves (Eulerian description of motion). Values of pressure at the nodal points of the element represent the pressures at the nodes rather than of the nodes (ref. 13).

For a three-dimensional region, a volume element is associated with a number of nodal points which define its shape. Within each fluid tetrahedral element the variation of the pressure p is prescribed by the values associated with the four element nodes through the expression:

$$p(x,y,z) = \lfloor L \rfloor \{p\}^e = \lfloor L_i \; L_j \; L_\ell \; L_m \rfloor \begin{Bmatrix} p_i \\ p_j \\ p_\ell \\ p_m \end{Bmatrix} \tag{1}$$

where the natural or volume coordinates L_i, L_j, L_ℓ, and L_m are defined mathematically by

$$\begin{Bmatrix} L_i \\ L_j \\ L_\ell \\ L_m \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_i & x_j & x_\ell & x_m \\ y_i & y_j & y_\ell & y_m \\ z_i & z_j & z_\ell & z_m \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} \tag{2}$$

and physically by figure 1.

The matrix form of the homogeneous acoustic wave equations for the discretized fluid continuum is given by

$$([F] - \omega^2[T])\{p\} = \{0\} \tag{3}$$

where F is the coefficient matrix derived from a consideration of the potential energy of a fluid element, and T is the coefficient matrix derived from a consideration of the kinetic energy of the fluid element. The natural circular frequency of the discretely modeled fluid continuum is designated as ω.

The matrix F is developed through a superpositioning of the individual symmetric 4x4 element matrices, f, with restraint of the appropriate d.o.f. A typical term of the f matrix is (ref. 14):

$$f^e_{rs} = \frac{1}{\rho}\left(\frac{\partial L_r}{\partial_x}\frac{\partial L_s}{\partial_x} + \frac{\partial L_r}{\partial_y}\frac{\partial L_s}{\partial_y} + \frac{\partial L_r}{\partial_z}\frac{\partial L_s}{\partial_z}\right)V \tag{4}$$

where r and s take the values i,j,ℓ,m cyclically. The fluid mass density and the volume of the tetrahedron are designated ρ and V, respectively.

The matrix T is likewise developed through the superpositioning of the individual symmetric 4x4 element matrices, t, with restraint of the same appropriate d.o.f. The t matrix has the form (ref. 14):

$$[t] = \frac{1}{B}\int \begin{bmatrix} L_i^2 & L_iL_j & L_iL_\ell & L_iL_m \\ & L_j^2 & L_jL_\ell & L_jL_m \\ & & L_\ell^2 & L_\ell L_m \\ \text{sym} & & & L_m^2 \end{bmatrix} dxdydz \tag{5}$$

where B is the bulk modulus of the fluid. A useful integration formula for evaluating the terms in the t matrix is:

$$\int L_i^a L_j^b L_\ell^c L_m^d \, dxdydz = \frac{a!b!c!d!}{(a+b+c+d+3)!}\, 6V \tag{6}$$

Solid Propellant

Within each solid tetrahedral element, the variation of displacement is prescribed by the displacement of each element node through the expression:

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = [L]\{\Delta\}^e = [L_iI \quad L_jI \quad L_\ell I \quad L_mI]\begin{Bmatrix} \Delta_i \\ \Delta_j \\ \Delta_\ell \\ \Delta_m \end{Bmatrix} \tag{7}$$

where I is the 3x3 identity matrix and where

$$\lfloor \Delta_i \rfloor = \lfloor u_i \quad v_i \quad w_i \rfloor, \text{ etc.} \tag{8}$$

The u,v,w terms are the displacement components in the x,y,z directions, respectively.

The matrix form of the homogeneous equations of motion for the discretized solid continuum is given by

$$\left(\begin{bmatrix} K \end{bmatrix} - \omega_S^2 \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right] \right) \left\{\begin{array}{c} \Delta \\ \hline H \end{array}\right\} = \{0\} \tag{9}$$

where K is the stiffness matrix and M is the mass matrix. The term ω_S is the natural circular frequency of vibration of the solid system. The linear variation of the mean pressure parameter

$$H(x,y,z) = \frac{1}{1-2\nu} (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \tag{10}$$

is prescribed in the same manner as were the pressure (in the acoustic cavity) and the displacement function, i.e.,

$$H(x,y.z) = \lfloor L \rfloor \{H\}^e = \lfloor L_i \; L_j \; L_\ell \; L_m \rfloor \begin{Bmatrix} H_i \\ H_j \\ H_\ell \\ H_m \end{Bmatrix} \tag{11}$$

The terms ε_{xx}, ε_{yy}, and ε_{zz} are the normal strain components and ν is Poisson's ratio.

The matrix K is developed through a superpositioning of the individual symmetric 16x16 element stiffness matrices, k, with restraint of the appropriate d.o.f. The form of the matrix k is given by (ref. 10):

$$[k] = \left[\begin{array}{c|c} D^T & 0 \\ \hline 0 & I \end{array}\right] \int [\Phi]^T [\bar{E}] [\Phi] \, dxdydz \left[\begin{array}{c|c} D & 0 \\ \hline 0 & I \end{array}\right] \tag{12}$$

where I is the 4x4 identity matrix and where

$$[D] = \left[\begin{array}{cccc|cccc|cccc}
\frac{\partial L_i}{\partial x} & \frac{\partial L_j}{\partial x} & \frac{\partial L_\ell}{\partial x} & \frac{\partial L_m}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial y} & \frac{\partial L_\ell}{\partial y} & \frac{\partial L_m}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial L_i}{\partial z} & \frac{\partial L_j}{\partial z} & \frac{\partial L_\ell}{\partial z} & \frac{\partial L_m}{\partial z} \\
\frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial y} & \frac{\partial L_\ell}{\partial y} & \frac{\partial L_m}{\partial y} & \frac{\partial L_i}{\partial x} & \frac{\partial L_j}{\partial x} & \frac{\partial L_\ell}{\partial x} & \frac{\partial L_m}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial L_i}{\partial z} & \frac{\partial L_j}{\partial z} & \frac{\partial L_\ell}{\partial z} & \frac{\partial L_m}{\partial z} & \frac{\partial L_i}{\partial y} & \frac{\partial L_j}{\partial y} & \frac{\partial L_\ell}{\partial y} & \frac{\partial L_m}{\partial y} \\
\frac{\partial L_i}{\partial z} & \frac{\partial L_j}{\partial z} & \frac{\partial L_\ell}{\partial z} & \frac{\partial L_m}{\partial z} & 0 & 0 & 0 & 0 & \frac{\partial L_i}{\partial x} & \frac{\partial L_j}{\partial x} & \frac{\partial L_\ell}{\partial x} & \frac{\partial L_m}{\partial x}
\end{array}\right] \tag{13}$$

$$[\Phi] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & L_i & L_j & L_\ell & L_m
\end{bmatrix} \tag{14}$$

and

$$[\bar{E}] = 2G\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \nu \\
0 & 1 & 0 & 0 & 0 & 0 & \nu \\
0 & 0 & 1 & 0 & 0 & 0 & \nu \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
\nu & \nu & \nu & 0 & 0 & 0 & -\nu(1-2\nu)
\end{bmatrix} \tag{15}$$

where G is the shear modulus of the material. The ordering of rows and columns in the k matrix corresponds to

$$\lfloor u_i \; u_j \; u_\ell \; u_m \; v_i \; v_j \; v_\ell \; v_m \; w_i \; w_j \; w_\ell \; w_m \; H_i \; H_j \; H_\ell \; H_m \rfloor$$

The matrix M is developed through the superpositioning of the individual symmetric 12x12 element mass matrices, m, with restraint of the same appropriate d.o.f. The consistent mass matrix, derived in references 15 and 16, is used in the formulation and is defined by:

$$[m] = \rho_S \; [L]^T[L] \; dxdydz \tag{16}$$

where ρ_S is the mass density of the solid propellant grain and where

$$[L] = [L_i I \quad L_j I \quad L_\ell I \quad L_m I] \tag{17}$$

Acoustic-Solid Coupling

The formulation of the coupled acoustic-solid equations through equilibrium considerations is found in references 5 and 6. Reference 5 deals with the transient response of a coupled system while, in reference 6 the eigenvalue problem is formulated. The matrix form of the homogeneous equations of equilibrium for the discretized coupled system is given by

$$\left(\left[\begin{array}{c|c} F & 0 \\ \hline -U^T & K \end{array}\right] - \omega_C^2 \left[\begin{array}{c|c} T & U \\ \hline 0 & \begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array} \end{array}\right] \right) \left\{\begin{array}{c} P \\ \hline \Delta \\ \hline H \end{array}\right\} = \{0\} \tag{18}$$

where ω_C is the natural circular frequency of the coupled system.

The matrix U of equation (18) has dimensions pxq, where p is the number of pressure d.o.f. and q is the number of displacement and mean pressure parameter d.o.f. The components of the U matrix are projections of the cavity-propellant interface area associated with each surface nodal point, the surface area distribution being physically defined in figure 3. The portion of the U matrix associated with the i,j,ℓ triangular surface area of a boundary tetrahedron is given by

$$\left[\begin{array}{ccc|ccc|ccc|ccc} \beta_{ix}S_i & 0 & 0 & \beta_{iy}S_i & 0 & 0 & \beta_{iz}S_i & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_{jx}S_j & 0 & 0 & \beta_{jy}S_j & 0 & 0 & \beta_{jz}S_j & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_{\ell x}S_\ell & 0 & 0 & \beta_{\ell y}S_\ell & 0 & 0 & \beta_{\ell z}S_\ell & 0 & 0 & 0 \end{array}\right]$$

where β_{ix}, ... , $\beta_{\ell z}$ are the direction cosines and S_i, S_j and S_ℓ are the contributing (to a node) surface areas.

SOLUTION

Solution of the system mathematically modeled by equation (18) is facilitated by the use of condensation (ref. 12). In applying the technique to this problem, the pressure d.o.f. are designated master and the displacement and mean pressure parameter d.o.f. are designated slave.

Example Application

The developed program is applied to the solution of the simple acoustic-solid system shown in figure 4. The plane defined by nodes 5678 is the acoustic-solid interface. The pressure is set equal to zero at nodes 1,2,3, and 4. Thus, for the corresponding solution, $p = 4$ and $q = 32$.

The properties of the cavity gas used are: a mass density ρ of 1.225 kg/m^3 (1.146×10^{-7} $lb\text{-}sec^2/in^4$) and a bulk modulus B of 1.42×10^5 Pa (20.59 lb/in^2). The properties of the solid propellant grain used are: a shear modulus G of 4.41×10^7 Pa (6400 lb/in^2), a Poisson's ratio ν of 0.5 and a mass density ρ_S of 1.770×10^3 kg/m^3 (1.656×10^{-4} $lb\text{-}sec^2/in^4$).

The solution for the coupled system yielded frequencies of 488 Hz, 1008 Hz, 1193 Hz, and 1804 Hz, compared to respectively corresponding values of 3670 Hz, 6934 Hz, 10,358 Hz, and 12,368 Hz obtained for the fluid system alone.

CONCLUDING REMARKS

It has been demonstrated through the use of a simple example that the developed computer code can be used to predict the effect of solid propellant grain structure on the natural frequency associated with the acoustic vibrations occurring in the rocket motor cavity. It can be concluded that the effect of certain suppression devices such as resonance rods can be predicted in the same manner.

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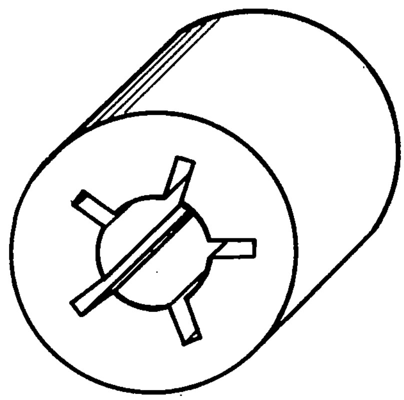

Figure 1.- Solid propellant configuration.

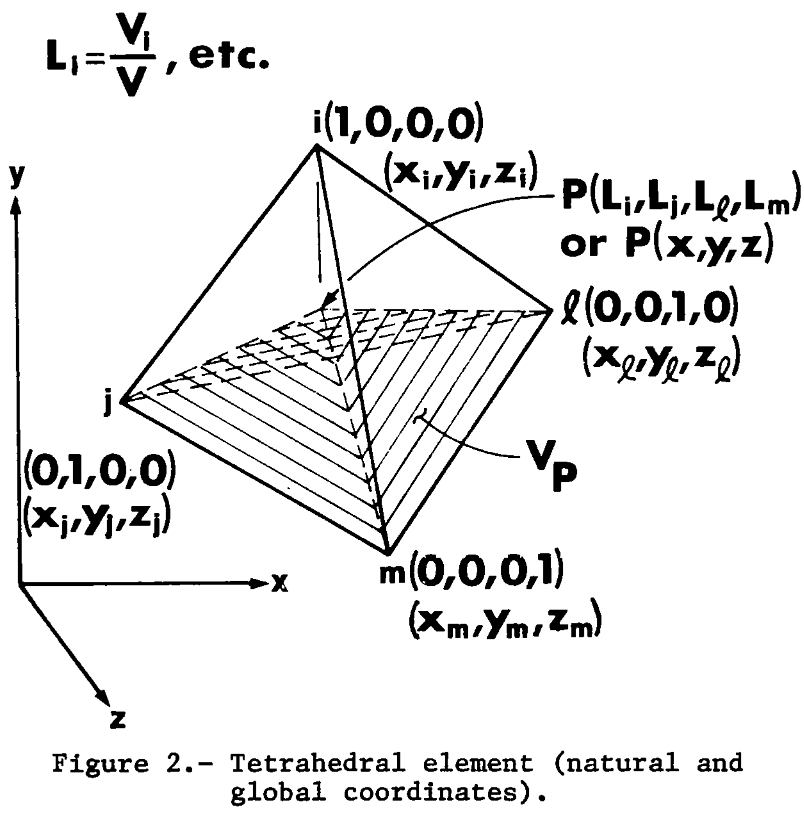

Figure 2.- Tetrahedral element (natural and global coordinates).

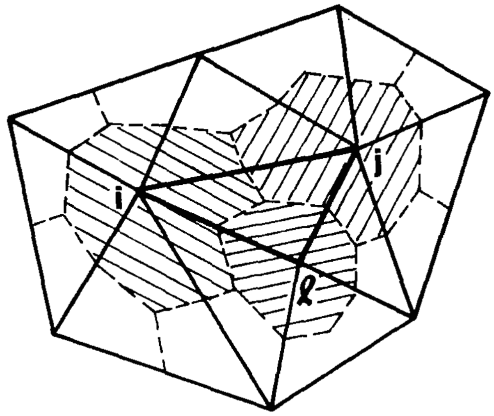

Figure 3.- Surface areas associated with interfacial nodal points.

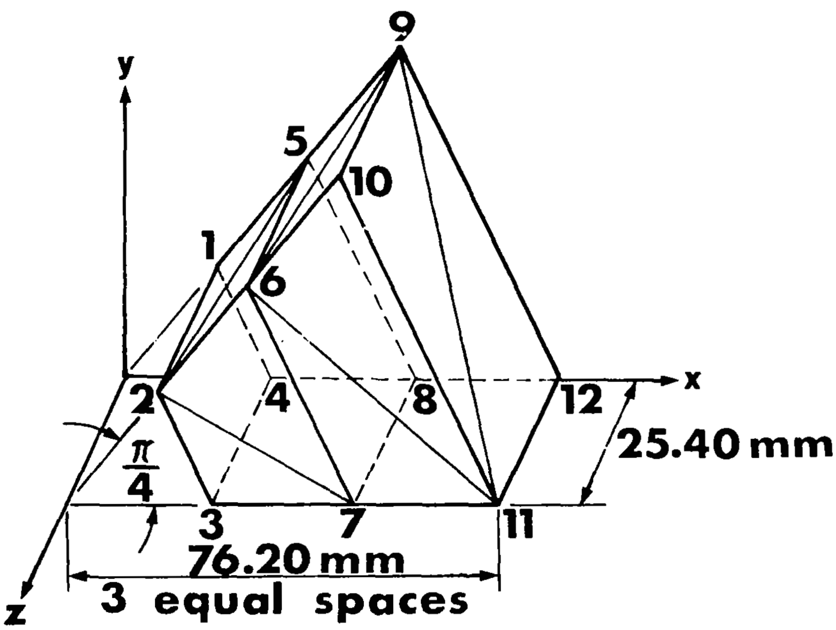

Figure 4.- Acoustic-solid system.