THE MULTIGRID METHOD: FAST RELAXATION FOR TRANSONIC FLOWS*

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SUMMARY

A multi-level grid method has been studied as a possible means of accelerating convergence in relaxation calculations for transonic flows. The method employs a hierarchy of grids, ranging from very coarse (e.g. 4 × 2 mesh cells) to fine (e.g. 64 × 32); the coarser grids are used to diminish the magnitude of the smooth part of the residuals, hopefully with far less total work than would be required with, say, optimal SLOR iterations on the finest grid. To date the method has been applied quite successfully to the solution of the transonic small-disturbance equation for the velocity potential in conservation form. Non-lifting transonic flow past a parabolic-arc airfoil is the example studied, with meshes of both constant and variable step size.

INTRODUCTION

The multi-level grid method, for accelerating convergence in relaxation calculations, has been shown to be very efficient for solving elliptic problems with Dirichlet boundary conditions. For background and historical material, see references 1 to 4. The idea of the method is based on the fact that in many typical elliptic boundary-value problems, the error is composed of a discrete spectrum of wave lengths, which range from the width of the region down to the width of a mesh cell. The short wave-length components of the error are usually diminished quite rapidly in a relaxation calculation, while the long wave-length components diminish very slowly. After only a few iterations the residual will be smooth, since the short wave-length error components have been eliminated; and thus the residual can be represented accurately on a coarser mesh. An equation called the "residual" equation is then solved on the coarser mesh, and the resulting correction is added to the last approximation on the fine mesh, yielding a significant improvement with very little work.

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Since relaxation methods are currently the most attractive for obtaining numerical solutions to transonic aerodynamics problems, the question arises as to whether a multi-level, or multi-grid (MG), method can be used in a mixed flow with shock waves. In this paper we report some early results using the MG method to solve a simple transonic problem: we consider the transonic small-disturbance equation for the velocity potential, for nonlifting flow past a parabolic-arc airfoil.

PROBLEM DESCRIPTION

The transonic small-disturbance equation for the velocity potential can be written in conservation form as:

$$p_x + q_y = 0 \tag{1}$$

where

$$p = \left[K - \frac{(\gamma + 1)}{2} M_\infty^2 \phi_x\right] \phi_x \tag{2}$$

$$q = \phi_y \tag{3}$$

$$K = (1 - M_\infty^2)/\tau^{2/3} \tag{4}$$

Equation (1) is to be solved subject to the boundary conditions that the disturbance potential, ϕ, vanishes at infinity and the flow is tangent to the airfoil surface, in the interval $|x| \leq 1/2$; i.e.,

$$\text{at } y = 0, \quad \phi_y = F'(x) \quad \text{for} \quad |x| \leq 1/2$$
$$= 0 \quad \text{for,} \quad |x| > 1/2 \tag{5}$$

where $F(x)$ is the (upper surface) thickness distribution function, τ is the usual thickness ratio, and γ, M_∞, and K are the ratio of specific heats, free-stream Mach number, and transonic similarity parameter, respectively. The form of equations (1) to (5) is a correctly scaled transonic similarity form, in that all quantities are of order 1. Equation (1) is of hyperbolic or elliptic type depending on whether

$$U = K - (\gamma + 1)M_\infty^2 \phi_x \tag{6}$$

is negative or positive, respectively.

Finite-Difference Equations

Murman's conservative difference scheme (ref. 5) can be conveniently presented in terms of Jameson's "switching function" (ref. 6) as follows:

$$\left(1 - \mu_{ij}\right) P_{ij} + \mu_{i-1,j} P_{i-1,j} + Q_{ij} = 0 \tag{7}$$

where

$$P_{ij} = U_{ij}\left(\frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{\Delta x^2}\right) \tag{8}$$

$$U_{ij} = K - (\gamma+1)M_\infty^2\left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}\right) \tag{9}$$

$$Q_{ij} = \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{i,j-1}}{\Delta y^2} \tag{10}$$

and where

$$\begin{aligned} \mu_{ij} &= 0 \quad \text{if} \quad U_{ij} > 0 \\ &= 1 \quad \text{if} \quad U_{ij} \leq 0 \end{aligned} \tag{11}$$

It should be noted here that, in the interest of simplicity, we have presented only the constant-step-size (unstretched grid) form of the difference equations. The actual computer program is written for a stretched grid, with the identity transformation (constant step size) included as a special case.

Vertical Line Relaxation

A vertical line relaxation scheme for solving equation (7) by iteration can be written as:

$$AT_{i,j-1} + BT_{ij} + CT_{i,j+1} = R_{ij} + DT_{i-1,j} + ET_{i-2,j} \tag{12}$$

where

$$T_{ij} = \phi_{ij}^{+} - \phi_{ij} \tag{13}$$

ϕ^+ denotes a "new" value of ϕ, obtained during the latest iteration sweep, while ϕ is the value from the previous sweep. R_{ij}, which is the left-hand side of equation (7), is evaluated with "old" values of ϕ_{ij}, as are the iteration coefficients A through E, which are given in the appendix.

Multi-Grid Approach

Residual equation.- Let us introduce a sequence of grids $G_1, G_2, \ldots, G_m$, where for simplicity, $h_k = 2h_{k+1}$, and h_k represents the step size of the G_k grid. We can represent the iteration operator (e.g., eq. (12)) on the finest grid G_M as:

$$L_M (\phi_M) = f_M \tag{14}$$

where ϕ_M is the exact discrete solution on the G_M grid. We can write

$$\phi_M = u_M + v_M \tag{15}$$

where u_M is the approximate solution and v_M is the error. Then we have the residual equation:

$$\begin{aligned} \bar{L}_M (v_M) &= f_M - L_M (u_M) \\ &= - R_M \end{aligned} \tag{16}$$

where R_M is the residual of the approximation u_M on the G_M grid. $\bar{L}_M$ is in general different from L_M in the nonlinear case, which complicates matters. Nevertheless, if R_M is smooth, the error will be smooth, and the residual equation (16) can be solved on a coarser grid. Thus, for example, we can write

$$\bar{L}_{M-1} (w_{M-1}) = I_M^{M-1} (R_M) \tag{17}$$

where w_{M-1} is an approximation to the error v_M on the G_{M-1} grid, and I_k^{ℓ} denotes interpolation from the G_k to G_{ℓ}. After solving the problem (17) (usually with homogeneous boundary conditions), we interpolate the function w_{M-1} back onto the G_M mesh, and thus form an improved approximation:

$$(u_M)_{\text{new}} = (u_M)_{\text{old}} + I_{M-1}^{M} (w_{M-1}) \tag{18}$$

In the complete MG algorithm, the solution of equation (17) is also performed by relaxation; and if the convergence rate falls below a prescribed level, we can apply a similar procedure, backing up to the G_{M-2} grid level, and so on, until we arrive at G_1, if necessary. The G_1 grid is so coarse that a direct solution could be used economically, but we have used iteration here also.

Full approximation.- In the general nonlinear case, the form of the operation $\bar{L}$ can be quite complicated — more so than the original operator, L — and thus applications to, say, the full potential equation may be tedious to program. It turns out that for the transonic small-disturbance equation, the job is simple, and our first program did use the exact expression for $\bar{L}$ in an efficient way. However, there is an equivalent, easier method for solving the residual equation, which we call the full approximation method, as follows:

Suppose we add to both sides of equation (17) the function

$$L_{M-1}\,(u_M) - f_{M-1} = \tilde{R}_{M-1} \tag{19}$$

Then, since

$$\bar{L}_{M-1}\,(w_{M-1}) + L_{M-1}\,(u_M) \simeq L_{M-1}\,(\phi_M),$$

we have

$$L_{M-1}\,(\phi_M) \simeq \tilde{R}_{M-1} - I_M^{M-1}\,(R_M) \tag{20}$$

We can now use the original operator on all the grids, which greatly simplifies the programming. The right-hand side of equation (20) is the difference between the residuals of u_M calculated with the coarse- and fine-grid operators. Note that when the solution converges on the G_M grid, then

$$R_M \rightarrow 0 \tag{21a}$$

$$I_M^{M-1}(R_M) \rightarrow 0 \tag{21b}$$

but $\tilde{R}_{M-1}$ will remain finite, since ϕ_M is a solution on the G_M grid; $\tilde{R}_{M-1}$ is essentially the truncation error of the L_{M-1} operator.

After equation (20) is solved to sufficient accuracy, we determine the function

$$w_{M-1} = \phi_M - I_M^{M-1} (u_M) \tag{22}$$

by subtraction at all points of the grid G_{M-1}, and then interpolate w_{M-1} to the G_M grid as before in equation (18).

RESULTS AND DISCUSSION

In order to estimate the efficiency of the method, a work unit can be defined as the amount of computational effort required for one relaxation sweep on the (finest) G_M grid. Thus a relaxation sweep on the G_k grid costs $n_w = (1/4)^{M-k}$ work units, for example. Likewise, when we calculate the residuals for the G_k grid, we perform these calculations at the points of the G_{k-1} grid, i.e., 1/4 as few points; hence each residual calculation costs less than 1/4 the effort of a relaxation sweep on the G_k grid, or approximately $(1/4)^{M-k+1}$. Note that this is an overestimate, since the tridiagonal system (12) is not inverted, nor do we calculate the iteration coefficients during the residual calculations. On the other hand we did not count the work of interpolation in equation (18), for example, or any other "overhead" of that type.

An overall estimate of efficiency can be given by the effective spectral radius

$$a = \left\{ ||R_{M,n_w}|| \; / \; ||R_{M,1}|| \right\}^{1/n_w} \tag{23}$$

where

$||R_{M,1}||$ = norm of R_M after first sweep on G_M

$||R_{M,n_w}||$ = norm of R_M after n_w work units

and

$$||R_M|| = \left(\Delta x \; \Delta y \sum_{ij} R_M^2\right)^{1/2} \tag{24}$$

Hence the norm we use is the root mean square of the residual on G_M. This number is typically about 5 to 10 times smaller than the maximum norm in transonic problems. We consider an <u>approximate solution to be converged when</u>

$$||R_M|| < C/(\text{no. of grid points}) \qquad (25)$$

where the prescribed constant C is typically chosen as 1 so as to estimate the nominal truncation error.

Unstretched Grids

In the case of a grid with constant steps in both directions, the present MG method performed quite well. Some typical results are summarized in table I and discussed briefly in the following.

In all cases, the MG runs were made with a relaxation factor $\omega = 1.0$ on all grids.

<u>Laplace's equation with smooth boundary conditions</u>.- To illustrate just how fast the MG method works for a nice, smooth, elliptic problem, we present in table I results for the solution of Laplace's equation with the prescribed normal derivative equal to $\sin \pi x$ along $y = 0$. Because of the smoothness of the boundary data, it could be expected that interpolating a converged G_4 (32×16 grid) solution onto G_5 will give a very good starting approximation for G_5. This is true, for although the convergence rate on G_5 yielded $a = .583$, the efficiency of the two combined levels is more like $a = .46$! In contrast, successive line overrelaxation (SLOR) achieved $a = .924$ on G_5, starting from the zero solution, and using a relaxation factor $\omega = 1.85$.

<u>Nonlinear airfoil flows</u>.- The next three entries in table I show the results for the nonlinear problem of flow over a parabolic-arc airfoil. In these cases, the Neumann boundary condition is an "N-wave" — far from smooth — but the $M_\infty = 0.7$ subcritical case (i.e., no supersonic flow) converged as well as the previous smooth problem; hence, it can be concluded that discontinuous boundary conditions do not deteriorate MG performance. The "combined" mode of operation, where the converged solution for G_4 is used to start G_5, was not helpful, since the truncation errors around boundary singularities and shock waves were so large. That is, the G_4-solution gives a large residual when interpolated on the G_5-mesh. The $M_\infty = 0.85$ (moderately supercritical) case had 124 supersonic points out of a total of 2145 mesh points on G_5, or 6%. The relative efficiency between MG and SLOR is still unaffected. In both of the aforementioned nonlinear cases, the SLOR runs were carried out with $\omega = 1.85$, which was found to be near optimal by experiment.

The last of the unstretched grid cases is $M_\infty = 0.95$ (highly supercritical), with 355 supersonic points. The flow pattern exhibited a weak oblique shock at the trailing edge, followed by a triangular region of nearly constant supersonic flow, which was terminated by a normal shock in the wake. The final number of supersonic points was established after 38 work units, and the solution converged after 67.6 work units, giving $a = 0.858$. The SLOR run was unstable with $\omega = 1.85$, and had to be "babied" by slowly increasing ω, using an interactive remote terminal. The best result achieved was $n_w = 228$, with $a = .957$.

Stretched Grids

An attractive way to satisfy the boundary condition at infinity is to transform the independent variables such that the infinite space is mapped onto a finite domain. However, it became quickly evident that vertical line relaxation alone is not the best way to relax the solution for a stretched grid, either in the MG mode or simple SLOR. Analysis of the difficulty shows that all the high-frequency error modes are not rapidly damped if the mesh aspect ratio differs significantly from 1.0; the success of the MG method, of course, hinges on this feature. The analysis, not given here, also indicates that a solution to this problem is to sweep in all directions alternately (forward, backward, up, and down).

The last entry in table I shows the results of a stretched-grid case, again for $M_\infty = 0.95$. The deterioration of the MG method is clear; some benefit over SLOR is achieved, however, by the MG method.

CONCLUDING REMARKS

The MG method for accelerating relaxation calculations has proved to be applicable to nonlifting transonic flows with embedded shock waves. The method appears to work from three to five times faster than optimal SLOR on unstretched grids of moderate size (64×32); the relative advantage of MG over SLOR increases as the grid gets finer, since the MG convergence rate is nearly independent of mesh size. It is probable that the gains in three-dimensional calculations would be even more impressive, since each coarser grid requires only 1/8 the work of the next finer grid.

On stretched grids, the present MG approach slows down, being only about twice as fast as SLOR. It is felt that a remedy is the use of alternating-direction relaxation sweeps.

In the future we hope to develop the MG method for flows with lift; for otherwise it will have limited usefulness in aerodynamics.

During the course of our work, Professor Antony Jameson of the Courant Institute of Mathematical Sciences, New York University, also carried out research on the multi-grid method. He showed independently that the "full approximation" approach would work, and some of his attempts at alternating-direction sweeps have been encouraging. Our many discussions have been beneficial.

APPENDIX

ITERATION COEFFICIENTS

We have used various choices for iteration coefficients in equation (12). The coefficients used to make the calculations presented in this paper are simply based on the Newton linearization of equations (7), (8), and (10). They are as follows:

First define: (dropping the j index, since all quantities are evaluated at the same j)

$$b_{i+\frac{1}{2}} = K - (\gamma+1)M_\infty^2\frac{(\phi_{i+1} - \phi_i)}{\Delta x} \tag{A1}$$

Then we have

$$\bar{U}_i = \frac{1}{2}\left(b_{i+\frac{1}{2}} + b_{i-\frac{1}{2}}\right) = U_i\Delta x^{-2} \tag{A2}$$

$$A = C = -\Delta y^{-2} \tag{A3}$$

$$B = 2\Delta y^{-2} + 2\ (1-\mu_i)\ \bar{U}_i/\omega - \mu_{i-1}\ b_{i-\frac{1}{2}} \tag{A4}$$

$$D = (1-\mu_i)\ b_{i-\frac{1}{2}} - 2\mu_{i-1}\ \bar{U}_{i-1} \tag{A5}$$

$$E = \mu_{i-1}\ b_{i-\frac{3}{2}} \tag{A6}$$

where

$$\begin{aligned} \mu_i &= 0 \quad \text{if} \quad U_i > 0 \\ &= 1 \quad \text{if} \quad U_i \le 0 \end{aligned} \tag{A7}$$

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TABLE I.- SUMMARY OF MULTI-GRID RESULTS, 64 × 32 CELLS

Problem description		Effective spectral radius[a] for —	
		MG	SLOR
Unstretched grid	Laplace's equation, smooth boundary conditions	0.583 (0.46 combined levels)	0.924
	Parabolic airfoil, $M_\infty = 0.70$	.549	.868
	Parabolic airfoil, $M_\infty = .85$	.593	.855
	Parabolic airfoil, $M_\infty = .95$	.858	.957
Stretched grid	Parabolic airfoil, $M_\infty = 0.95$	0.936	0.974

[a]See equation (23).