LIFTING SURFACE THEORY FOR RECTANGULAR WINGS*

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SUMMARY

A new incompressible lifting-surface theory is developed for thin rectangular wings. The solution requires the downwash equation to be in the form of Cauchy-type integrals. Lan's method is employed for the chordwise integrals since it properly accounts for the leading-edge singularity, Cauchy singularity and Kutta condition. The Cauchy singularity in the spanwise integral is also accounted for by using the midpoint trapezoidal rule and theory of Chebychev polynomials. The resulting matrix equation, formed by satisfying the boundary condition at control points, is simplier and quicker to compute than other lifting surface theories. Solutions were found to converge with only a small number of control points and to compare favorably with results from other methods.

INTRODUCTION

Numerous subsonic lifting surface theories have been developed for thin wings over the past thirty years. A comparison of three of the more prominent numerical methods in 1968 is given in reference 1. Although all these methods give essentially the same results, they require tedious integration techniques and consume considerable computational time. Vortex lattice methods are simpler and can be applied to more complex configurations; however, they are generally less accurate than lifting surface methods. This author developed a lifting surface method for thin rectangular wings (ref. 2) which can also be interpreted as a vortex lattice method. The present paper examines the convergence of solutions and compares results with those of the NRL (National Aerospace Laboratory, Netherlands) method given in reference 1.

SYMBOLS

A aspect ratio

b wing span

c wing chord

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c_ℓ	sectional lift coefficient
c_d	sectional drag coefficient
C_{D_i}	far-field induced-drag coefficient
$C_{D_{ii}}$	near-field induced-drag coefficient
C_L	wing lift coefficient
C_M	wing pitching-moment coefficient, about leading edge
C_S	leading-edge suction parameter, see eq. (13)
G	parameter defined by eq. (10)
$K_{ijk\ell}$	parameter defined by eq. (15)
M-1	number of spanwise control points over whole span
N	number of chordwise control points
NLR	National Aerospace Laboratory, Netherlands
S	wing planform area
V_∞	freestream velocity
w	non-dimensional downwash velocity, referred to V_∞ and positive upwards
x	chordwise coordinate measured from leading edge in direction of V_∞
x_{ac}, X_{ac}	sectional and wing aerodynamic-center locations, respectively
y	spanwise coordinate, positive to the right
α	angle of attack
γ	non-dimensional circulation per unit chord
Γ	circulation
θ	transformed chordwise coordinate, see eq. (7)

ϕ transformed spanwise coordinate, see eq. (8)

Subscripts:

i chordwise control point, see eq. (5)

j spanwise control point, see eq. (17)

k chordwise integration point, see eq. (4)

ℓ spanwise integration point, see eq. (16)

p evaluated at spanwise position $\phi_p = p\pi/M$

ANALYSIS

For simplicity, the present method is developed for rectangular wings. The downwash equation from lifting surface theory is usually given by one of the two following forms (ref. 3):

$$w(x,y) = \frac{1}{4\pi} \int\int_S \frac{\gamma(x_1, y_1)}{(y - y_1)^2} \left[1 + \frac{(x - x_1)}{\sqrt{(x - x_1)^2 + (y - y_1)^2}}\right] dx_1 \, dy_1 \qquad (1)$$

or

$$w(x,y) = -\frac{1}{4\pi} \int\int_S \frac{\partial\gamma}{\partial y_1} \frac{1}{(y - y_1)} \left[1 + \frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{(x - x_1)}\right] dx_1 \, dy_1 \qquad (2)$$

Equation (1) is the form used by the three methods compared in reference 1. It contains the Mangler-type integral due to the term $(y - y_1)^2$ in the denominator. Equation (2), however, contains Cauchy-type integrals since the terms $(y - y_1)$ and $(x - x_1)$ in the denominator are linear. The present method requires the downwash to be in the form of equation (2).

In order to understand the development of the present method, consider the two-dimensional problem first. Lan (ref. 4) developed an ingenious method for thin airfoils by using the midpoint trapezoidal rule and the theory of Chebychev polynomials to reduce the two-dimensional downwash integral (Cauchy-type integral) to a finite sum. The remarkable accuracy of this technique is due to the following summational result:

$$\sum_{k=1}^{N} \frac{1}{\cos\theta_k - \cos\theta_i} = \begin{cases} -N^2 & (i = 0) \\ 0 & (i \neq 0,N) \\ N^2 & (i = N) \end{cases} \qquad (3)$$

where

$$\theta_k = \frac{(2k - 1)\pi}{2N} \qquad \left(k = 1, \ldots, N\right) \tag{4}$$

are the integration points and

$$\theta_i = i\pi/N \qquad \left(i = 1, \ldots, N\right) \tag{5}$$

are the control points (positions where the boundary condition is applied). Note the similarity of equation (3) to the integral result

$$\int_0^\pi \frac{d\theta_1}{\cos\theta_1 - \cos\theta} = 0 \tag{6}$$

Lan used these equations to develop the two-dimensional downwash summation which properly accounts for the Cauchy singularity, the leading-edge square-root singularity, and the Kutta condition at the trailing edge.

Since equation (2) contains Cauchy-type integrals, Lan's approach will be applied to both the chordwise and spanwise integrals. First, transform the chordwise coordinate by

$$2x/c = 1 - \cos\theta \tag{7}$$

and the spanwise coordinate by

$$2y/b = -\cos\phi \tag{8}$$

Then, use Multhopp's interpolation formula to represent the circulation per unit chord by the trigonometric sum

$$\gamma(\theta,\phi) = \frac{2}{M}\sum_{p=1}^{M-1} \gamma_p(\theta) \sum_{n=1}^{M-1} \sin n\phi_p \sin n\phi \tag{9}$$

where the subscript "p" refers to the spanwise position $\phi_p = p\pi/M$ and $(M - 1)$ represents the number of spanwise control points. Substitute equations (7), (8), and (9) into equation (2) and consider the chordwise integral first. With

$$G(\theta,\theta_1,\phi,\phi_1) \equiv \sqrt{(\cos\theta_1 - \cos\theta)^2 + A^2(\sin\phi_1 - \sin\phi)^2} \tag{10}$$

the chordwise integrals are

$$\int_0^\pi \gamma_p(\theta_1)\sin\theta_1\left[1 + \frac{G}{\cos\theta_1 - \cos\theta}\right]d\theta_1 = \frac{2\Gamma_p}{cV_\infty} + \int_0^\pi \frac{\gamma_p(\theta_1)\sin\theta_1\, G\, d\theta_1}{\cos\theta_1 - \cos\theta} \tag{11}$$

The integral on the right side can be reduced to a finite sum by the midpoint trapezoidal rule; and equations (3) and (6) can be used to account for the leading-edge singularity, Cauchy singularity, and Kutta condition ($\gamma_p(\pi) = 0$) as shown below:

$$\int_0^{\pi} \frac{\gamma_p(\theta_1) \sin\theta_1 \, G \, d\theta_1}{\cos\theta_1 - \cos\theta} = \int_0^{\pi} \frac{[\gamma_p(\theta_1) \sin\theta_1 \, G - \gamma_p(\theta) \sin\theta \, A|\cos\phi_1 - \cos\phi|]d\theta_1}{\cos\theta_1 - \cos\theta}$$

$$\approx \frac{\pi}{N} \sum_{k=1}^{N} \frac{\gamma_p(\theta_k) \sin\theta_k \, G_{ik}(\phi,\phi_1)}{\cos\theta_k - \cos\theta_i} + \begin{cases} 4\pi N C_{S_p} A|\cos\phi_1 - \cos\phi| & (i=0) \\ 0 & (i\neq 0) \end{cases} \qquad (12)$$

where θ_k are the chordwise integration points given by equation (4) and θ_i are the chordwise control points given by equation (5). The leading-edge suction parameter is defined as

$$4C_{S_p} = \lim_{\theta\to 0} \gamma_p(\theta) \sin\theta \qquad (13)$$

Now substitute equation (12) into equation (2) and perform the spanwise integration in a somewhat similar manner, accounting for the Cauchy singularity, to obtain the final form of the downwash as

$$w_{i,j} = \frac{-\pi c}{2bM^2N} \sum_{\ell=1}^{M} \sum_{k=1}^{N} \sum_{p=1}^{M-1} \gamma_{p,k} \sum_{n=1}^{M-1} \frac{n \sin n\phi_p \cos n\phi_\ell \, K_{ijk\ell} \sin\theta_k}{(\cos\phi_\ell - \cos\phi_j)} + \begin{cases} -2NC_{S_j} & (i=0) \\ 0 & (i\neq 0) \end{cases} \qquad (14)$$

where

$$K_{ijk\ell} \equiv 1 + \frac{\sqrt{(\cos\theta_k - \cos\theta_i)^2 + A^2(\cos\phi_\ell - \cos\phi_j)^2}}{\cos\theta_k - \cos\theta_i} \qquad (15)$$

The spanwise integration points are

$$\phi_\ell = \frac{(2\ell - 1)\pi}{2M} \qquad (\ell = 1, \ldots, M) \qquad (16)$$

and the spanwise control points are

$$\phi_j = j\pi/M \qquad (j = 1, \ldots, M - 1) \qquad (17)$$

The tangent-flow boundary condition for flat wings requires that $w_{i,j} = -\alpha$.

The N(M - 1) values of $\gamma_{p,k}$ are calculated by solving the matrix equation formed by applying equation (14) for $i \neq 0$ at the chordwise and spanwise control points given by equations (5) and (17). Then after the $\gamma_{p,k}$ are calculated, the (M - 1) leading-edge suction parameters C_{S_j} can be computed by successively applying equation (14) with i = 0 (control point at the leading edge) at the spanwise positions j = 1, ..., M - 1. Regardless of the number (N) of chordwise control points used, there is always a control point at the trailing edge which accounts for the Kutta condition, and another control point at the leading edge which gives the leading-edge suction parameter, if desired.

The sectional and wing aerodynamic characteristics may now be calculated by using the midpoint trapezoidal rule to reduce the integrals to finite sums, as illustrated below:

$$(c_\ell)_p = \frac{2\Gamma_p}{cV_\infty} = \frac{2}{c}\int_0^c \gamma_p(x_1)\, dx_1 \simeq \frac{\pi}{N}\sum_{k=1}^{N} \gamma_{p,k} \sin\theta_k$$

$$C_L = \int_{-b/2}^{b/2} c_\ell\, c\, dy/S \simeq \frac{\pi}{2M}\sum_{p=1}^{M-1} (c_\ell)_p \sin\phi_p$$

$$C_{D_i} = \frac{C_L^2}{\pi A}\sum_{n=1}^{M-1} n\left[\sum_{p=1}^{M-1}\Gamma_p \sin n\phi_p\right]^2 \Bigg/ \left[\sum_{p=1}^{M-1}\Gamma_p \sin\phi_p\right]^2$$

$$C_{D_{ii}} = C_L\,\alpha - \frac{\pi^2}{M}\sum_{j=1}^{M-1} C_{S_j}^{\;2} \sin\phi_j$$

The spanwise loading can be made continuous by equation (9), and the chordwise loading can also be made continuous by fitting C_{S_j} and $\gamma_{j,k}$ to the chordwise loading functions for thin airfoil theory.

RESULTS AND DISCUSSION

For one control point (N = 1, M = 2) the present method yields

$$C_L/\alpha = \pi A/(1 + \sqrt{1 + A^2/2}) \quad \text{and} \quad C_{D_i} = C_L^2/\pi A$$

These results give the correct limit as $A \to 0$, but just as Lan found for airfoils, at least two chordwise control points are needed to get an accurate pitching moment. Reference 2 showed that the present spanwise integration method gives the exact classical solution to Prandtl's lifting line equation.

Table 1 gives a detailed comparison of the present method with the NLR method (ref. 1) for an A = 2 rectangular wing with N = 4 and M = 16. Although the NLR method used 15 spanwise loading functions, 127 spanwise integration points were employed. Excellent agreement is obtained between the two methods for the lift, pitching moment, aerodynamic center, far-field drag, and spanwise lift distribution. The spanwise variation of section drag and aerodynamic center compares well except near the wing tips, and the near-field drag values differ. As noted in reference 1, at least 8 chordwise control points are needed to get convergence of the section drag near the wing tips and the near-field drag. The computational time required for the results in Table 1 was 22 minutes for the NLR method on a CDC 3300 computer, whereas the present method required less than 10 seconds on an IBM 370/165 computer.

The effects of the number of control points on the convergence of lift and aerodynamic center are shown in figure 1 for rectangular wings with A = 2 and 7. These results indicate that good results are obtained for the A = 7 wing with N = 2 and M = 10, whereas more chordwise and less spanwise control points are needed for the A = 2 wing. Figure 2 illustrates the convergence of the near- and far-field induced drag for the same two wings. Note that the far-field induced drag is insensitive to both M and N for the A = 2 and A = 7 wings. On the other hand, the near-field induced drag depends on both M and N, particularly for the A = 2 wing.

CONCLUDING REMARKS

The present lifting-surface method for rectangular wings was found to compare favorably with other methods, but it is simpler and requires smaller computational times. The number of control points required for convergence of the aerodynamic characteristics is dependent on both the wing aspect ratio and the aerodynamic parameter. Convergence is fast for lift, pitching moment, aerodynamic center, far-field drag, and spanwise lift distribution. For the A = 7 wing two chordwise and about 10 spanwise control points gave good results, whereas at least 4 chordwise and 8 spanwise control points are needed for the A = 2 wing. The far-field induced drag is particularly insensitive to the number of control points, with good results for N = 2 and M = 4 on both wings. Convergence is slow for the section drag and aerodynamic center near the wing tips. The near-field drag converges more slowly than any of the other parameters. Other planforms are presently being investigated.

REFERENCES

1. Garner, H. C.; Hewitt, B. L.; and Labrujere, T. E.: Comparison of Three Methods for the Evaluation of Subsonic Lifting-Surface Theory. Reports and Memoranda 3597, Aeronautical Research Council, London, England, June 1968.

2. DeJarnette, Fred R.: Arrangement of Vortex Lattices on Subsonic Wings. Vortex-Lattice Utilization, NASA SP-405, 1976, pp. 301-323.

3. Ashley, Holt; and Landahl, M. T.: Aerodynamics of Wings and Bodies. Addison-Wesley Publishing Co., Inc., Mass., 1965.

4. Lan, C. E.: A Quasi-Vortex Lattice Method in Thin Wing Theory. J. Aircraft, vol. 11, no. 9, Sept. 1974, pp. 518-527.

TABLE 1. RESULTS FOR RECTANGULAR PLANFORM, A = 2

Overall Values

	Present N=4, M=16	NLR (ref. 1) N=4, M=16
C_L/α	2.4732	2.4744
$-C_M/\alpha$	0.5187	0.5182
X_{ac}/C	0.2097	0.2094
$\pi AC_{D_i}/C_L^2$	1.0007	1.0007
$\pi AC_{D_{ii}}/C_L^2$	0.9951	1.0108

Values of c_d/C_L^2

2y/b	Present	NLR
0	0.1847	0.1848
0.1951	0.1832	0.1832
0.3827	0.1784	0.1781
0.5556	0.1693	0.1686
0.7071	0.1548	0.1541
0.8315	0.1331	0.1353
0.9239	0.0988	0.1131
0.9808	0.0394	0.0770

Values of $c\ /C_L$

2y/b	Present	NLR
0	1.2543	1.2543
0.1951	1.2331	1.2331
0.3827	1.1692	1.1692
0.5556	1.0625	1.0625
0.7071	0.9137	0.9137
0.8315	0.7257	0.7257
0.9239	0.5045	0.5044
0.9808	0.2588	0.2587

Values of x_{ac}/c

2y/b	Present	NLR
0	0.2200	0.2199
0.1951	0.2187	0.2187
0.3827	0.2150	0.2149
0.5556	0.2087	0.2085
0.7071	0.1999	0.1996
0.8315	0.1896	0.1886
0.9239	0.1798	0.1773
0.9808	0.1731	0.1685

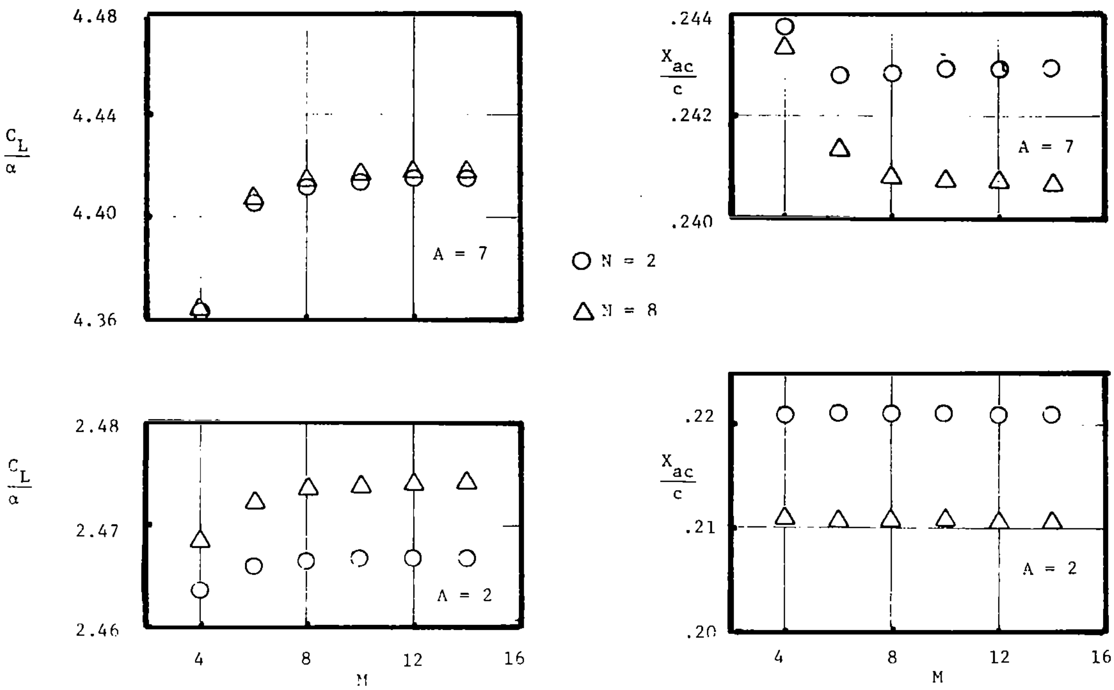

Figure 1.- Convergence of lift and aerodynamic center on rectangular wings. A = 2 and 7.

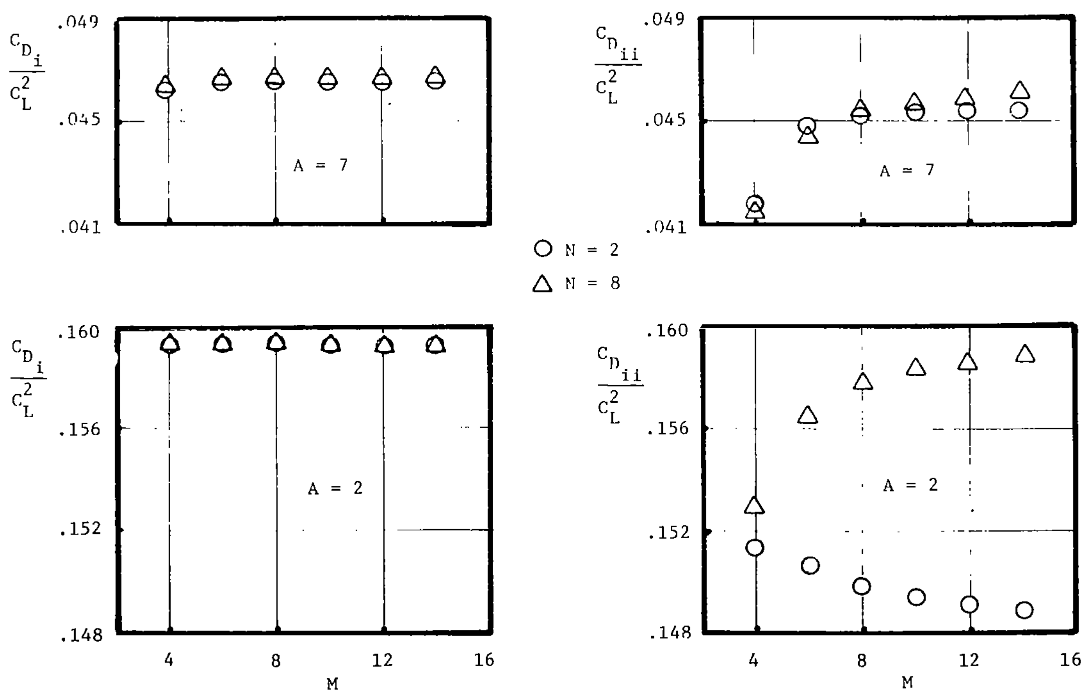

Figure 2.- Convergence of near- and far-field induced drag on rectangular wings. A = 2 and 7.