

ONE-DIMENSIONAL WAVE PROPAGATION IN  
PARTICULATE SUSPENSIONS

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SUMMARY

One-dimensional small-amplitude wave motion in a two-phase system consisting of an inviscid gas and a cloud of suspended particles is analyzed using a continuum theory of suspensions. Laplace transform methods are used to obtain several approximate solutions. From these solutions are inferred some of the interesting properties of acoustic wave motion in particulate suspensions.

INTRODUCTION

This paper is concerned with small-amplitude wave propagation in a particulate suspension contained within a semi-infinite tube. Small-amplitude wave propagation in particulate suspensions is of interest because of applications to problems involving sound attenuation in fogs, flow visualization, nuclear reactor cooling systems, and combustion instabilities in rocket motors. Most previous work is devoted to various aspects of the problem of harmonic wave propagation in a suspension of infinite extent. Representative of early papers on this subject are those by Sewell (ref. 1), Epstein (ref. 2), and Epstein and Carhart (ref. 3). In these papers the flow past each particle was considered in detail, at least in principle. More recent calculations have employed two-phase continuum models of suspension behavior. In general, these models are appropriate when a representative volume element of the suspension, which is small compared to the characteristic dimensions of the flow field, contains an amount of fluid and an amount of particles sufficiently large to allow the formation of meaningful averages of the properties of the two phases within the volume element. Then the volume is treated as a differential element (a point) and the averages are treated as continuous variables. Representative of this approach to problems of small-amplitude wave propagation in suspensions are the work of Temkin and Dobbins (ref. 4), Morfey (ref. 5), Schmitt von Schubert (ref. 6), Marble and Wooten (ref. 7), Goldman (ref. 8), Mecredy and Hamilton (ref. 9), and the review articles by Marble (ref. 10), and Rudinger (ref. 11). Marble (ref. 10) points out that comparison of the predictions of continuum theories with the more detailed analysis given by Epstein and Carhart (ref. 3) shows that the continuum approach is completely adequate for wavelengths that are long compared to the particle dimensions.

In the present paper a simple continuum theory of particulate suspension behavior is applied to the problem of small-amplitude wave motion of a suspension in a semi-infinite tube. In contrast to the large amount of work on

harmonic wave propagation, there appears to be little (if any) work available on the propagation of non-harmonic waves. In order to focus attention on the basic relaxation mechanism inherent in such two-phase flows, several simplifying assumptions are made. These are: the motion is one-dimensional, the fluid phase can be modeled as an inviscid gas obeying a linear pressure-density relationship, the interphase force is directly proportional to the vector difference between the velocities of the two phases (thus, contributions due to added mass, history, etc. are neglected), and the volume fraction of the particle phase is small. The linear acoustic equations which follow from these assumptions are solved by the Laplace transform method for a step input of velocity at the end of the tube.

### GOVERNING EQUATIONS

Let  $\rho_0$  be the initial gas-phase density,  $\gamma_0$  be the initial particle-phase density,  $a$  be the clean-gas speed of sound,  $U$  be the inlet gas velocity,  $M = U/a$  be a Mach number, and  $\tau$  be the relaxation time of the suspension (see Marble, reference 10). If the usual acoustic linearizations are made, the balance equations for mass and linear momentum and the equation of state take the dimensionless forms

$$\partial_t \rho + \partial_x u = 0, \quad \partial_t u = -\partial_x p + \kappa(v-u), \quad p = \rho \quad (1)$$

for the gas phase, and

$$\partial_t \gamma + \partial_x v = 0, \quad \partial_t v = u - v \quad (2)$$

for the particle phase. In equations (1) and (2)  $ax$  is the axial coordinate,  $\tau t$  is time,  $Uu$  is the gas-phase velocity,  $Uv$  is the particle-phase velocity,  $\rho_0 M p$  is the difference between the current and initial gas densities,  $\gamma_0 M \gamma$  is the difference between the current and initial particle-phase densities,  $\rho_0 U a p$  is the difference between the current and initial pressures, and  $\kappa = \gamma_0 / \rho_0$ . It can be seen that equations (1) and (2) are five equations involving five unknowns. Thus it is not necessary to consider the balance-of-energy equations for the two phases in order to determine the mechanical behavior. This is the reason for the second simplifying assumption discussed in the previous section.

Equations (1a), (1b), and (1c) can be combined to yield the modified wave equation

$$\partial_{tt} u = \partial_{xx} u + \kappa(\partial_t v - \partial_t u) \quad (3)$$

Equations (2b) and (3) can be solved simultaneously for  $u$  and  $v$ . Then equation (1a) can be solved for  $\rho$  and equation (2a) can be solved for  $\gamma$ .

It should be noted that the dimensional form of the equation of state is

$$(\rho_0 U a p) = a^2 (\rho_0 M p) \quad (4)$$

Thus the dimensional clean-gas speed of sound is

$$d(\rho_0 U_{ap})/d(\rho_0 M\rho) = a^2 \quad (5)$$

as originally stated. Because of the way  $a$  was used in the nondimensionalization process it can be seen from equation (1c) that the dimensionless clean-gas speed of sound is

$$dp/d\rho = 1 \quad (6)$$

#### LAPLACE TRANSFORM OF SOLUTION

The suspension is contained in a semi-infinite pipe beginning at  $x = 0$  and extending along the positive  $x$  axis. The suspension is at rest until  $t = 0$  when a constant gas inlet velocity is suddenly created. Thus

$$u(0,t) = \Delta(t) \quad (7)$$

where the symbol  $\Delta(\xi)$  is used to denote a unit step function. That is

$$\Delta(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases} \quad (8)$$

Taking the Laplace transforms of equations (1a), (1b), (2a), (2b), (3), and (7) one obtains

$$s\bar{\rho} + \bar{u}' = 0, \quad s\bar{\gamma} + \bar{v}' = 0$$

$$s^2\bar{u} = \bar{u}'' + \kappa s(\bar{v} - \bar{u}), \quad s\bar{v} = \bar{u} - \bar{v} \quad (9)$$

$$\bar{u}(0) = 1/s \quad (10)$$

where  $s$  is the Laplace transform parameter, a superposed bar denotes a Laplace transform, and a prime denotes differentiation with respect to  $x$ . Equations (9c) and (9d) can be combined to yield

$$\bar{u}'' - s^2 b^2 \bar{u} = 0 \quad (11)$$

where

$$b(s) = (1 + \kappa/(1 + s))^{\frac{1}{2}} \quad (12)$$

Solving equation (11) subject to equation (10) and the condition that  $\bar{u}(x)$  should remain bounded for all  $x > 0$ , and substituting this solution into equations (9a), (9b), and (9d) leads to

$$\bar{u} = \exp(-sbx)/s, \quad \bar{v} = \exp(-sbx)/(s(1 + s))$$

$$\bar{\rho} = b \exp(-sbx)/s, \quad \bar{\gamma} = b \exp(-sbx)/(s(1 + s)) \quad (13)$$

It appears that no exact inversions of equations (13) can be obtained in terms of elementary functions. In subsequent sections several simple approximate inversions will be found and used to illustrate some of the properties of the solution to this problem.

#### INVERSION FOR SMALL TIMES

Approximate solutions for  $t \ll 1$  can be obtained by expanding various functions appearing in equation (13) for  $s \gg 1$ . Expanding  $b$  (eq. (12)) in this way and retaining the first two terms leads to

$$b \doteq 1 + \kappa/(2s) \tag{14}$$

If only the first term in equation (14) is retained the corresponding inversions of equations (13) are (see Roberts and Kaufman, reference 12)

$$\begin{aligned} u &\doteq \rho \doteq \Delta(t - x) \\ v &\doteq \gamma \doteq (1 - \exp(-(t - x)))\Delta(t - x) \end{aligned} \tag{15}$$

Equations (15a) and (15b) represent the solution for a clean gas. Thus immediately after the beginning of the motion, the motion is independent of the presence of the particles.

If the first two terms in equation (14) are retained the corresponding inversions are found to be

$$\begin{aligned} u &\doteq \exp(-\kappa x/2)\Delta(t - x) \\ \rho &\doteq \exp(-\kappa x/2)(1 + \kappa(t - x)/2)\Delta(t - x) \\ v &\doteq \exp(-\kappa x/2)(1 - \exp(-(t - x)))\Delta(t - x) \\ \gamma &\doteq \exp(-\kappa x/2)((1 - \kappa/2)(1 - \exp(-(t - x))) + \kappa(t - x)/2)\Delta(t - x) \end{aligned} \tag{16}$$

Equations (16) illustrate the coupling between the motions of the two phases which manifests itself as the time since the beginning of the motion increases. To interpret these results most easily it is useful to remember that nonzero results are obtained only for  $t > x$ . Thus the condition  $t \ll 1$  implies that equations (16) are valid only for  $x \ll 1$  and  $(t-x) \ll 1$ . Simplifying equations (16c) and (16d) for  $(t-x) \ll 1$  leads to

$$\begin{aligned} u &\doteq \exp(-\kappa x/2)\Delta(t - x) \\ \rho &\doteq \exp(-\kappa x/2)(1 + \kappa(t - x)/2)\Delta(t - x) \\ v &\doteq \exp(-\kappa x/2)(t - x)\Delta(t - x) \\ \gamma &\doteq \exp(-\kappa x/2)(t - x)\Delta(t - x) \end{aligned} \tag{17}$$

Some observations based on equations (17) are as follows. For small  $t$  all disturbances propagate with the clean-gas wave speed  $l$ . The amplitudes of all variables decrease with increasing  $x$ . For large values of the particle loading  $\kappa$  this effect can be significant. For a given value of  $t-x$  (the time since the wave front passed position  $x$ ) the degree of spatial attenuation increases with  $x$ . For a given value of  $x$ , the terms  $\rho$ ,  $v$ , and  $\gamma$  are increasing functions of the time since passage of the wave front.

#### INVERSION FOR LARGE TIMES

Approximate solutions for  $t \gg 1$  can be found by expanding the various functions appearing in equation (13) for  $s \ll 1$ . Expanding  $b$  (eq. (12)) in this way and retaining the first two terms gives

$$b \doteq (1 + \kappa) - \kappa s \quad (18)$$

Retaining only the first term in equation (18), substituting into equations (13), and inverting yields

$$\begin{aligned} u &\doteq \Delta(t - (1+\kappa)^{\frac{1}{2}}x), & \rho &\doteq (1+\kappa)^{\frac{1}{2}}\Delta(t - (1+\kappa)^{\frac{1}{2}}x) \\ v &\doteq (1 - \exp(-(t - (1+\kappa)^{\frac{1}{2}}x)))\Delta(t - (1+\kappa)^{\frac{1}{2}}x) \\ \gamma &\doteq (1+\kappa)^{\frac{1}{2}}(1 - \exp(-(t - (1+\kappa)^{\frac{1}{2}}x)))\Delta(t - (1+\kappa)^{\frac{1}{2}}x) \end{aligned} \quad (19)$$

It can be seen from equations (19) that for  $t \gg 1$  all quantities propagate with wave speed  $l/(1+\kappa)^{\frac{1}{2}}$ . For values of  $x$  away from the wave front  $u$  and  $v$  are essentially equal as are  $\rho$  and  $\gamma$ . For values of  $x$  near the wave front differences between the velocities and densities remain for arbitrarily large values of  $t$ . In contrast to equation (17a), equation (19a) predicts that the gas velocity has a value of unity for all  $x$ . Thus the amplitude of  $u$  at a given  $x$  must increase with time. More insight into this matter will be provided by the results obtained in the next section.

It was attempted to invert equations (13) using the first two terms of the expansion of  $b$  for small  $s$  (eq. (18)). No inversion in terms of elementary functions could be found.

#### INVERSION FOR SMALL PARTICLE LOADING

Expanding equation (12) for  $\kappa \ll 1$  and retaining the first two terms one gets

$$b \doteq 1 + \kappa/(2(1 + s)) \quad (20)$$

If equations (13) are inverted using only the first term of equation (20) the results are equations (15). For the important special case of negligible

particle loading equations (15) represent the exact solution. To find a correction for finite values of  $\kappa$  two terms of equation (20) must be retained. If equation (20) is substituted into equation (13) no simple inversion of the resulting expressions appears possible. Further simplification is achieved by expanding the exponentials involving  $b$  for small  $\kappa$  and keeping the first two terms. This results in

$$\begin{aligned}\bar{u} &\doteq (1 - \kappa s x / (2(1 + s))) \exp(-s x) / s \\ \bar{\rho} &\doteq (1 - \kappa (s x - 1) / (2(1 + s))) \exp(-s x) / s \\ \bar{v} &\doteq (1 - \kappa s x / (2(1 + s))) \exp(-s x) / (s(1 + s)) \\ \bar{\gamma} &\doteq (1 - \kappa (s x - 1) / (2(1 + s))) \exp(-s x) / (s(1 + s))\end{aligned}\quad (21)$$

Inverting equations (21) yields

$$\begin{aligned}u &\doteq (1 - \kappa x \exp(-(t - x)) / 2) \Delta(t - x) \\ \rho &\doteq (1 + \kappa (1 - (1 + x) \exp(-(t - x))) / 2) \Delta(t - x) \\ v &\doteq (1 - \exp(-(t - x)) - \kappa x (t - x) \exp(-(t - x)) / 2) \Delta(t - x) \\ \gamma &\doteq (1 - \exp(-(t - x)) + \kappa (1 - \exp(-(t - x)) \\ &\quad - (1 + x)(t - x) \exp(-(t - x))) / 2) \Delta(t - x)\end{aligned}\quad (22)$$

These expressions appear to be computationally useful for small  $x$  and all  $t$ . Equation (22a) shows that for a given  $x$  the value of  $u$  at the time of passage of the wave front is  $1 - \kappa x / 2$ . (Note that this is the two-term expansion for small  $\kappa$  of the  $\exp(-\kappa x / 2)$  appearing in equation (16a)). As the time  $t - x$  since passage of the wave front increases, the amplitude of  $u$  increases to unity. Similarly it can be seen that the value of  $v$  for large  $t - x$  is unity while the values of both  $\rho$  and  $\gamma$  for large  $t - x$  are  $1 + \kappa / 2$  which is the two-term expansion for small  $\kappa$  of the  $(1 + \kappa)^{\frac{1}{2}}$  appearing in equations (19b) and (19d). Thus the large-time limiting values predicted by equations (22) are consistent with those predicted by equations (19). Equations (22) do not predict the change in wave speed indicated by equations (19). It can be shown that this is due to the process of expanding the arguments of the exponentials appearing in equations (13) before inversion. For  $\kappa \ll 1$  the difference between  $(1 + \kappa)^{\frac{1}{2}}$  and unity is small so this is not a serious matter.

#### DISCUSSION OF RESULTS

From the three sets of approximate solutions developed in the previous sections (eqs. (16), (19), and (22)) it is possible to put together a fairly complete picture of the wave motion produced by a step velocity input at  $x = 0$ . All waves travel with the same wave speed. For small times this is unity and for large times it is  $1 / (1 + \kappa)^{\frac{1}{2}}$ . The former is called the frozen wave speed. It is

the wave speed associated with the clean gas. The latter is called the equilibrium wave speed. It is the wave speed associated with wave motion in a gas having initial density equal to the initial suspension density. These results are to be expected on physical grounds. For small times the motion of the gas (through which the waves propagate) is independent of the presence of the particles as indicated by equations (15). For large times the velocities of both phases are essentially equal. Thus the suspension behaves like a gas with effective dimensionless density  $1+\kappa$ . The exact manner by which the transition from the frozen to the equilibrium wave speed is accomplished is not revealed by the approximate solutions obtained in this work.

The gas velocity  $u$  has the prescribed value of unity at the inlet and decreases to a minimum value at the wave front. The value of  $u$  at each point behind the wave front increases with time and eventually approaches unity. The particle velocity  $v$  and the fluid- and particle-density perturbations,  $\rho$  and  $\gamma$  respectively, are also decreasing functions of  $x$ . Their values for all values of  $x$  (including  $x = 0$ ) increase with time. Finally  $\rho$  approaches a constant value throughout the region of motion while  $v$  and  $\gamma$  approach constant values except near the wave front. Because the step increase in gas velocity must be transmitted to the particles through the interphase-momentum-transfer mechanism the particles in the immediate vicinity of the wave front can never quite catch up to the gas.

It should be pointed out that for  $t > 0$  a particle-free zone exists adjacent to the inlet. It can be shown that the speed of the forward boundary of this region is  $O(M)$ . Since the acoustic equations are valid only for small Mach numbers, and since the speed of the wave front is  $O(1)$ , the length of the particle-free zone is negligible compared to the length of the region of motion. For this reason the particle-free zone was neglected in this analysis. For waves of finite amplitude this could not be done. The position of the forward boundary of the particle-free zone would have to be computed as part of the solution. This would greatly increase the complexity of the analysis.

#### CONCLUSION

In this paper the problem of small-amplitude wave propagation in a particulate suspension was analyzed using a continuum theory of suspensions. The governing equations were solved approximately by the Laplace transform method. Three approximate inversions were developed and from these were inferred some of the properties of the wave motion.

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