THE RELATIONSHIP BETWEEN EDDY-TRANSPORT AND SECOND-ORDER CLOSURE MODELS FOR STRATIFIED MEDIA AND FOR VORTICES*

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INTRODUCTION

A question which invariably arises when one considers the calculation of turbulent shear flows is, "How complex a model should be used to calculate such motions?" Available at the present time are models varying in complexity from very simple eddy-transport models to models in which all the equations for the nonzero second-order correlations are solved simultaneously with the equations for the mean variables. For this reason, it might be instructive to present a discussion of the relationship between these two models of turbulent shear flow. Two types of motion will be discussed: first, turbulent shear flow in a stratified medium and, second, the motion in a turbulent line vortex. These two cases are instructive because in the first example eddy-transport methods have proven reasonably effective, whereas in the second, they have led to erroneous conclusions.

It is not generally appreciated that the simplest form of eddy-transport theory can be derived from second-order closure models of turbulent flow by a suitably limiting process. This paper will discuss this limiting process and the suitability of eddy-transport modeling for stratified media and line vortices.

SYMBOLS

a,b	model parameters
c_p	specific heat at constant pressure
D	operator
g	gravitational acceleration
g_i, g_k	general acceleration vectors

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i,j,k	indices
m,n	proportionality constants
N	stability parameter (eq. (90))
N_{Ri}	Richardson number
P	parameter (eq. (49))
p	pressure
Q	turbulent energy, $(UU + VV + WW)^{1/2}$
q	scalar velocity, $\left[(u^i)'u_i'\right]^{1/2}$
r,ϕ,z	cylindrical coordinates
r_c	vortex core radius (fig. 7)
T	temperature
t	time
U,V,W	nondimensional second-order velocity correlations
$\overline{u},\overline{v},\overline{w}$	mean velocity in r-, ϕ-, z-direction, respectively, for two-dimensional line vortex; mean velocity in x-, y-, z-direction, respectively, for atmospheric motion
x,y,z	Cartesian coordinates
α	length scale proportionality constant
Γ	vortex strength
Δ	difference
δ	breadth of layer under consideration

δ_{ik}	Kronecker delta
ϵ	local deformation in vortex
κ	Von Kármán's constant
$\Lambda_1, \Lambda_2, \Lambda_3$	length scales
λ	dissipative scale
μ	viscosity
ν	kinematic viscosity
ρ	density
τ_{ij}	stress tensor
τ_t	turbulent shear stress

Subscripts:

char	characteristic
crit	critical
max	maximum
o	undisturbed, adiabatic atmosphere

Bars over a quantity indicate mean values. Primes indicate the instantaneous fluctuation of the quantity from its mean value.

A SECOND-ORDER CLOSURE MODEL FOR TURBULENT SHEAR FLOW

In reference 1 the author presented a discussion of the development of an invariant second-order closure model for turbulent shear flow in an incompressible medium. The basic equations of this model are

$$\frac{\partial \bar{u}_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}^j \bar{u}_{i,j} = -\frac{1}{\rho} \bar{p}_{,i} + \left[\bar{\tau}_i{}^j - \overline{\left(u^j\right)' u_i'}\right]_{,j} \tag{2}$$

$$\frac{\partial \overline{u_i' u_k'}}{\partial t} + \bar{u}^j \left[\overline{u_i'\left(u^j\right)'}\right]_{,j} = -\overline{u_j' u_k'} \frac{\partial \bar{u}_i}{\partial x_j} - \overline{u_j' u_i'} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\Lambda_2 q \left(\frac{\partial}{\partial x_j} \overline{u_i' u_k'} + \frac{\partial}{\partial x_i} \overline{u_j' u_k'} + \frac{\partial}{\partial x_k} \overline{u_j' u_i'}\right)\right] + \frac{\partial}{\partial x_k}\left(\Lambda_3 q \frac{\partial \overline{u_j' u_i'}}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(\Lambda_3 q \frac{\partial \overline{u_j' u_k'}}{\partial x_j}\right) - \frac{q}{\Lambda_1}\left(\overline{u_i' u_k'} - \delta_{ik} \frac{q^2}{3}\right) + \nu \frac{\partial^2}{\partial x_j^2} \overline{u_i' u_k'} - 2\nu \frac{\overline{u_i' u_k'}}{\lambda^2} \tag{3}$$

with

$$\bar{\tau}_{ij} = \mu\left(\bar{u}_{i,j} + \bar{u}_{j,i}\right) \tag{4}$$

and

$$q = \left[\overline{(u^i)' u_i'}\right]^{1/2} \tag{5}$$

In this model, the length scales Λ_1, Λ_2, and Λ_3 are assumed to be proportional to one another. The dissipative scale λ is given by

$$\lambda^2 = \frac{\Lambda_1{}^2}{a + \left(b\rho q \Lambda_1/\mu\right)} \tag{6}$$

From a rather lengthy parameter search to determine the values of the quantities a, b, Λ_1, Λ_2, and Λ_3 that are to be used in the calculations, it was found that good results were obtained by using

$$\left.\begin{aligned} a &= 2.5 \\ b &= 0.125 \\ \frac{\Lambda_2}{\Lambda_1} &= 0.1 \\ \frac{\Lambda_3}{\Lambda_1} &= 0.1 \end{aligned}\right\} \tag{7}$$

It was further found that the remaining free parameter Λ_1 was approximately equal to 0.6 times the longitudinal integral scale of the motions studied in the parameter search.

In reference 2, this model was extended to the case of turbulent motion and transport in the earth's boundary layer. The case considered is that when the scales of the mean distributions of velocity and temperature are not greatly different and the Prandtl and Schmidt numbers are 1. The resulting equations, written in Cartesian tensor notation, are

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0 \tag{8}$$

$$\frac{D\overline{u}_i}{Dt} = -\frac{\partial \overline{p}}{\partial x_i} + \nu_o \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \frac{1}{T_o} g_i \overline{T} \tag{9}$$

$$\frac{D\overline{T}}{Dt} = \nu_o \frac{\partial^2 \overline{T}}{\partial x_j^2} - \frac{\partial \overline{u_j' T'}}{\partial x_j} \tag{10}$$

$$\begin{aligned}\frac{D\overline{u_i' u_k'}}{Dt} = {} & -\overline{u_j' u_k'} \frac{\partial \overline{u}_i}{\partial x_j} - \overline{u_j' u_i'} \frac{\partial \overline{u}_k}{\partial x_j} + \frac{1}{T_o}\left(g_i \overline{u_k' T'} + g_k \overline{u_i' T'}\right) + \frac{\partial}{\partial x_j}\left[\Lambda_2 q \left(\frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \frac{\partial \overline{u_j' u_k'}}{\partial x_i}\right.\right. \\ & \left.\left. + \frac{\partial \overline{u_k' u_i'}}{\partial x_j}\right)\right] + \frac{\partial}{\partial x_i}\left(\Lambda_3 q \frac{\partial \overline{u_j' u_k'}}{\partial x_j}\right) + \frac{\partial}{\partial x_k}\left(\Lambda_3 q \frac{\partial \overline{u_j' u_i'}}{\partial x_j}\right) - \frac{q}{\Lambda_1}\left(\overline{u_i' u_k'} - \frac{\delta_{ik}}{3} q^2\right) \\ & + \nu_o \frac{\partial^2 \overline{u_i' u_k'}}{\partial x_j^2} - 2\nu_o \frac{\overline{u_i' u_k'}}{\lambda^2}\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{D\overline{u_k' T'}}{Dt} = {} & -\overline{u_j' u_k'} \frac{\partial \overline{T}}{\partial x_j} - \overline{u_j' T'} \frac{\partial \overline{u}_k}{\partial x_j} + \frac{1}{T_o} g_k \overline{(T')^2} + \frac{\partial}{\partial x_j}\left[\Lambda_2 q \left(\frac{\partial \overline{u_j' T'}}{\partial x_k} + \frac{\partial \overline{u_k' T'}}{\partial x_j}\right)\right] \\ & + \frac{\partial}{\partial x_k}\left(\Lambda_3 q \frac{\partial \overline{u_j' T'}}{\partial x_j}\right) - \frac{q}{\Lambda_1} \overline{u_k' T'} + \nu_o \frac{\partial^2 \overline{u_k' T'}}{\partial x_j^2} - 2\nu_o \frac{\overline{u_k' T'}}{\lambda^2}\end{aligned} \tag{12}$$

$$\frac{D\overline{(T')^2}}{Dt} = -2\overline{u_j' T'} \frac{\partial \overline{T}}{\partial x_j} + \frac{\partial}{\partial x_j}\left[\Lambda_2 q \frac{\partial \overline{(T')^2}}{\partial x_j}\right] + \nu_o \frac{\partial^2 \overline{(T')^2}}{\partial x_j^2} - 2\nu_o \frac{\overline{(T')^2}}{\lambda^2} \tag{13}$$

In these equations,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j} \tag{14}$$

T_o and ν_o are the local temperature and kinematic viscosity, respectively, in an undisturbed adiabatic atmosphere; $\overline{T}$ is the departure of the mean temperature from the adiabatic temperature T_o; and T' is the instantaneous fluctuation of the temperature about its mean.

For the two cases of motion considered herein, equations (1) to (14) reduce to the following:

For the case of a two-dimensional line vortex in cylindrical coordinates r,ϕ,z with velocities $\bar{u},\bar{v},\bar{w}$, equations (2) and (3) result in

$$\frac{\partial \bar{v}}{\partial t} = -\frac{\partial \overline{u'v'}}{\partial r} - \frac{2}{r}\overline{u'v'} + \nu\left(\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) \tag{15}$$

$$\begin{aligned}\frac{\partial \overline{u'u'}}{\partial t} = {} & \frac{4\bar{v}}{r}\overline{u'v'} + 3\frac{\partial}{\partial r}\left(\Lambda_2 q \frac{\partial \overline{u'u'}}{\partial r}\right) - \frac{2\Lambda_2 q}{r}\frac{\partial \overline{v'v'}}{\partial r} + \frac{3\Lambda_2 q}{r}\frac{\partial \overline{u'u'}}{\partial r} + \frac{4\Lambda_2 q}{r^2}\left(\overline{v'v'} - \overline{u'u'}\right) \\ & + 2\frac{\partial}{\partial r}\left(\Lambda_3 q \frac{\partial \overline{u'u'}}{\partial r}\right) - \frac{2}{r}\frac{\partial\left(\Lambda_3 q \overline{v'v'}\right)}{\partial r} + \frac{2}{r}\frac{\partial\left(\Lambda_3 q \overline{u'u'}\right)}{\partial r} + \frac{2}{r^2}\Lambda_3 q\left(\overline{v'v'} - \overline{u'u'}\right) \\ & - \frac{q}{\Lambda_1}\left(\overline{u'u'} - \frac{q^2}{3}\right) + \nu\left[\frac{\partial^2 \overline{u'u'}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u'u'}}{\partial r} + \frac{2}{r^2}\left(\overline{v'v'} - \overline{u'u'}\right) - \frac{2}{\lambda^2}\overline{u'u'}\right]\end{aligned} \tag{16}$$

$$\begin{aligned}\frac{\partial \overline{v'v'}}{\partial t} = {} & -\left(\frac{2\bar{v}}{r} + \frac{\partial \bar{v}}{\partial r}\right)\overline{u'v'} + \frac{\partial}{\partial r}\left(\Lambda_2 q \frac{\partial \overline{v'v'}}{\partial r}\right) + \frac{3\Lambda_2 q}{r}\frac{\partial \overline{v'v'}}{\partial r} - \frac{2}{r}\frac{\partial\left(\Lambda_2 q \overline{v'v'}\right)}{\partial r} \\ & + \frac{2}{r}\frac{\partial\left(\Lambda_2 q \overline{u'u'}\right)}{\partial r} + \frac{4\Lambda_2 q}{r^2}\left(\overline{u'u'} - \overline{v'v'}\right) + \frac{2\Lambda_3 q}{r}\frac{\partial \overline{u'u'}}{\partial r} + \frac{2\Lambda_3 q}{r^2}\left(\overline{u'u'} - \overline{v'v'}\right) \\ & - \frac{q}{\Lambda_1}\left(\overline{v'v'} - \frac{q^2}{3}\right) + \nu\left[\frac{\partial^2 \overline{v'v'}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v'v'}}{\partial r} + \frac{2}{r^2}\left(\overline{u'u'} - \overline{v'v'}\right) - \frac{2\overline{v'v'}}{\lambda^2}\right]\end{aligned} \tag{17}$$

$$\frac{\partial \overline{w'w'}}{\partial t} = \frac{\partial}{\partial r}\left(\Lambda_2 q \frac{\partial \overline{w'w'}}{\partial r}\right) + \frac{\Lambda_2 q}{r}\frac{\partial \overline{w'w'}}{\partial r} - \frac{q}{\Lambda_1}\left(\overline{w'w'} - \frac{q^2}{3}\right)$$

$$+ \nu\left(\frac{\partial^2 \overline{w'w'}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{w'w'}}{\partial r} - \frac{2\overline{w'w'}}{\lambda^2}\right) \tag{18}$$

$$\frac{\partial \overline{u'v'}}{\partial t} = \frac{\bar{v}}{r}\left(2\overline{v'v'} - \overline{u'u'}\right) - \frac{\partial \bar{v}}{\partial r}\overline{u'u'} + 2\frac{\partial}{\partial r}\left(\Lambda_2 q \frac{\partial \overline{u'v'}}{\partial r}\right) - \frac{2}{r}\frac{\partial\left(\Lambda_2 q \overline{u'w'}\right)}{\partial r} + \frac{4\Lambda_2 q}{r}\frac{\partial \overline{u'v'}}{\partial r}$$

$$- \frac{8\Lambda_2 q}{r^2}\overline{u'v'} + \frac{\partial}{\partial r}\left(\Lambda_3 q \frac{\partial \overline{u'v'}}{\partial r}\right) + \frac{2}{r}\frac{\partial\left(\Lambda_3 q \overline{u'v'}\right)}{\partial r} - \frac{\Lambda_3 q}{r}\frac{\partial \overline{u'v'}}{\partial r} - \frac{4\Lambda_3 q \overline{u'v'}}{r^2}$$

$$- \frac{q}{\Lambda_1}\overline{u'v'} + \nu\left(\frac{\partial^2 \overline{u'v'}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u'v'}}{\partial r} - \frac{4\overline{u'v'}}{r^2} - \frac{2\overline{u'v'}}{\lambda^2}\right) \tag{19}$$

For the atmospheric motion considered herein, in which only a mean velocity $\bar{u}$ in the horizontal direction x exists and in which the mean lateral and vertical velocities $\bar{v}$ and $\bar{w}$ in the directions y and z, respectively, are zero, the appropriate equations derived from equations (8) to (13) are

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \bar{p}}{\partial x}\nu_o \frac{\partial^2 \bar{u}}{\partial z^2} - \frac{\partial \overline{u'w'}}{\partial z} \tag{20}$$

$$\frac{\partial \bar{T}}{\partial t} = \nu_o \frac{\partial^2 \bar{T}}{\partial z^2} - \frac{\partial \overline{T'w'}}{\partial z} \tag{21}$$

$$\frac{\partial \overline{u'u'}}{\partial t} = -2\overline{u'w'}\frac{\partial \bar{u}}{\partial z} + \frac{\partial}{\partial z}\left(\Lambda_2 q \frac{\partial \overline{u'u'}}{\partial z}\right) - \frac{q}{\Lambda_1}\left(\overline{u'u'} - \frac{q^2}{3}\right) + \nu_o \frac{\partial^2 \overline{u'u'}}{\partial z^2} - 2\nu_o \frac{\overline{u'u'}}{\lambda^2} \tag{22}$$

$$\frac{\partial \overline{v'v'}}{\partial t} = \frac{\partial}{\partial z}\left(\Lambda_2 q \frac{\partial \overline{v'v'}}{\partial z}\right) - \frac{q}{\Lambda_1}\left(\overline{v'v'} - \frac{q^2}{3}\right) + \nu_o \frac{\partial^2 \overline{v'v'}}{\partial z^2} - 2\nu_o \frac{\overline{v'v'}}{\lambda^2} \tag{23}$$

$$\frac{\partial \overline{w'w'}}{\partial t} = \frac{2g}{T_o}\overline{w'T'} + 3\frac{\partial}{\partial z}\left(\Lambda_2 q \frac{\partial \overline{w'w'}}{\partial z}\right) + \frac{2}{\rho_o}\frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q \frac{\partial \overline{w'w'}}{\partial z}\right)$$

$$- \frac{q}{\Lambda_1}\left(\overline{w'w'} - \frac{q^2}{3}\right) + \nu_o \frac{\partial^2 \overline{w'w'}}{\partial z^2} - 2\nu_o \frac{\overline{w'w'}}{\lambda^2} \tag{24}$$

$$\frac{\partial \overline{u'w'}}{\partial t} = -\overline{w'w'}\,\frac{\partial \overline{u}}{\partial z} + \frac{g}{T_o}\,\overline{u'T'} + 2\,\frac{\partial}{\partial z}\left(\Lambda_2 q\,\frac{\partial \overline{u'w'}}{\partial z}\right) + \frac{1}{\rho_o}\,\frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q\,\frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$- \frac{q}{\Lambda_1}\,\overline{u'w'} + \nu_o\,\frac{\partial^2 \overline{u'w'}}{\partial z^2} - 2\nu_o\,\frac{\overline{u'w'}}{\lambda^2} \qquad (25)$$

$$\frac{\partial \overline{u'T'}}{\partial t} = -\overline{u'w'}\,\frac{\partial \overline{T}}{\partial z} - \overline{w'T'}\,\frac{\partial \overline{u}}{\partial z} + \frac{\partial}{\partial z}\left(\Lambda_2 q\,\frac{\partial \overline{u'T'}}{\partial z}\right) - \frac{q}{\Lambda_1}\,\overline{u'T'} + \nu_o\,\frac{\partial^2 \overline{u'T'}}{\partial z^2} - 2\nu_o\,\frac{\overline{u'T'}}{\lambda^2} \qquad (26)$$

$$\frac{\partial \overline{w'T'}}{\partial t} = -\overline{w'w'}\,\frac{\partial \overline{T}}{\partial z} + \frac{g}{T_o}\,\overline{(T')^2} + 2\,\frac{\partial}{\partial z}\left(\Lambda_2 q\,\frac{\partial \overline{w'T'}}{\partial z}\right) + \frac{1}{\rho_o}\,\frac{\partial}{\partial z}\left(\rho_o \Lambda_3 q\,\frac{\partial \overline{w'T'}}{\partial z}\right)$$

$$- \frac{q}{\Lambda_1}\,\overline{w'T'} + \nu_o\,\frac{\partial^2 \overline{w'T'}}{\partial z^2} - 2\nu_o\,\frac{\overline{w'T'}}{\lambda^2} \qquad (27)$$

$$\frac{\partial \overline{(T')^2}}{\partial t} = -2\overline{w'T'}\,\frac{\partial \overline{T}}{\partial z} + \frac{\partial}{\partial z}\left[\Lambda_2 q\,\frac{\partial \overline{(T')^2}}{\partial z}\right] + \nu_o\,\frac{\partial^2 \overline{(T')^2}}{\partial z^2} - 2\nu_o\,\frac{\overline{(T')^2}}{\lambda^2} \qquad (28)$$

For most people who practice the art of predicting turbulent flows, the equations for the line vortex (eqs. (15) to (19)) and for the flow in the atmospheric boundary layer (eqs. (20) to (28)) are of a familiar form. If distributions on the dependent variables are given at time $t = 0$, the development of the motion at subsequent times can be computed by simultaneous solution of the appropriate coupled sets of partial differential equations. This is the general approach of second-order modeling.

The older and still widely used method of treating these problems is to consider only the equations for the mean variables and to assume that the second-order correlations which appear in them might be represented by empirically determined eddy-transport models patterned after the transport of the appropriate quantity by molecular means. The section which follows will examine what information can be gleaned from the equations for the second-order correlations about the nature of such eddy-transport models.

SUPEREQUILIBRIUM MODELS

To determine how to obtain information about the nature of eddy-transport models from the model or rate equations for the appropriate second-order correlations, one must consider what is implied when it is assumed that a turbulent flow can exhibit an eddy viscosity or an eddy diffusivity.

First, it is apparent that if the turbulent transport of a quantity depends only on the local gradient of that quantity and a scale length associated with the mean flows at the location under consideration, the turbulent transport cannot have a "memory" of its past history along the streamline. This is tantamount to the assumption that at each point in the flow the turbulent-transport correlations can track their local equilibrium values. These local equilibrium values can be obtained from the rate equations for the correlations by setting the left-hand sides of the equations, as they are given in the preceding section, equal to zero. Thus it is assumed that the rate of change of a transport correlation as it follows the mean motion is small compared with the production, dissipation, and diffusion terms which occur at the point in question.

Second, the notion of an eddy-transport coefficient is one which does not allow the behavior of the turbulent transport at one point in the flow to affect directly the turbulent transport at another point. This notion is equivalent to the neglect of the diffusion terms in the equations for the second-order correlations, for it is these terms which link the generation of transport correlations at one point in the flow to the transport correlations at another point.

Finally, the use of an eddy-transport model is a practice generally restricted to flows with high Reynolds numbers. Therefore, the high Reynolds number limit of the equations for the second-order correlations can be taken if it is desired to derive a simple form of eddy-transport model from these equations.

If the three rules set forth above are followed, it should be possible to derive from the equations for the second-order correlations a simple theory of eddy transport. As discussed above, this theory represents the equilibrium, nondiffusive, high Reynolds number limit of a second-order closure model. For reasons of brevity, this limit has for some time been referred to by the author as the "superequilibrium" limit.

By following the three rules set forth, the following equations are found to be the superequilibrium equations for a line vortex:

$$0 = \frac{4\overline{v}}{r}\,\overline{u'v'} - (1 + 2b)\frac{q}{\Lambda}\,\overline{u'u'} + \frac{q^3}{3\Lambda} \tag{29}$$

$$0 = -2\left(\frac{\partial\overline{v}}{\partial r} + \frac{\overline{v}}{rz}\right)\overline{u'v'} - (1 + 2b)\frac{q}{\Lambda}\,\overline{v'v'} + \frac{q^3}{3} \tag{30}$$

$$0 = -(1 + 2b)\frac{q}{\Lambda}\,\overline{w'w'} + \frac{q^3}{3} \tag{31}$$

$$0 = -\left(\frac{\partial\overline{v}}{\partial r} + \frac{\overline{v}}{r}\right)\overline{u'u'} + \frac{2\overline{v}}{r}\,\overline{v'v'} - (1 + 2b)\frac{q}{\Lambda}\,\overline{u'v'} \tag{32}$$

For atmospheric motion, one finds

$$0 = -2\overline{u'w'}\,\frac{\partial\bar{u}}{\partial z} - \frac{q}{\Lambda}(1 + 2b)\overline{u'u'} + \frac{q^3}{\Lambda} \tag{33}$$

$$0 = -\frac{q}{\Lambda}(1 + 2b)\overline{v'v'} + \frac{q^3}{\Lambda} \tag{34}$$

$$0 = \frac{2g}{T_o}\,\overline{w'T'} - \frac{q}{\Lambda}(1 + 2b)\overline{w'w'} + \frac{q^3}{\Lambda} \tag{35}$$

$$0 = -\overline{w'w'}\,\frac{\partial\bar{u}}{\partial z} + \frac{g}{T_o}\,\overline{u'T'} - \frac{q}{\Lambda}(1 + 2b)\overline{u'w'} \tag{36}$$

$$0 = -\overline{u'w'}\,\frac{\partial\bar{T}}{\partial z} - \overline{w'T'}\,\frac{\partial\bar{u}}{\partial z} - \frac{q}{\Lambda}(1 + 2b)\overline{u'T'} \tag{37}$$

$$0 = -\overline{w'w'}\,\frac{\partial\bar{T}}{\partial z} + \frac{g}{T_o}\overline{(T')^2} - \frac{q}{\Lambda}(1 + 2b)\overline{w'T'} \tag{38}$$

$$0 = -2\overline{w'T'}\,\frac{\partial\bar{T}}{\partial z} - 2b\,\frac{q}{\Lambda}\overline{(T')^2} \tag{39}$$

In writing these equations, Λ_1 was taken equal to Λ.

EDDY TRANSPORT IN THE ATMOSPHERE

It is instructive to carry out the solution of equations (33) to (39). These equations are algebraic for all the nonzero correlations. The solution of equations (33) to (39) can be obtained if the following definitions are introduced: Let (for $\partial\bar{u}/\partial z > 0$)

$$\left.\begin{aligned}
\overline{u'u'} &= UU\Lambda_1^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 & \overline{u'T'} &= UT\Lambda_1^2\,\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{T}}{\partial z}\\
\overline{v'v'} &= VV\Lambda_1^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 & \overline{w'T'} &= WT\Lambda_1^2\,\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{T}}{\partial z}\\
\overline{w'w'} &= WW\Lambda_1^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 & \overline{(T')^2} &= TT\Lambda_1^2\left(\frac{\partial\bar{T}}{\partial z}\right)^2\\
\overline{u'w'} &= UW\Lambda_1^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 & q^2 &= QQ\Lambda_1^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2
\end{aligned}\right\} \tag{40}$$

and note that

$$QQ = Q^2 = UU + VV + WW \tag{41}$$

Substitution of the definitions given in equations (40) and (41) into equations (33) to (39) results in

$$Q(1 + 2b)UU = \frac{Q^3}{3} - 2UW \tag{42}$$

$$Q(1 + 2b)VV = \frac{Q^3}{3} \tag{43}$$

$$Q(1 + 2b)UW = -WW + N_{Ri}UT \tag{44}$$

$$Q(1 + 2b)UT = -UW - WT \tag{45}$$

$$Q(1 + 2b)WT = -WW + N_{Ri}TT \tag{46}$$

$$Q(2b)TT = -WT \tag{47}$$

In these equations, N_{Ri} is the Richardson number given by

$$N_{Ri} = \frac{\frac{g}{T_o}\frac{\partial \overline{T}}{\partial z}}{\left(\frac{\partial \overline{u}}{\partial z}\right)^2} \tag{48}$$

It is immediately obvious from equations (42) to (47) that all the nondimensional second-order correlations are a function only of the Richardson number and the parameter b from the second-order closure model. It will be remembered that the value of b determined in the parameter search reported previously is 0.125.

It is convenient to express the solution of equations (42) to (47) in terms of the parameter

$$P = \frac{1 - (4 + 15b)N_{Ri} + \left[1 + 2(2 - 9b)N_{Ri} + (4 + 9b)^2\left(N_{Ri}\right)^2\right]^{1/2}}{6} \tag{49}$$

In terms of this parameter, the various correlations may be written

$$Q^2 = \frac{1}{b(1 + 2b)^2} P \tag{50}$$

$$UU = \frac{\left(P + bN_{Ri}\right)\left[P + (1 + 4b)N_{Ri}\right] + 2b\left[P + (1 + b)N_{Ri}\right]}{3(1 + 2b)\left(P + bN_{Ri}\right)\left[P + (1 + 4b)N_{Ri}\right]} Q^2 \tag{51}$$

$$VV = \frac{1}{3(1 + 2b)} Q^2 \tag{52}$$

$$WW = \frac{P + (1 + 2b)N_{Ri}}{3(1 + 2b)\left[P + (1 + 4b)N_{Ri}\right]} Q^2 \tag{53}$$

$$UW = -\frac{b}{3} \frac{P + (1 + b)N_{Ri}}{\left(P + bN_{Ri}\right)\left[P + (1 + 4b)N_{Ri}\right]} Q^3 \tag{54}$$

$$UT = \frac{b}{3(1 + 2b)} \frac{2P + (1 + 2b)N_{Ri}}{\left(P + bN_{Ri}\right)\left[P + (1 + 4b)N_{Ri}\right]} Q^2 \tag{55}$$

$$WT = -\frac{b}{3\left[P + (1 + 4b)N_{Ri}\right]} Q^3 \tag{56}$$

$$TT = \frac{1}{3\left[P + (1 + 4b)N_{Ri}\right]} Q^2 \tag{57}$$

It is clear from these equations that when the parameter $P = 0$, there is no turbulence $\left(Q^2 = 0\right)$ and all the second-order correlations vanish. The critical value of the Richardson number for which this occurs is a function of b and is given by

$$\left(N_{Ri}\right)_{crit} = \frac{1 + b}{4b(1 + 3b)} \tag{58}$$

For $b = 0.125$, the critical Richardson number is

$$\left(N_{Ri}\right)_{crit} = 1.636 \tag{59}$$

All the nondimensional second-order correlations as functions of the Richardson number are plotted in figures 1 to 5. From these figures, the profound difference between turbulence and turbulent transport in stable and unstable atmospheres is obvious. Note particularly that the nondimensional vertical transport of matter and heat falls off far more rapidly than do the nondimensional turbulent energy components when a stable atmospheric situation is approached. In fact, above a Richardson number of 1, vertical turbulent transport has almost ceased to exist although there is still some atmospheric turbulence.

It should be noted that the superequilibrium results just obtained specify the nondimensional values of second-order correlations. For example, if the value of $b = 0.125$ is assumed to be correct, a Richardson number of 0.10 would give

$$UW = -0.2712 \tag{60}$$

$$WT = -0.2631 \tag{61}$$

The transports of momentum and heat would then be given by (for $\partial\bar{u}/\partial z > 0$)

$$-\rho_o \overline{u'w'} = 0.2712 \rho_o \Lambda^2 \left(\frac{\partial\bar{u}}{\partial z}\right)^2 \tag{62}$$

$$-\rho_o (c_p)_o \overline{w'T'} = 0.2631 \rho_o \Lambda^2 \frac{\partial\bar{u}}{\partial z} \frac{\partial\bar{T}}{\partial z} \tag{63}$$

It is clear from these expressions that the actual transport is not defined until the length scale Λ is known. This is a difficulty with atmospheric flows, for unless Λ is determined at a given altitude and the local Richardson number specified there, the transports are not known. In general, Λ will depend at a given altitude on the Richardson number but can assume a range of values depending on the past history of the motion. Although this range of values is limited so that the order of magnitude of the transport might be determined, there will always be a variation in transport proportional to the square of the variation in Λ at any fixed Richardson number.

For classical laboratory flows, this problem does not exist. In this case, it is generally found that Λ is proportional to the characteristic breadth of the layer under consideration while the gradients are proportional to a characteristic velocity, temperature, or concentration difference divided by this characteristic breadth. Thus, for the classical shear flows,

$$\Lambda_1{}^2 \left(\frac{\partial\bar{u}}{\partial z}\right)^2 = \delta^2_{char} \left(\frac{\Delta\bar{u}_{char}}{\delta_{char}}\right)^2 = \text{Const} \left(\Delta\bar{u}_{char}\right)^2 \tag{64}$$

and, likewise,

$$\Lambda_1{}^2 \frac{\partial\bar{u}}{\partial z} \frac{\partial\bar{T}}{\partial z} = \text{Const}\ \Delta\bar{u}_{char}\ \Delta\bar{T}_{char} \tag{65}$$

For each type of flow, these constants are well defined. This type of simplicity is, alas, not true of the atmosphere.

It is instructive to compare the results of superequilibrium theory with certain well-known results from classical turbulent-transport theory for the case when no gravitational effects are involved. To do this, the Richardson number is placed equal to zero in the expressions given in equations (49) to (57), and for b = 0.125, the following expressions are obtained:

$$P = 0.3333 \tag{66}$$

$$Q^2 = \frac{1}{3b(1 + 2b)^2} = 1.7066 \tag{67}$$

$$UU = \frac{1 + 6b}{9b(1 + 2b)^3} = 0.7964 \tag{68}$$

$$VV = WW = \frac{1}{9b(1 + 2b)^3} = 0.4551 \tag{69}$$

$$UW = TW = -\frac{\sqrt{3/b}}{9(1 + 2b)^3} = -0.2786 \tag{70}$$

$$UT = \frac{2}{3(1 + 2b)^3} = 0.3413 \tag{71}$$

$$TT = \frac{1}{3b(1 + 2b)^2} = 1.7066 \tag{72}$$

Some interesting results are noted from the above comparison. First, superequilibrium theory indicates that $\overline{v'v'} = \overline{w'w'}$ and, further, that

$$\frac{\overline{u'u'}}{\overline{v'v'}} = \frac{\overline{u'u'}}{\overline{w'w'}} = \frac{UU}{VV} = \frac{UU}{WW} = 1 + 6b = 1.75 \tag{73}$$

Second, the value of $-\overline{u'w'}/q^2$, which Bradshaw, Ferriss, and Atwell (ref. 3) assume to be a constant equal to 0.15, is defined by superequilibrium theory to be

$$-\frac{UW}{q^2} = \frac{1}{1 + 2b}\sqrt{b/3} = 0.163 \tag{74}$$

This is a rather surprisingly accurate result in view of the fact that the value of b was determined from very different considerations in the development of the second-order closure model.

The value of Von Kármán's constant κ in his expression for the turbulent shear near a surface may also be derived from superequilibrium theory:

$$\tau_t = -\rho\overline{u'w'} = \rho\kappa^2 z^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 \tag{75}$$

From the present results,

$$\tau_t = -\rho UW\Lambda_1^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 = \frac{\sqrt{3/b}}{9(1+2b)^3}\,\rho\Lambda_1^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 \tag{76}$$

In reference 1 it was found that near a surface, Λ is of the form $\Lambda = \alpha z$, where α in the parameter search is found to be 0.7. Letting

$$\Lambda = 0.7z \tag{77}$$

equation (76) gives

$$\tau_t = \frac{0.49\sqrt{3/b}}{9(1+2b)^3}\,\rho z^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2 \tag{78}$$

Comparison of equations (75) and (78) reveals that

$$\kappa^2 = \frac{0.49\sqrt{3/b}}{9(1+2b)^3} = 0.137 \tag{79}$$

or

$$\kappa = 0.37 \tag{80}$$

The value of Von Kármán's constant is actually 0.4. Again, the agreement between results obtained by taking the equilibrium, nondiffusive limit of the present second-order closure model of turbulent shear flow and the classical mixing-length theory is rather remarkable.

EDDY TRANSPORT IN A VORTEX?

If a scheme such as that pursued in the previous section for the superequilibrium equation for a line vortex is followed, the following definitions are introduced into equations (29) to (32):

$$\overline{u'w'} = UU\Lambda^2 \left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right)^2 \tag{81}$$

$$\overline{v'v'} = VV\Lambda^2 \left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right)^2 \tag{82}$$

$$\overline{w'w'} = WW\Lambda^2 \left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right)^2 \tag{83}$$

$$\overline{u'v'} = UV\Lambda^2 \left|\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right| \left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) \tag{84}$$

$$q = Q\Lambda \left|\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right| \tag{85}$$

From this substitution, the following equations are obtained:

$$Q(1 + 2b)UU = \frac{Q^3}{3} + 4UVN \tag{86}$$

$$Q(1 + 2b)VV = \frac{Q^3}{3} - 2UV - 4UVN \tag{87}$$

$$Q(1 + 2b)WW = \frac{Q^3}{3} \tag{88}$$

$$Q(1 + 2b)UV = -UU + 2(VV - UU)N \tag{89}$$

In these equations N is a stability number defined as

$$N = \frac{\bar{v}/r}{\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}} \tag{90}$$

The solution of equations (86) to (89) in terms of the parameters b and N is

$$UU = \frac{Q^2}{1 + 2b}\left(\frac{1}{3} - 4bN\right) \tag{91}$$

$$VV = \frac{Q^2}{1 + 2b}\left(\frac{1}{3} + 2b + 4bN\right) \tag{92}$$

$$WW = \frac{Q^2}{3(1 + 2b)} \tag{93}$$

$$UV = -bQ^3 \tag{94}$$

with

$$Q^2 = UU + VV + WW = \frac{1}{b(1 + 2b)^2}\left(\frac{1}{3} - 8bN - 16bN^2\right) \tag{95}$$

It is clear, since Q^2 is positive definite, that under the assumptions made here turbulence is impossible if

$$N < -\frac{1}{4}\left(1 + \sqrt{1 + \frac{1}{3b}}\right) \tag{96}$$

or if

$$N > \frac{1}{4}\left(\sqrt{1 + \frac{1}{3b}} - 1\right) \tag{97}$$

For the value of b used herein (0.125), these limits become $N < -0.729$ and $N > 0.229$.

Figure 6 shows the behavior of the quantities UU, VV, WW, UV, and Q^2 with variations of the stability parameter N for $b = 0.125$. The results are plotted in terms of the ratios of the quantities to their values for $N = 0$, namely, $(UU)_o$, $(VV)_o$, and so forth. Thus, figure 6 shows the ratios of $\overline{u'u'}$, $\overline{v'v'}$, $\overline{w'w'}$, $\overline{u'v'}$, and q^2 in a vortex to these quantities in a parallel shearing motion having the same mean deformation rate and scale.

It may be seen from figure 6 that the turbulent energy and shear have the same value for $N = -1/2$ as they do for $N = 0$. Between $N = -1/2$ and $N = 0$, the turbulent energy and shear are larger than they are in a parallel shearing motion. For $N < -0.729$ and $N > 0.229$, as mentioned previously, no locally sustained turbulent flow is possible. Thus, for $-0.729 < N < -0.5$, locally self-sustained turbulence is possible, although the turbulence is damped by centrifugal effects. For $0 < N < 0.229$, turbulence is also possible, but here again it is damped by the action of centrifugal forces.

What sort of flows does each of these regions represent? First, note that when $\partial\overline{v}/\partial r = 0$, $N = -1$. Thus at the core radius of a vortex (defined here as the radius where $\partial\overline{v}/\partial r = 0$), a turbulent vortex is stable. Near the center of a free vortex, the tangential velocity $\overline{v}$ is of the form $\overline{v} = mr - 2nr^2$ so that as $r \rightarrow 0$, $N \rightarrow -\infty$. Also, for a free vortex, $\overline{v} \rightarrow \Gamma/2\pi r$ as $r \rightarrow \infty$ and one finds then that as $r \rightarrow \infty$, $N \rightarrow -1/2$. Thus for the classical vortex distribution,

$$\bar{v} = \frac{\Gamma}{2\pi r}\left(1 - e^{-r^2/4\nu t}\right) \tag{98}$$

The flow in the outer regions of the vortex exhibits an eddy diffusivity similar to a parallel flow. As the core of the vortex is approached, the flow becomes more and more stable. It becomes completely stable somewhat outside the core of the vortex. Indeed, the flow is stable at the point of maximum deformation $\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}$. This behavior of the stability parameter N for the classical vortex is shown in figure 7.

The region of increased turbulence and shear between $N = -1/2$ and $N = 0$ can be understood if it is noted that the stability parameter may be written

$$N = \frac{\bar{v}/r}{\frac{1}{2\pi}\frac{\partial \Gamma}{\partial r} - \frac{2\bar{v}}{r}} \tag{99}$$

Thus the region $-1/2 < N < 0$ represents flows for which $d\Gamma/dr$ is negative. These are, of course, flows which exhibit the well-known Taylor instability (ref. 4).

The region $0 < N < 0.229$ is representative of flows occurring between two cylinders rotating in the same direction, so that Γ at the outer cylinder is larger than Γ at the inner cylinder when the centrifugal forces due to the general level rotation cannot completely stabilize the flow.

For a free vortex, it may be surmised from this analysis that the core regions of vortices are locally stable. Regions outside the core are unstable and can generate turbulence. If the core regions of vortices are to exhibit a turbulent shear, this must be caused by turbulence which has diffused into the core region from outer regions which are unstable or by turbulence which has been generated by a shear in the axial direction that is not considered in this analysis. This fact, namely, that the turbulent shear $-\rho\overline{u'v}$ in a vortex is not directly related to the local deformation $\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}$, would lead one to believe that it would be impossible to establish any general rules for determining an eddy viscosity for a vortex. To calculate such flows reliably, it will probably be necessary to use the full power of second-order closure methods.

It might be noted, in this connection, that if one were to use an energy method on such flows, much of the physics of the problem would be lost. This may be seen by considering the sum of equations (86) to (89) with $UU = VV$ as the governing equation of the flow. In this case, the parameter N disappears from the equations and the essential physics of the problem have been lost.

CONCLUDING REMARKS

This short paper has tried to exhibit the relationship between a full second-order closure model for turbulent flow and the older eddy-viscosity models of such flows. It has been shown that classical eddy-transport theory can be obtained from a consideration of the equilibrium, nondiffusive, high Reynolds number limit of the equations of a second-order closure model.

The nature of such limits has been discussed for the flow in sheared stratified media and for a line vortex. The limitations of an eddy-transport model of turbulence in a line vortex have been discussed through the use of the equations derived by the limiting process described herein.

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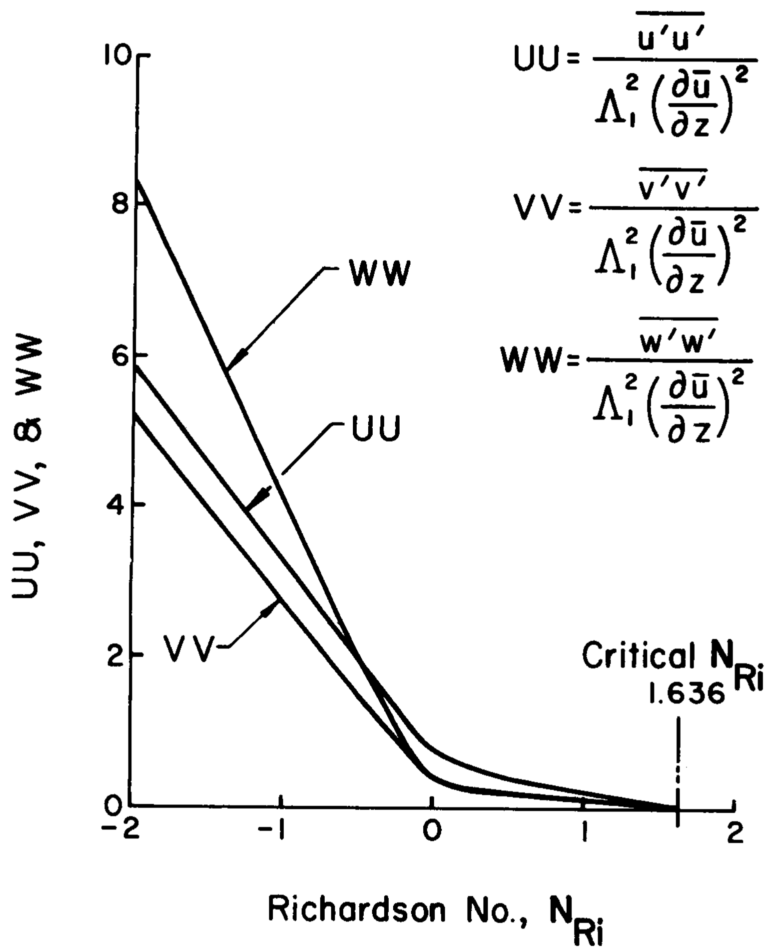


Figure 1.- Superequilibrium values of the turbulence components as a function of the Richardson number for b = 0.125.

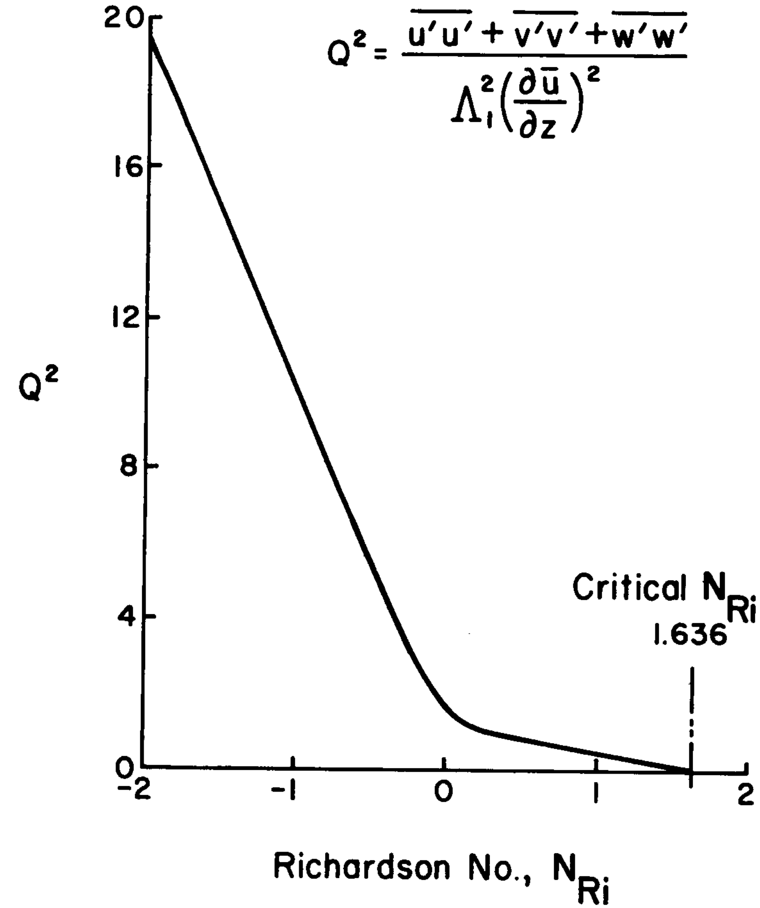
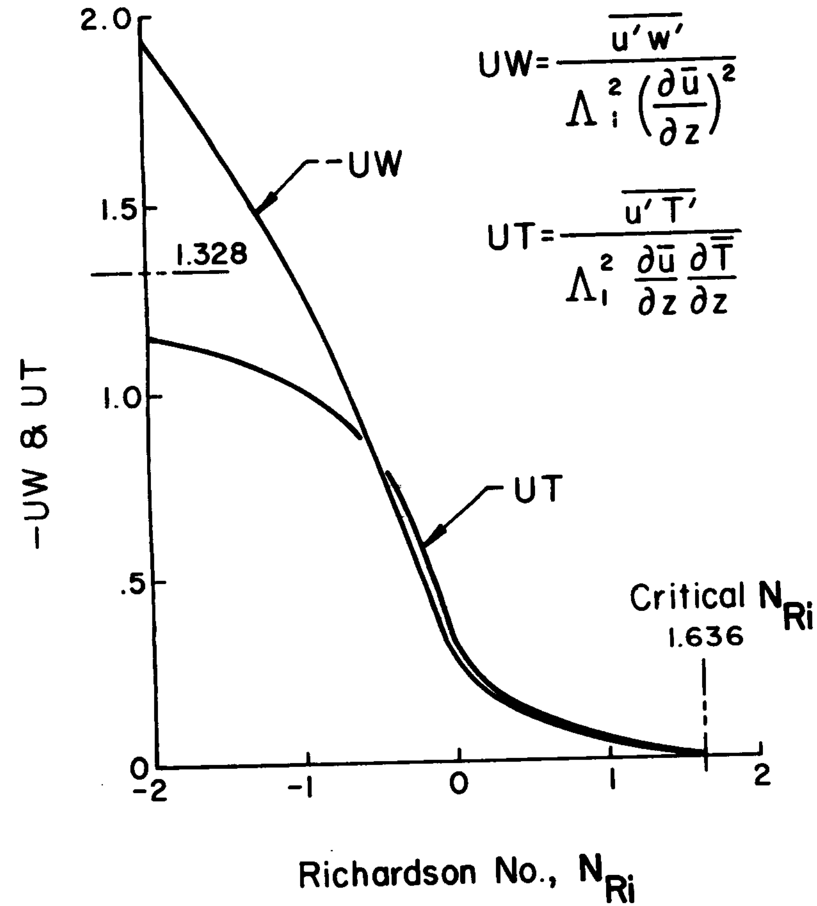


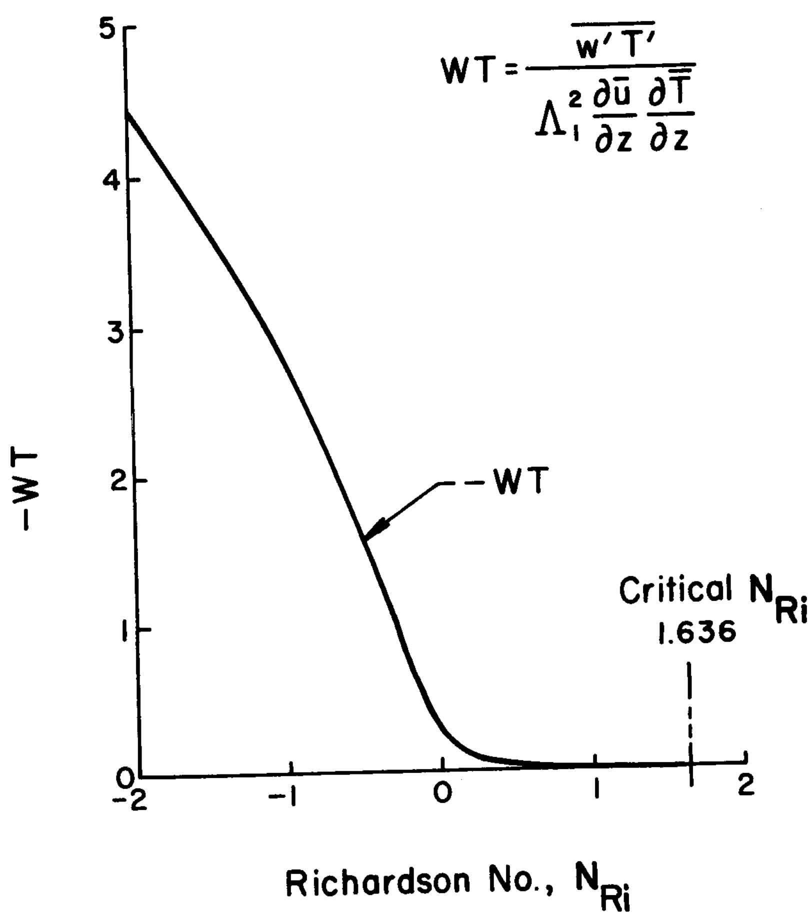
Figure 2.- Superequilibrium value of the sum of the squares of the turbulent velocity fluctuations as a function of the Richardson number for b = 0.125.



$$UW = \frac{\overline{u'w'}}{\Lambda_1^2 \left(\frac{\partial \bar{u}}{\partial z}\right)^2}$$

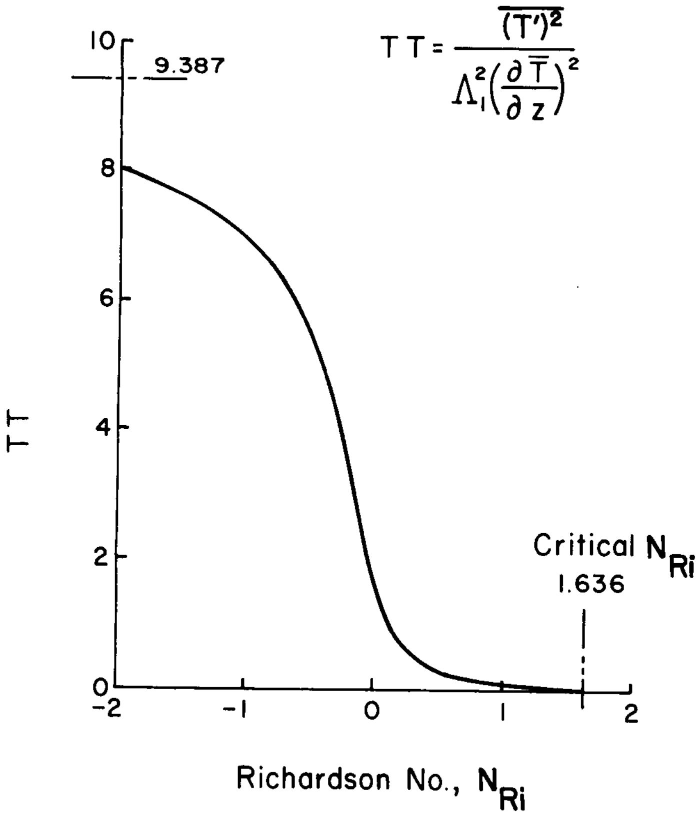
$$UT = \frac{\overline{u'T'}}{\Lambda_1^2 \frac{\partial \bar{u}}{\partial z} \frac{\partial \bar{T}}{\partial z}}$$

Figure 3.- Superequilibrium values for shear and longitudinal heat- and mass-transfer correlations as functions of the Richardson number for b = 0.125. Note that UT approaches the value 1.328 for $N_{Ri} = -\infty$.



$$WT = \frac{\overline{w'T'}}{\Lambda_1^2 \frac{\partial \bar{u}}{\partial z} \frac{\partial \bar{T}}{\partial z}}$$

Figure 4.- Superequilibrium values for vertical heat-transfer correlation as a function of the Richardson number for b = 0.125.



$$TT = \frac{\overline{(T')^2}}{\Lambda_1^2\left(\frac{\partial \overline{T}}{\partial z}\right)^2}$$

Figure 5.- Superequilibrium values for TT as a function of the Richardson number for b = 0.125. Note that this function approaches the limit 9.387 for $N_{Ri} = -\infty$.

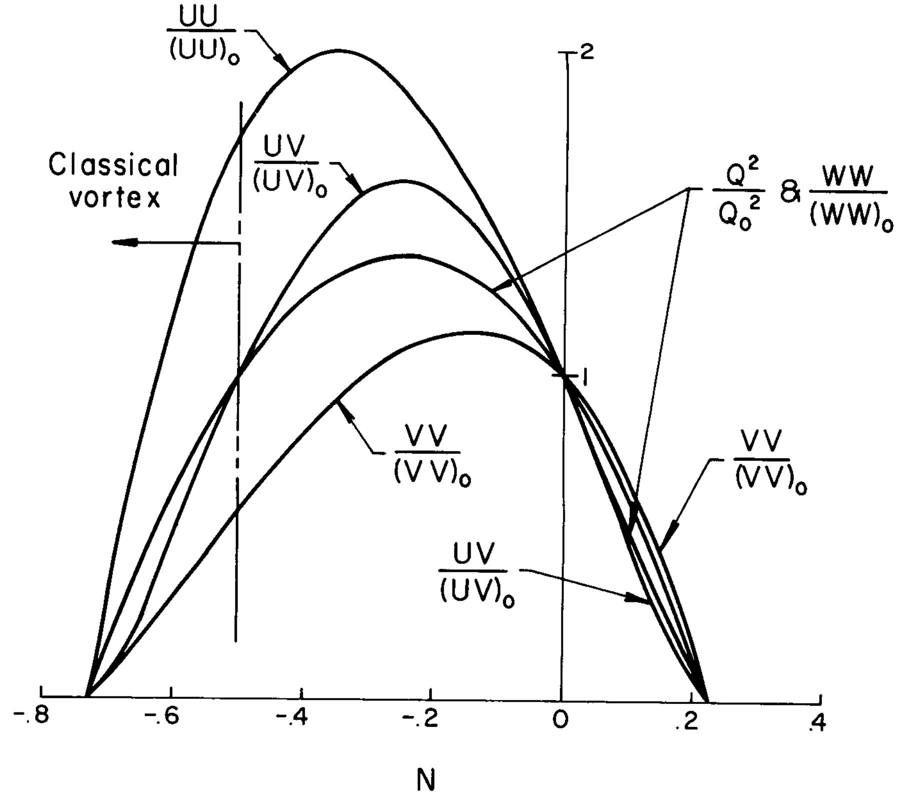


Figure 6.- Behavior of UU, VV, WW, Q^2, and UV as functions of the stability parameter N for b = 0.125.

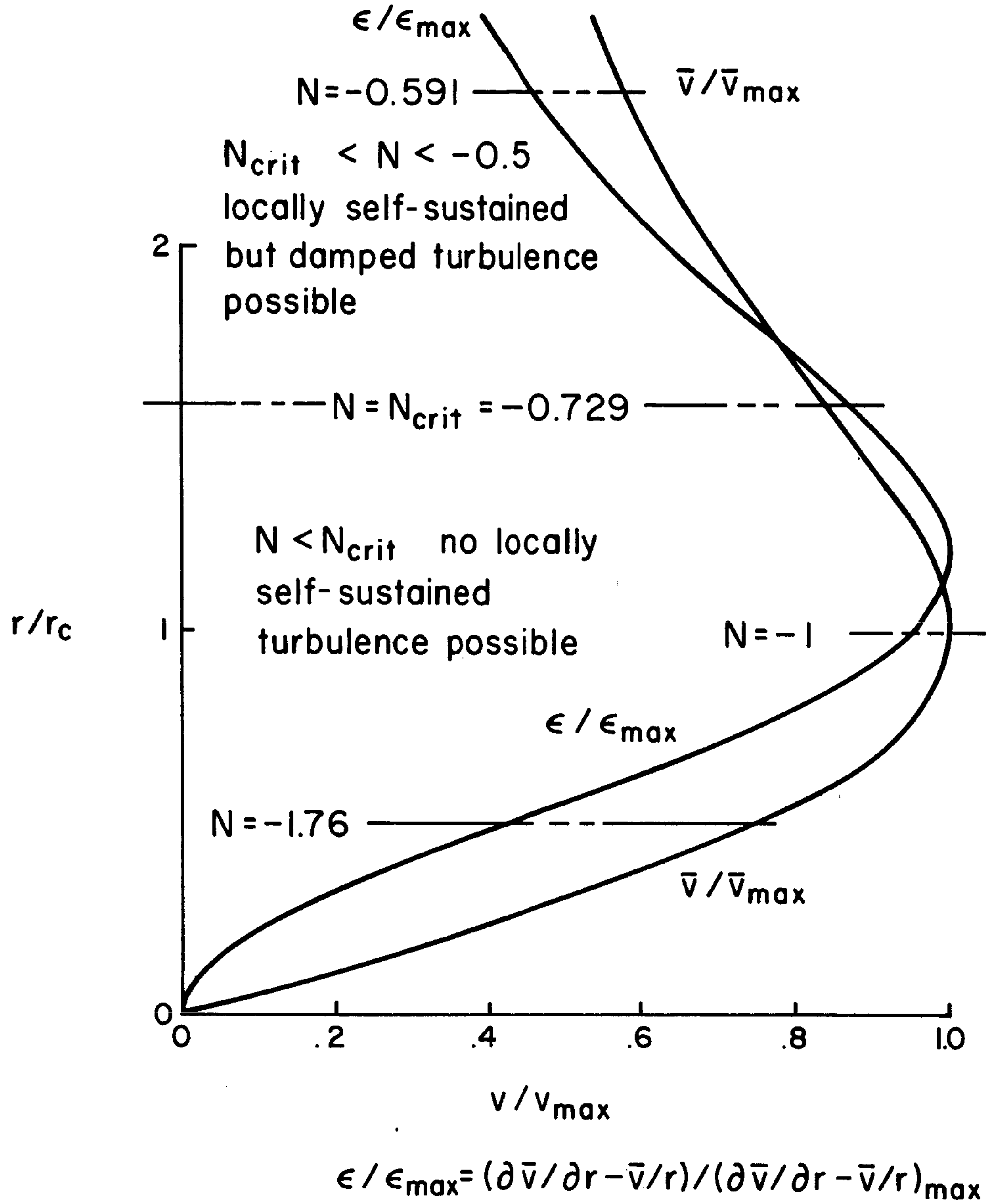


Figure 7. - Behavior of the stability parameter N for a classical line vortex (see eq. (98)).

DISCUSSION

G. L. Mellor: We just finished an experiment at Princeton a year or so ago which is a boundary layer on a flat wall followed by a curved wall and we did a complete set of turbulent measurements. It was very dramatic indeed; we could see the turbulence nearly shut off on the convex side of the wall. And then we did try to develop an expression for the eddy viscosity by balancing production with dissipation using these equations. You can, as you did, find a correction for the eddy viscosity in terms of the proper curvature parameter and it works very well. So I'll just put in two cents here that it seems to work well and it seems to compare with the rather dramatic measurements that we've developed in the last couple of years.

C. duP. Donaldson: One of the points to be made here is that if you are going to do some trick flow like this, it is best not to shortcut and just use, say, the energy and stress equations with some trick for guessing what is missing because you may overlook the physics of the problem. If you use all the equations, you get the physics right.

S. I. Pai: I notice that you assume the mixing lengths for velocity and for temperature are the same. My questions are (1) Do you make this assumption to simplify the analysis and to obtain some essential features of this problem? and (2) If these mixing lengths are different, would you expect that your results would be modified considerably?

C. duP. Donaldson: Yes, the same lengths were chosen to simplify the analysis. In this case, which is the superequilibrium limit, you will begin to see the nature of the problem. No matter what you choose for your eddy-viscosity model, if you want to see whether you should really use such a model or not, you can make this kind of limiting argument. It is true that if you have vastly different scales of the temperature and velocity fields, you are going to have to do something different. As an example, take a large turbulent pipe flow in which you know all the pertinent mean quantities, and with the turbulence model you have, compute all the turbulent characteristics in this flow. Then, if you assume a tiny pencil of heated air is placed in the center of the tube so as to form a very thin hot jet in that region and you use the same scale in the $\overline{u_i'T'}$ and $\overline{(T')^2}$ equations as you use in the $\overline{u_i'u_k'}$ equation, you will find a remarkable result. The general spread of the hot material is about as it should be, but a hot spot stays near the center of the tube. This is a result of using the wrong scale in the $\overline{u'T'}$ and $\overline{(T')^2}$ equations. When the scales of the mean temperature and mean velocity fields are so disparate, one may not use the same scales in the equations for the various second-order correlations.

S. C. Lee: I have two questions. First, I haven't seen any of your models compared with the suggested cases. My question is, have you compared any, and if you did, would we be able to see any of your comparisons?

C. duP. Donaldson: The only comparisons made to date were for the cases which were used to construct the model. In these cases, model parameter searches were made to try to obtain a best fit to the existing data for a free jet, a two-dimensional shear layer, and a flat-plate boundary layer. The model was determined to be that one that gave the best fit to all three cases.

I think I should point out here something that I've said ever since I started making such calculations; namely, it's pretty hard to calculate better at first blush something that somebody has been calculating empirically for the last 20 years. That really wasn't my reason for getting into these second-order closure methods – my real reason was to get at the calculation of some problem which just can't be done by conventional methods and for which you just don't known what the answer is at all, such as the vortex and the behavior of turbulence in stratified media.

S. C. Lee: My second question is related to your particular model with which you are interested in atmospheric conditions for stable and unstable atmospheric conditions. Have you calculated any of those?

C. duP. Donaldson: Yes.

S. C. Lee: Would those be in the paper?

C. duP. Donaldson: Not in the paper to be published in these proceedings. Some results have been published elsewhere. I have just finished writing a paper which is to be mailed out soon to many of the people at this meeting. I think you are on the list.

S. Corrsin: You have identified one necessary condition for the use of an eddy-viscosity model and you of course know that also there is another kind of necessary condition; that is, the characteristics length of the mechanism transporting the property you are interested in must be very small compared with any distance over which the mean property changes appreciably and this is violated by almost all flows that we ever talk about. So there is quite a different class of reasons why eddy-viscosity models may be wrong in principle but work in practice.

C. duP. Donaldson: I understand – valid comment.

G. L. Mellor: I ask Stan, who states this necessary condition – I know it has something to do with kinetic theory but who states it for turbulent?

S. Corrsin: Well as far as ordinary transport models go there's a book on Transport by a fellow named Bosworth[1] that was published in the 1940's which is the only textbook I have seen that actually mentions it. But as far as turbulence phenomena go, in general the gradient transport term is the first term in an infinite-series approximation and this is

[1] Bosworth, Richard Charles Leslie: Heat Transfer Phenomena – The Flow of Heat in Physical Systems. John Wiley & Sons, Inc., [1952].

why, for instance, the Prandtl modified-mixing-length theory with the second-order term could be better. In fact, if your computers get big enough, it may be better to just put on higher and higher derivatives in an attempt to make something which is physically meaningful as well as computable.

C. duP. Donaldson: Yes, that's certainly true for those cases that you can do like that. When, indeed, the flow is completely stable in the superequilibrium sense in the region where the deformations are largest, you can't do that.