JPL Quarterly Technical Review Volume 2, Number 3 October 1972

Index: mathematical sciences, mechanics, power sources, solid-state physics, structural engineering

An Algorithm for Synthesizing Mass and Stiffness Matrices From Experimental Vibration Modes

R. G. Ross, Jr.

Engineering Mechanics Division

It is sometimes desirable to derive a dynamic model of highly complex structures from experimental vibration data. This article presents an algorithm for synthesizing the mass and stiffness matrices from experimentally derived modal data in a way that preserves the physical significance of the individual mass and stiffness elements. The mass and stiffness matrices are derived for a rollup solar array example, and are then used to define the modal response of a modified array.

Introduction

Structural design often requires an estimate of the response of a conceived structure to dynamic excitation before the design is finalized. The predicted response is generally obtained using a finite element discretization of the governing differential equations to calculate the matrices $\mathbf{K}$ and $\mathbf{M}$ that characterize the structure's stiffness and mass properties. The solution to the algebraic eigenvalue problem $\mathbf{K}x = \omega^2\mathbf{M}x$ defines the predicted natural frequencies ω and vibration mode shapes of the structure.

To verify the analytical model, the low-order natural frequencies and mode shapes may also be determined experimentally in a modal test. These tests are usually run on the final structure or on a structural simulation of the final structure, often at a time when it is difficult to incorporate anomalies into the design cycle. For this reason, there is a growing desire to incorporate modal test results of early prototypes or skeletal systems into the analytical models of these systems so that extrapolation to final designs can be made with greater confidence.

In addition to space applications as described in this article, one area of current interest is the modal testing of multistory building structures that is conducted after the completion of the primary structure but before the addition of interior walls, trim, and furnishings. It is desired to use the results

of preliminary modal tests to improve the analytical model of the primary structure so that extrapolation to the dynamic earthquake response of the finished building can be made by adding the elemental stiffness and mass matrices describing the structural modifications directly to the synthesized matrices.

Several methods for synthesizing the governing mass and stiffness matrices from experimental vibration modes have been recently described (Reference 1). This article describes an improved matrix synthesis technique that is shown to allow elemental stiffness and mass matrices describing structural modifications to be added directly to the synthesized matrices.

Definition of Problem

Following the rationale outlined in Reference 1, the problem is the following: with a set of p eigenvalues λ_i ($\lambda_i = \omega_i^2$) and corresponding eigenvectors ϕ_i (mode shapes) of the unknown structural system, we wish to determine the n^{th} order mass and stiffness matrices so that the eigensystem

$$\mathbf{K}x = \lambda \mathbf{M}x \tag{1}$$

will possess eigenvalues and eigenvectors that are either equal to or as close as possible to the measured ones. Since the most significant characteristics of the mass and stiffness matrices are their representation of the system's kinetic and strain energies (Reference 1), **M** and **K** are further required to accurately represent known kinetic and strain energy characteristics of the system. This is necessary if **M** and **K** are to be compatible with the analytical model of the structure.

To determine the requirements for **M** and **K**, we note that if **M** and **K** are required to be symmetric and to define the measured eigensolution, then the eigenvectors must be orthogonal with respect to them; i.e., if Φ is the modal matrix of eigenvectors, then

$$\Phi^T \mathbf{M} \Phi = \mathbf{D} \quad \text{and} \quad \Phi^T \mathbf{K} \Phi = \Lambda \mathbf{D} \tag{2}$$

where **D** is an arbitrary diagonal matrix and Λ is a diagonal matrix of the λ_i. If we temporarily assume that n eigenvectors are available so that Φ is nonsingular, then Equation 2 requires

$$\mathbf{M} = {\Phi^{-1}}^T \mathbf{D} \Phi^{-1} \tag{3}$$

and

$$\mathbf{K} = {\Phi^{-1}}^T \mathbf{D} \Lambda \Phi^{-1}$$

However, because the number of coordinates used is assumed to be much larger than the number p of natural modes determined, only the first p out of n eigenvectors are available, and the inverse of the modal matrix is not defined. To alleviate the problem, we consider the inverse matrices

$$\mathbf{M}^{-1} = \Phi \mathbf{D}^{-1} \Phi^T$$

$$\mathbf{K}^{-1} = \Phi \Lambda^{-1} \mathbf{D}^{-1} \Phi^T \tag{4}$$

which are singular, rank p matrices when only p modes are available. Since ultimately **M** and **K** are required, the inversion of these rank deficient matrices will be necessary. However, first consider the problem of determining the diagonal normalization matrix **D** so as to preserve as much as possible the known kinetic and strain energy characteristics.

Determination of the Normalization Matrix D

In most cases, knowledge of the system energies is limited to that represented by the analytical mass and stiffness matrices. Allowing for other known energy characteristics, $\tilde{\mathbf{M}}$ is defined as the approximate mass matrix that represents knowledge of the system kinetic energy, and similarly $\tilde{\mathbf{K}}$ is defined as the approximate stiffness matrix that represents knowledge of the system strain energy.

Recalling that the strain and kinetic energies associated with an arbitrary displacement q_i or velocity distribution v_i are given by $1/2 q_i^T \mathbf{K} q_i$ and $1/2 v_i^T \mathbf{M} v_i$, respectively, the normalization **D** can be defined by requiring that

$$v_i^T \mathbf{M} v_i = v_i^T \tilde{\mathbf{M}} v_i \tag{5}$$

and

$$q_i^T \mathbf{K} q_i = q_i^T \tilde{\mathbf{K}} q_i \tag{6}$$

for a total of p arbitrary vectors q_i and v_i. If more than p important energy relationships exist, **D** can be alternatively defined by requiring that Equations 5 and 6 be satisfied in the least squares sense.

To preserve the important energies defined by **M** and **K**, the arbitrary vectors should be chosen as meaningful displacement and velocity distributions for the structural system under examination. Though the optimum vectors will depend on the specific characteristics of $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{M}}$, a logical choice for most problems can be made by noting that **M** will approximate

the kinetic energies of the low-order modes reasonably well; i.e., $\phi_i^T \mathbf{M} \phi_i$ is well described by $\phi_i^T \tilde{\mathbf{M}} \phi_i$ for low-order ϕ_i. For such problems, the measured ϕ_i are therefore a logical choice for the arbitrary vectors. With this selection, and the use of Equation 3 for **M**, the p d_i are defined by

$$d_i = \phi_i^T \tilde{\mathbf{M}} \phi_i \quad [i = 1, p] \tag{7}$$

Similarly, if a measured mode ϕ_i is used as a q_i, the i^{th} diagonal element is defined by

$$d_i = (\phi_i^T \tilde{\mathbf{K}} \phi_i)/\lambda_i \tag{8}$$

Inversion of the Rank Deficient Inverse Matrices

If the number p of vibration modes determined is equal to the number of measured coordinates n, then Equations 4, and 7 and/or 8 define nonsingular inverse mass and stiffness matrices that, when inverted, define the measured frequencies and mode shapes exactly. When p is less than n, the inverse matrices are singular, rank p, and Equations 7 and 8 generally only approximate the equalities defined by Equations 5 and 6.

In Reference 1 it was suggested that Equations 4 could be made invertable by filling out the modal matrix with n-p arbitrary linearly independent vectors. Though these vectors will be eigenvectors of the resulting eigensystem, choosing larger λ_i for them will make them higher order modes out of the range of interest. The advantage of this approach is that the resulting mass and stiffness matrices define the measured frequencies and mode shapes exactly, and Equations 5 and 6 are satisfied exactly for a total of n vectors. The disadvantage of this technique is the difficulty sometimes encountered in choosing linearly independent vectors to fill out Φ that are not nearly dependent on the measured mode shapes, and that do not lead to ill-conditioned mass and stiffness matrices.

An alternate inversion approach, which is considered here, is a modified spectral inversion of the rank difficient inverse mass and stiffness matrices. Recall that these symmetric matrices can be described in terms of their spectral decompositions (eigenvalues and eigenvectors) as follows

$$\mathbf{M}^{-1} = \sum_{i=1}^{n} \beta_i \xi_i \xi_i^T = \mathbf{X} \mathbf{B} \mathbf{X}^T \tag{9}$$

and

$$\mathbf{K}^{-1} = \sum_{i=1}^{n} \alpha_i \zeta_i \zeta_i^T = \mathbf{Z}\mathbf{A}\mathbf{Z}^T \tag{10}$$

where

β_i = i^{th} eigenvalue of $\mathbf{M}^{-1}$ (ordered smallest to largest)

ξ_i = i^{th} eigenvector of $\mathbf{M}^{-1}$ (normalized $\xi_i^T \xi_i = 1$)

α_i = i^{th} eigenvalue of $\mathbf{K}^{-1}$ (ordered smallest to largest)

ζ_i = i^{th} eigenvector of $\mathbf{K}^{-1}$ (normalized $\zeta_i^T \zeta_i = 1$)

$\mathbf{B}$ = diagonal matrix of β_i

$\mathbf{X}$ = modal matrix of ξ_i

$\mathbf{A}$ = diagonal matrix of α_i

$\mathbf{Z}$ = modal matrix of ζ_i

Since the eigenvectors of symmetric matrices are orthonormal, $\mathbf{X}^T = \mathbf{X}^{-1}$, $\mathbf{Z}^T = \mathbf{Z}^{-1}$ and thus

$$\mathbf{M} = \mathbf{X}\mathbf{B}^{-1}\mathbf{X}^T \tag{11}$$

and

$$\mathbf{K} = \mathbf{Z}\mathbf{A}^{-1}\mathbf{Z}^T \tag{12}$$

However, because $\mathbf{K}^{-1}$ and $\mathbf{M}^{-1}$ are rank p, $\alpha_i = 0$ and $\beta_i = 0$ [$i = 1, n - p$] and $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ are undefined. This is overcome by requiring

$$\xi_i^T \mathbf{M} \xi_i = \xi_i^T \widetilde{\mathbf{M}} \xi_i \quad [i = 1, n - p]$$

and

$$\zeta_i^T \mathbf{K} \zeta_i = \zeta_i^T \widetilde{\mathbf{K}} \zeta_i \quad [i = 1, n - p]$$

Substituting into Equations 11 and 12 gives

$$\alpha_i^{-1} = \zeta_i^T \widetilde{\mathbf{K}} \zeta_i \quad [i = 1, n - p] \tag{13}$$

$$\beta_i^{-1} = \xi_i^T \tilde{M} \xi_i \; [i = 1, n-p] \tag{14}$$

Using $\alpha_i^{-1} = 1/\alpha_i$ and $\beta_i^{-1} = 1/\beta_i$ for $i = n-p+1, n$ in conjunction with Equations 13 and 14 completely defines $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

The complete algorithm for synthesizing the mass and stiffness matrices from experimental modal data is defined by Equations 4, 7 and/or 8, 11, 12, 13, and 14.

Example—Application to Rollup Solar Array

As a demonstration, the matrix synthesis technique just described is used to synthesize the mass and stiffness matrices for the rollup solar array shown in Figure 1. The array consists of two solar-cell blankets tensioned between the base and a relatively stiff leading edge beam that is attached to the tip of the deployed boom (Reference 2). For the purposes of this example, the out-of-plane dynamic behavior of the array is described by the seven-degree-of-freedom model in Figure 2. The parameter values associated with the array are given in Table 1. Note that the actual array is assumed to have unequal blanket tensions on the two sides and an uneven mass distribution. The first few eigenvalues (natural frequencies squared) and mode shapes defined by this model will be used as the "experimentally determined modal data" and are given in Table 2.

The idealized equal tension uniform mass values in Table 1 are assumed in the approximate analytical model that defines $\tilde{K}$ and $\tilde{M}$. For comparison, the eigenvalues and mode shapes defined by $\tilde{K}$ and $\tilde{M}$ are given in Table 2. Note that this "approximate" model leads to a pair of equal eigenvalue pure blanket resonances in addition to the pure symmetric and antisymmetric

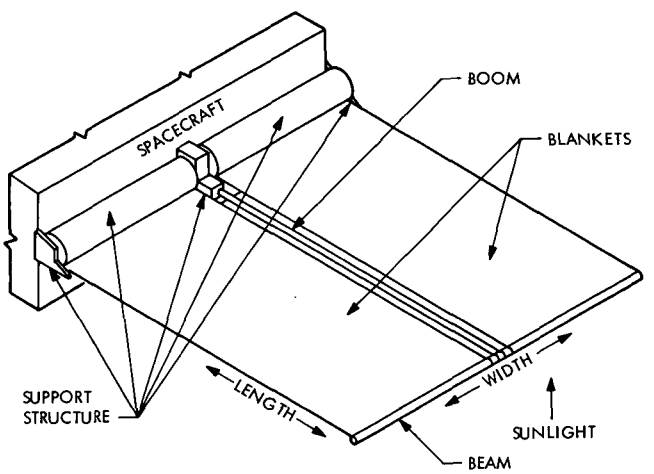

Figure 1. Deployed array

Table 1. Solar array parameter values

Parameter	Units	Actual array	Approximate model
Length	m	1.22	1.22
Width	m	0.61	0.61
Boom mass	kg	0.56	0.54
Boom bending stiffness (EI)	Nm^2	18.6	16.5
Boom torsional stiffness (GJ)	Nm^2	0.	0.
Leading edge beam mass	kg	0.41	0.45
Boom tip mass	kg	0.04	0.0
Blanket mass:			
Outer half left side	kg	0.95	0.91
Inner half left side	kg	0.86	0.91
Outer half right side	kg	0.95	0.91
Inner half right side	kg	0.86	0.91
Blanket tension:			
Left side	N	9.3	8.9
Right side	N	7.1	8.9

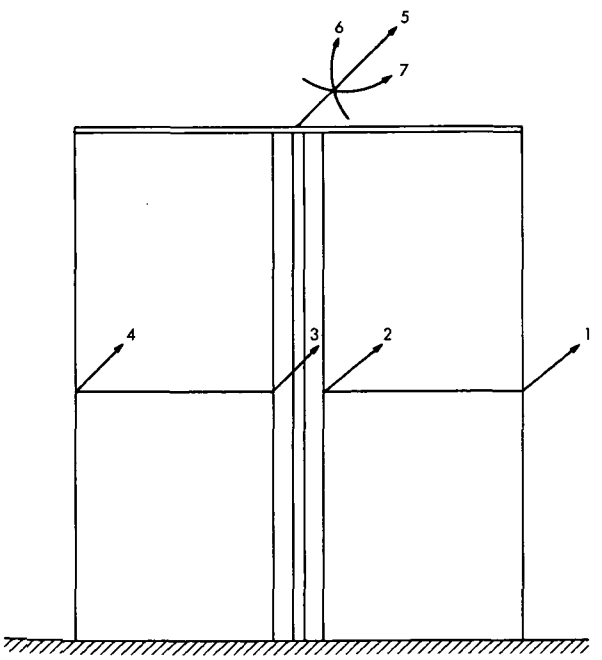

Figure 2. Finite element model of array

Table 2. Eigensolutions for example array

"Measured" eigensolution of actual array						
0.787+01	0.137+02	0.392+02	0.493+02	0.758+02	0.924+02	0.801+04
1	2	3	4	5	6	7
0.504	0.414	0.589	−0.228	0.660	−0.157	0.018
0.002	0.455	−0.797	−0.084	0.237	0.184	0.018
0.002	0.388	0.045	0.785	0.408	0.272	0.018
−0.462	0.423	0.123	−0.567	−0.319	0.777	0.018
0.003	0.503	0.015	0.029	−0.236	−0.240	−0.075
0.001	0.192	0.006	0.011	−0.088	−0.089	−0.996
0.730	−0.045	−0.026	0.050	−0.421	0.445	0.000
Eigensolution from approximate model						
0.811+01	0.121+02	0.483+02	0.483+02	0.878+02	0.885+02	0.720+04
1	2	3	4	5	6	7
−0.479	−0.407	0.130	−0.587	0.455	−0.603	0.018
−0.000	−0.407	0.094	0.789	0.455	−0.000	0.018
−0.000	−0.407	−0.801	−0.021	0.455	−0.000	0.018
0.479	−0.407	0.577	−0.182	0.455	0.603	0.018
−0.000	−0.542	0.000	−0.000	−0.389	0.000	−0.072
−0.000	−0.208	0.000	−0.000	−0.145	0.000	−0.997
−0.735	0.000	−0.000	−0.000	0.000	0.523	−0.000
Eigensolution from synthesized model						
0.787+01	0.137+02	0.392+02	0.493+02	0.537+02	0.881+03	0.121+04
1	2	3	4	5	6	7
0.504	0.414	−0.589	0.228	0.506	0.189	0.012
0.002	0.455	0.797	0.084	0.331	0.055	0.006
0.002	0.388	−0.045	−0.785	−0.013	−0.352	−0.010
−0.462	0.423	−0.123	0.567	−0.344	−0.590	−0.020
0.003	0.503	−0.015	−0.029	−0.387	0.457	−0.350
0.001	0.192	−0.006	−0.011	−0.212	0.158	0.936
0.730	−0.045	0.026	−0.050	−0.567	−0.505	−0.022

[a]The eigenvalues are listed horizontally using scientific notation and the eigenvectors are listed vertically beneath them.

structural modes. These are different from the modes of the "actual" array as given in Table 2.

The problem is to synthesize new mass and stiffness matrices that better approximate the assumed "measured" modal data, and that remain compatible with the finite element analysis used to define the approximate model. Using the previously derived algorithm in conjunction with the first four measured modes ($p = 4$) gives new matrices for the structure. The eigenvalues and mode shapes defined by these synthesized mass and stiffness

matrices are given in Table 2, and compare quite favorably with the "measured" modal data.

To demonstrate the compatibility with the approximate finite element model, the effect of assuming a 1.36 Nm/rad (1 ft-lb/rad) torsional stiffness for the boom and adding a 0.45 kg (1 lb) lumped mass to the tip of boom is considered. Adding the appropriate mass and stiffness terms to the "actual," approximate, and synthesized matrices leads to the modified eigenvalues and mode shapes given in Table 3. From these results, it is clear that the

Table 3. Eigensolutions for modified array

"Measured" eigensolution of modified array						
0.116+02	0.240+02	0.392+02	0.495+02	0.702+02	0.117+03	0.782+04
1	2	3	4	5	6	7
0.443	0.707	−0.552	−0.199	0.533	−0.400	0.013
0.435	−0.049	0.820	−0.071	0.336	0.022	0.013
0.383	−0.033	−0.052	0.757	0.636	0.029	0.013
0.377	−0.548	−0.138	−0.616	0.264	0.591	0.013
0.534	−0.029	−0.017	0.025	−0.294	−0.036	−0.054
0.204	−0.011	−0.006	0.009	−0.110	−0.013	−0.998
0.009	0.442	0.029	0.045	−0.172	0.699	0.000
Eigensolution from modified approximate model						
0.101+02	0.246+02	0.483+02	0.483+02	0.752+02	0.117+03	0.704+04
1	2	3	4	5	6	7
0.397	0.619	0.500	−0.316	0.474	−0.524	0.013
0.397	0.000	−0.500	0.632	0.474	−0.000	0.013
0.397	−0.000	−0.500	−0.632	0.474	−0.000	0.013
0.397	−0.619	0.500	0.316	0.474	0.524	0.013
0.568	0.000	0.000	0.000	−0.297	0.000	−0.052
0.218	0.000	0.000	0.000	−0.111	0.000	−0.998
−0.000	0.483	−0.000	−0.000	−0.000	0.671	0.000
Eigensolution from modified synthesized model						
0.116+02	0.237+02	0.392+02	0.495+02	0.627+02	0.705+03	0.113+04
1	2	3	4	5	6	7
−0.411	0.733	−0.557	0.188	0.406	0.174	0.076
−0.426	0.048	0.818	0.055	0.228	0.060	0.023
−0.401	−0.092	−0.055	−0.778	−0.120	−0.398	−0.110
−0.415	−0.446	−0.126	0.596	0.146	−0.583	−0.206
−0.525	−0.099	−0.024	−0.001	−0.281	0.374	−0.120
−0.201	−0.056	−0.011	0.004	−0.153	−0.192	0.944
−0.005	0.491	0.012	−0.018	−0.803	−0.539	−0.186

[a]The eigenvalues are listed horizontally using scientific notation and the eigenvectors are listed vertically beneath them.

modified synthesized model provides a much better description of the modified structure than does the modified approximate model. Though the agreement between the modified synthesized and modified exact models is not perfect, it is considered good in light of the size of the error in the starting approximate model and the size of modification that was made.

References

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