NASA TECH BRIEF
Ames Research Center

Algorithm for Liapunov Stability Analysis

An algorithm has been developed for automatic computation of a quadratic estimate of the domain of stability for the stable equilibrium states of nonlinear systems of ordinary differential equations. Standard linear stability analysis techniques failed to predict adequately the stability of the current NASA Orbiting Astronomical Observatory coarse-pointing-mode control system.

In developing the algorithm, various minimization and random search techniques were utilized to solve the min–max problem which yields the estimate; gradient search and penalty function techniques were found to be generally inapplicable to high order nonlinear problems. The new methods are the first known example of solving a min–max problem via two random searches.

The algorithm is based on Liapunov stability theory, the only available sufficiently general approach to nonlinear system stability analysis. The primary difficulty in application of the theory is the construction of an appropriate Liapunov function. This difficulty is eliminated by restricting consideration to positive definite quadratic-form Liapunov functions. This restriction is substantial, but it results in an estimate that is always an ellipsoid in *n*-space, and is often a better estimate than those obtained with more complex functions. The fact that the estimate is always a hyperellipsoid means it is easier to visualize and interpret than other estimates, and some of the computations are simplified.

The algorithm is applicable to systems that are described by a set of quasilinear differential equations of the form:

$$\dot{x} = Ax + g(x)$$

where x is the *n* vector of state variables, A is a stable matrix, g(x) is at least of order x^2, and $x = 0$ is the equilibrium point of interest (i.e., $g(0) = 0$). Most systems that are designed by techniques currently used can be treated as indicated above. The objective is to determine or estimate the set of initial states from which the system will return to the equilibrium state $x = 0$, i.e., to estimate the domain of attraction of the equilibrium state.

The objective is achieved by using LaSalle's theorem on the extent of asymptotic stability. Basically, the theorem states that if a Liapunov function V(x) and its total time derivative have certain properties within the set Ω_l of states x such that $V(x) < l$, where l is a constant, then all trajectories beginning in Ω_l tend toward the equilibrium $x = 0$. The problem of finding such a Liapunov function V(x) is resolved by restricting consideration to positive definite quadratic forms. The largest value of l for which the required conditions hold gives the best estimate relative to that particular quadratic form. This value is found to be the solution of a constrained minimum problem.

In developing the algorithm for application to this nonacademic complex problem, it was necessary to solve four major computational problems efficiently. These problems are: 1) generation of arbitrary positive definite matrices; 2) solution of the Liapunov matrix equation; 3) solution of a nine-variable constrained minimum problem; 4) solution of a maximization problem involving 45 variables. The first was accomplished by developing a parameterization of the set of positive definite matrices. The second was solved by selecting from the four available techniques the one with the least error growth with increased system dimension. The third was solved

(continued overleaf)

by devising an efficient random search tailored to the geometry of the problem. This was done subsequent to determining that the widely used gradient search techniques and penalty function methods are totally inapplicable to this problem. The fourth problem was partially solved via an "accelerated random search." Again, gradient techniques are totally inapplicable to this complex high-dimensional problem.

The algorithm is shown to be feasible for solution of this complex nonlinear stability analysis problem. However, the state of the art of search techniques for complex high-dimensional problems severely limits its immediate application as an analytical or operational tool. It is shown that, compared to simulation (the only other currently available tool), it promises to produce a lower cost solution with specified confidence or conversely to produce a higher confidence result for a given cost.

The algorithm developed and examined in this study is a feasible solution to the problem of estimating the domain of attraction of an equilibrium state of a complex nonlinear physical system. Its further development into a practical tool is limited, however, in two ways. (1) Since the nonlinearities are not accounted for in the geometric shape of the estimate, it is conceivable that there are systems for which one cannot find a suitably large family of ellipsoids within the domain of attraction such that all trajectories cut them in the inward direction as required by the problem formulation. (2) The state of the art in search techniques must be substantially advanced so that the maximization problem can be effectively and efficiently solved. For problems of reasonable complexity, gradient techniques must be abandoned in favor of random-search techniques. The problem is further complicated because the objective function is stated in terms of the solution of a minimum problem which is dependent upon the variables over which the maximization occurs. Thus, the problem is one of high-dimension with unknown, complex geometry.

Despite these hurdles, the continued development of the algorithm remains attractive because it promises to provide a tool that can provide higher confidence stability information for complex nonlinear systems at a lower cost than conventional simulation methods.

Notes:

1. The following documentation may be obtained from:

   National Technical Information Service
   Springfield, Virginia 22151
   Single document price $3.00
   (or microfiche $0.95)
   References:
   NASA CR-73383 (N70-10614), Further Development of an Algorithm for the Nonlinear Stability Analysis of the Orbiting Astronomical Observatory "Paired-Tracker" Control System.
   NASA CR-1729 (N71-26916), An Algorithm for Liapunov Stability Analysis of Complex Nonlinear Systems with Application to the Orbiting Astronomical Observatory.

2. No additional documentation is available. Specific questions, however, may be directed to:

   Technology Utilization Officer
   Ames Research Center
   Moffett Field, California 94035
   Reference: B72-10023

Patent status:

No patent action is contemplated by NASA.

Source: G. R. Geiss, V. D. Cohen, and D. S. Rothschild of Grumman Aerospace Corporation under contract to Ames Research Center
(ARC-10498)