THE STABILITY OF A ROTATING LIQUID MASS

Lecture 5:

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The problem of the fission of a rotating liquid mass is one which draws on investigations going back some 200 years. The problem has been most extensively treated on the basis of the assumption that the mass is a homogeneous fluid. It is quite clear that the earth is not now a homogeneous fluid; it is even conceivable that the earth never was a homogeneous fluid. Even if the latter is true, it is worthwhile to discuss the case of the homogeneous fluid because it gives us the best-explored road into the problem. Starting from this road we can made such changes as are required to account for the actual heterogeneity of the earth. We follow the treatment of Jeans (1919), and our equations are numbered like his, in his chapter III. New equations which we have inserted are followed by small letters.

We begin by asking about the forms which would be taken by a rotating fluid body which is constrained to be an ellipsoid. We shall show that certain ellipsoids are in fact equilibrium configurations. Here again we have simplified the problem and we must later justify the choice of an ellipsoid by showing that it is, in fact, the stable configuration for certain velocity ranges. Note that we are here interested in an exact solution to the approximate problem, rather than, as heretofore, in an approximate solution of the real problem.

In preparation for our problem we note that the equation of the boundary of an ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{51}$$

where the semiaxes of the ellipsoid are a, b, c. If we wish to consider a range of possible ellipsoids then it is useful in many cases, and in particular in the present problem, to consider the family of confocal ellipsoids given by the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1 \tag{52}$$

where λ ranges from 0 to ∞. Following Jeans, we put

$$\left.\begin{aligned} a^2+\lambda &= A;\ b^2+\lambda = B;\ c^2+\lambda = C \\ \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)} &= (ABC)^{1/2} = \Delta . \end{aligned}\right\} \tag{53}$$

We take the quantity $abc = r_0^3$ and the mass of the ellipsoid as given by

$$M = \frac{4}{3}\pi\rho abc = \frac{4}{3}\pi\rho r_0^3 .$$

Now the potential of this mass at an internal point with coordinates x, y, z is given by (Thomson and Tait, 1962)

$$V_i = -\pi\rho abc \int_0^\infty \left(\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - 1\right)\frac{d\lambda}{\Delta} \tag{55}$$

if we take the units such that the absolute constant of gravitation G is 1. For practical use, we should multiply ρ by G wherever it appears. Notice that the integration is over λ ; thus the potential can be considered as composed of a part which increases proportionally to x^2, another which increases with y^2 and a third which increases with z^2 as we move about in the interior of the ellipsoid.

For an exterior point the famous theorem of Ivory asserts that the potential is the same as that which would have been obtained for an ellipsoid whose surface passed through this exterior point and which had the same mass. This result is summed up in Jeans' equation

$$V_0 = -\pi\rho abc \int_\lambda^\infty \left(\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} - 1\right)\frac{d\lambda}{\Delta} \tag{54}$$

where λ is the parameter of the ellipsoid which passes through the given external point. Fuller discussions of this problem are to be found in Moulton's "Celestial Mechanics" and in standard treatises on potential theory.

Now Jeans introduces a set of abbreviated notations. He writes

$$\int_0^\infty \frac{d\lambda}{\Delta} = J$$

and also

$$\int_0^\infty \frac{d\lambda}{A^m B^n C^p \Delta} = J_{A^m B^n C^p} \; . \qquad (56)$$

With these notations the equation for the interior potential assumes the form

$$V_i = -\pi\rho abc \; (x^2 J_A + y^2 J_B + z^2 J_C - J \; . \qquad (57)$$

In this form it is easy to see that the potential is the sum of a constant term and terms dependent on x^2, y^2, and z^2 as previously mentioned. In addition, we find that $J_A + J_B + J_C = 2/abc$ because

$$\nabla^2 V_i = -4\pi\rho \; .$$

We can also verify by a fairly simple manipulation the formula that

$$J_B - J_A = (a^2 - b^2) \; J_{AB} \qquad (59)$$

and similarly his equation

$$J_{A^m B^n C^p} - J_{A^{m+1} B^n C^p} = (a^2 - b^2) \; J_{A^{m+1} B^{n+1} C} \; . \qquad (60)$$

With these preliminaries we remark that on a rotating body the potential referred to the rotating axes is given by

$$V_i + \frac{1}{2}\omega^2 (x^2 + y^2) \; . \qquad (62)$$

Here the true gravitational potential V_i has been augmented by the potential from centrifugal force $\frac{1}{2}\omega^2(x^2 + y^2)$. The form $\frac{1}{2}\omega^2(x^2 + y^2)$ is a routine result of the elementary theory of dynamics.

On a figure of equilibrium the above potential must be constant over a whole boundary. If we also require that the boundary shall be an ellipsoid then we have an equation of the form (51). The normal way of combining these two equations is to multiply one of them by undetermined multiplier, say θ, and add to form a new function, M, as follows:

$$M = V_i + \frac{1}{2}\omega^2(x^2 + y^2) + \theta\pi\rho abc\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

When this is done we can regard x and y, for example, as independent variables on the surface, so that we can legitimately ask that the partial derivative of M with respect to x and y following the surface shall be 0. When we perform the necessary differentiations we must include z as a function of x and y. We shall have, therefore,

$$\frac{\partial M\ (x,y)}{\partial x} = \frac{\partial M\ (x,y,z)}{\partial x} + \frac{\partial M\ (x,y,z)}{\partial z}\,\frac{\partial z}{\partial x},$$

$$\frac{\partial M\ (x,y)}{\partial y} = \frac{\partial M\ (x,y,z)}{\partial y} + \frac{\partial M\ (x,y,z)}{\partial z}\,\frac{\partial z}{\partial y},$$

The second terms on the right are rather ugly, and since we have not yet decided what we are going to do with θ it is permitted, since the equations are linear, to say that we will choose θ in such a way that

$$\frac{\partial M\ (x,y,z)}{\partial z} = 0 .$$

When we do so we have three similar equations in x, y, and z, since the ugly terms on the right-hand side have now been disposed of.

$$J_A - \frac{\omega^2}{2\pi_\rho abc} = \frac{\theta}{a^2}\ ; \tag{65}$$

$$J_B - \frac{\omega^2}{2\pi_\rho abc} = \frac{0}{b^2}\ ; \tag{66}$$

$$J_C = \frac{\theta}{c^2} \; . \tag{67}$$

Two of them simply express the condition that M is constant over the surface; but the third equation in effect defines θ. Naturally it makes no difference which of the equations we consider to be the one which defines θ. If we add all three equations we obtain:

$$J_A + J_B + J_C - \frac{2\omega^2}{2\pi\rho abc} = \theta \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)$$

$$\left(\frac{2}{abc} - \frac{2\omega^2}{2\pi\rho abc} \right) \frac{1}{\frac{1}{a^2} \;\; \frac{1}{b^2} \;\; \frac{1}{c^2}} = \theta = \frac{2\left(1 - \frac{\omega^2}{2\pi\rho}\right)}{abc \left(\frac{1}{a^2} \;\; \frac{1}{b^2} \;\; \frac{1}{c^2} \right)} \; . \tag{64}$$

Jeans gets the same result by taking advantage of the special property of the combined equation. He obtains the divergence of M and notes that if the divergence vanishes the function is a spherical harmonic. He can find a value for θ which will make the divergence vanish. The function is now a spherical harmonic and constant over the boundary of the ellipsoid, hence it must also be constant throughout the mass of the ellipsoid. Under these circumstances he can obtain the three important equations simply by equating the coefficients of x^2, y^2 and z^2 since the function must be independent of the coordinates.

From these equations Jeans proceeds to obtain the conditions for the existence of rotating homogeneous ellipsoids. He first subtracts corresponding sides of (66) and (67) and obtains:

$$J_B - J_A = (a^2 - b^2)\, J_{AB} = \frac{\theta}{b^2} - \frac{\theta}{a^2} = (a^2 - b^2) \frac{\theta}{a^2 b^2} \; . \tag{67a}$$

Theta is then eliminated between this equation and (67) which gives us

$$(a^2 - b^2)\ [\, a^2b^2J_{AB} - c^2J_C \,] = 0 \ . \tag{68}$$

Now it will be clear that it is possible to satisfy the three fundamental equations either by taking

$$a^2 = b^2 \tag{69}$$

or

$$a^2b^2J_{AB} = c^2J_C \ . \tag{70}$$

These two cases correspond respectively to the Maclaurin ellipsoids and the Jacobi ellipsoids. The Maclaurin ellipsoids, it will be shown later on, are stable for small values of the angular velocity of rotation. All known planets are in the region of stability of the Maclaurin ellipsoids. They are oblate ellipsoids of revolution. The Jacobi ellipsoids are produced only, it turns out, when the velocity of rotation is such that a breakup is being approached. We, therefore, begin by discussing the Maclaurin ellipsoids. Clearly these include the case of the sphere for which a = b = c and the angular velocity of rotation is 0. It is important to see that we have shown that these ellipsoids are <u>equilibrium</u> figures, whether or not they are figures of stable equilibrium.

For the Maclaurin ellipsoids we can omit equation (66) which is identical to (65) and we can eliminate θ between equation (65) and equation (67) which gives at once

$$a^2J_A - c^2J_C = \frac{\omega^2 a^2}{2\pi\rho abc} \ . \tag{70a}$$

We next substitute in equation 70a for J_A and J_C and get

$$a^2 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta} - c^2 \int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta} = \frac{\omega^2 a^2}{2\pi\rho abc} \tag{70b}$$

$$\int_0^{\infty} a^2(c^2 + \lambda\ ) \ - \ c^2(a^2 \ + \lambda\ ) \ ) \ \frac{d\lambda}{(a^2 \ + \lambda)\ (c^2 \ + \lambda)\Delta} \ = \frac{\omega^2 a^2}{2\pi\rho abc}$$

which is easily transformed into

$$\frac{(a^2 - c^2)}{a^2} \int_0^{\infty} \frac{\lambda\, d\lambda}{AC\Delta} = \frac{\omega^2}{2\pi\rho abc} \ . \tag{71}$$

The integration of (71) offers some difficulties. See Thomson and Tait (1912), Vol. II, p. 71. According to Moulton, page 13, we have that X^1, the force component in the x direction, is, in Jeans's notation,

$$\frac{X^1}{x} = 2 \ oabc \ \int_{\varkappa}^{\infty} \frac{d\lambda}{A\Delta} \ .$$

Now Moulton tells us that when the lower limit of integration, which he calls $\varkappa$ is 0, then in the case of an oblate spheroid we have

$$\frac{X^1}{x} = -2\pi\rho \frac{\sqrt{1 - e^2}}{e^3} \ [\ -e\sqrt{1 \ - e^2} \ + \ \sin^{-1} e]$$

which must equal

$$- \ 2\pi\rho a^2 c \int_0^{\infty} \frac{d\lambda}{A\Delta} = - \ 2 \ oa^2 cJ_A$$

and from this it follows that

$$a^2 cJ_A = \frac{\sqrt{1 \ - \ e^2}}{e^3} \ [- \ e\sqrt{1 \ - \ e^2} + \ \sin^{-1} e] \ .$$

In the same way we can use the z coordinate data of Moulton

$$\frac{Z^1}{z} = - \ 2 \ oabc \int_{\varkappa}^{\infty} \frac{d\lambda}{C\Delta} \ .$$

For $\varkappa = 0$

$$\frac{Z^1}{z} = -\frac{4\pi\rho}{e^3}\left[e - \sqrt{1-e^2}\tan^{-1}\frac{e}{\sqrt{1-e^2}}\right]$$

so that

$$c\,a^2 J_C = \frac{2}{e^3}\left[e - \sqrt{1-e^2}\sin^{-1}e\right].$$

Combining these two we form the equation

$$\frac{J_A}{c} - \frac{cJ_C}{a^2} = \frac{1}{a^2}\left\{\frac{1}{c^2}\frac{\sqrt{1-e^2}}{e^3}\left[-e\sqrt{1-e^2} + \sin^{-1}e\right]\right.$$

$$\left. - \frac{2}{a^2}\frac{1-e^2}{e^3}\left[\frac{e}{\sqrt{1-e^2}} - \sin^{-1}e\right]\right\}$$

which reduces, after some trouble, using (70a) to the result

$$\frac{\omega^2}{2\pi\rho} = \frac{1}{e^3}(3-2e^2)(1-e^2)^{1/2}\sin^{-1}e - 3\left(\frac{1}{e^2} - 1\right) \tag{72}$$

where e is the eccentricity defined by $e^2 = (a^2 - c^2)/a^2$. From this equation it is possible to calculate values of the quantity $\frac{\omega^2}{2\pi\rho}$ as a function of e. These values are tabulated on page 39 of Jeans. The critical value is 0.81267 for e which is the value at which the Maclaurin spheroids cease to be stable and make the transition to the Jacobi ellipsoids.

A calculation of the Jacobi ellipsoids is considerably more difficult. Numerical values have been obtained for the use of elliptic integrals by Darwin. Although the Jacobi ellipsoids and the Maclaurin ellipsoids can be calculated past the point of junction the Maclaurin spheroids will be unstable if they are more oblate than this critical value. The situation with the Jacobi ellipsoids is different. They form a continuous sequence which goes from ellipsoids with a large value of a through those where $a = b$, to values with a large value of b relative to a. The Jacobi ellipsoids for which a = b coincides with one of the Maclaurin ellipsoids and represents the junction between the Maclaurin ellipsoids and the Jacobi ellipsoids as in the diagram. The series is entirely symmetrical so that those with increasing a and those with increasing b are effectually identical.

The situation which has arisen here is typical of that in the study of rotating liquid masses. A sequence of configurations, in this case the Maclaurin ellipsoids, can be traced up to its intersection with another series. Beyond this point the first series becomes unstable and the stability is transferred to the second series.

When we pursue these studies by considering a further addition of angular momentum we find that the Jacobi ellipsoid becomes elongated. When the long axis comes to be something like 1.9 x r_0 a new deformation begins. In place of the Jacobi ellipsoid we have an asymmetrical figure which is generally called the pear-shaped figure of equiliorium because one end is narrower than the other. The calculated forms of the pear-shaped figure show, however, that it is more like the shape of a tenpin, that is to say relatively long as compared with a pear.

A series of pear-shaped configurations can be calculated going to higher and higher values of the angular momentum. These configurations, however, unlike the Jacobi ellipsoids, cannot represent the actual path of evolution of a rotating liquid mass. It turns out that the pear-shaped configurations are unstable. They are unstable not only in the sense that the effects of tidal friction will gradually tend to modify the body but in the more drastic sense that as soon as the Jacobi ellipsoid has received enough angular momentum to begin the formation of the pear-shaped body then it must continue catastrophically to change in some way which it has not yet been possible to follow mathematically. Although the pear-shaped configurations do not give us the actual path over which the body moves as it breaks up yet we may be sure that the breakup begins at the point where the pear-shaped configurations begin to be possible and we can further be sure that the path of evolution is tangent to the path of the series of pear-shaped bodies at the moment when breakup begins. This can probably be interpreted as meaning that the breakup begins with the formation of a neck around one end of the body. It is reasonable to suppose that further evolution proceeds by the deepening of this constriction until one end of the body is separated. In order to validate the above chain of reasoning for actual application to the problem of the earth it is necessary first of all to show that the ellipsoidal configurations are stable not only if we introduce the constraint that only ellipsoids' configurations will be possible but also if this constraint is removed. This point has been discussed by Poincaré.

The fact that we are able with a single value of θ to satisfy these equations means that the ellipsoid is actually an equilibrium figure in the problem of a self gravitating liquid. We notice that θ is not a function of the coordinates but only of the angular velocity ω. Tracing this fortunate fact backwards we see that it is a consequence of the fact that the potential can be expressed in the very simple form shown in Equation 57 or perhaps we might equally well say that it is a consequence of the fact that the laplacian ∇^2 takes a very simple form shown in Equation 63a. Suppose for instance that the equilibrium figure had not been an exact ellipsoid but something near it. In this case, when we went to solve for θ we would not have been able to find a single numerical constant but instead some kind of a function.

Poincaré showed that there is a method of investigating the stability of a series of bodies like the Maclaurin ellipsoids which greatly diminishes the effort involved. Poincarê begins by considering the general problem of equilibrium. Stability in a static system implies that the potential energy W is a minimum for a particular configuration as compared to all adjacent configurations. In a rotating system, it can be shown that the same is true. if we add a term as in (62).* We might think of a space of many dimensions, each dimension representing one of the parameters which describe the configuration. In this space of many dimensions, we consider a set of surfaces of constant potential energy. Each of these surfaces must form a hill whose top is at the given configuration. Let us choose one of the parameters (in our case the angular momentum) and let us think of the set of surfaces W = constant which exist for a sequence of values of the angular momentum say μ_1, μ_2 and so on. In the figure we plot μ against one of these variables which describes the configuration, say θ. We draw the surface W = constant; this surface must be concave downward. The configuration which we are thinking of, if it is really an equilibrium configuration, must have the value of θ which brings us up to

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*We must note, however, that the convention is to take the potential energy W as increasing <u>outward</u> from a gravitating body, while the potential V increases <u>inward.</u> If the volume is Ω, and an element of Ω is $d\Omega$,

$$W = -\iiint_{\Omega} V\rho \, d\Omega$$

the top of the surface = constant. The value of θ which corresponds to equilibrium will be the value at the top of the bulge. The reason for this is that we are assuming that W increases as μ increases. We chose this sense for plotting μ. If we were to plot two of these variables, θ^1 and θ^2 we would see that the curves of varying W (always for a given fixed value of μ) would degenerate into ellipses near the equilibrium configuration. The lowest value of W would be at the center of these ellipses and would represent the peak of a bulge coming up from below the plane of the diagram.

Now if we consider a series of configurations of equilibrium then we are in effect considering the series of points which are at the peaks of the surfaces W = constant for varying values of μ. Let us suppose that one of these values is stable. Then we cannot reach an unstable configuration as we follow along this sequence of states unless in one of the parameters, θ, these curves become concave upwards instead of concave down. When this happens it may be true that the curves when extended outwards continue to curl up. Or it may be true that when extended outwards they turn down again after having gone a sufficient distance. In the latter case it is clear that we can trace out a new set of crests (or rather two new sets of crests) which start out at the point where the first sequence becomes unstable and spread out from it in both directions through the new set of peaks. In the opposite case, when the surfaces beyond the point of stability turn up then we shall ordinarily expect that before reaching the point of instability there existed in the surfaces W = constant dips on either side of the set of humps which formed our original linear sequence. These configurations can also be represented by a line which passes through the point of instability of our original linear sequence. The third possibility is of course the limiting case where the point of instability is represented by a flat surface extending indefinitely in all directions and corresponding to neutral equilibrium. Setting this case aside for the moment, as trivial and as included in the other cases if monor changes of wording are made, we say that a linear sequence of configurations can only pass from stable to unstable when it encounters another linear sequence. This is a topological result. It is not in any way a consequence of the special properties of rotating ellipsoids.

In our particular case the sequence of Maclaurin ellipsoids must surely be considered stable at its initial point, where we are dealing with a sphere and zero rotation. As the angular momentum of this sphere increases we will be passing along a series of stable configurations until this is interesected by another set. It has been shown, by methods which I am not giving here, that the first sequence of forms which intersects the sequence of Maclaurin spheroids is the sequence of Jacobi ellipsoids. From this it follows that the Maclaurin spheroids will be stable up to the point where they encounter the series of Jacobi ellipsoids.

We can also see that the question whether the Jacobi ellipsoids are stable or not in this sequence depends on whether the curve which represents the sequence of Jacobi ellipsoids turns up or turns down in these diagrams. That is to say it depends on whether the Jacobi ellipsoids with higher values of the angular momentum are also ellipsoids with higher values of energy or not. Numerical computations have shown that in fact the Jacobi ellipsoids with higher energy are also those or higher angular momentum so that the curve does in fact turn upwards and the Jacobi ellipsoids are stable. From this it follows that a sequence of bodies of progressively increasing angular momentum will pass through a series of Maclaurin ellipsoids and then through a series of Jacobi ellipsoids. The stability of the Jacobi ellipsoids is terminated by a set of non-ellipsoidal pear-shaped figures, which has been found to be unstable. This second intersection takes place not far beyond the point at which the Jacobi ellipsoids begin to form. As a consequence in most discussions of stability, the appearance of the Jacobi ellipsoids is taken as an index of the approaching catastrophe.

In this discussion we have spoken as if the angular momentum could increase steadily. This is, of course, unrealistic; the angular momentum is constant. It turns out, however, that the quotient of the angular momentum divided by the density is the parameter which enters this discussion. Hence we may treat problems which are rally those of increasing density as though they were problems of increasing angular momentum. The problems of increasing density, however, are exactly those which would be expected in a liquid mass which has newly condensed and is in the process of cooling. We may expect that in the early days of the earth the density increased as the heat was lost. It is against this background that the above discussions of stability become relevant. Up to this point we have been considering a mass of liquid of constant density. We have done so because this is the only case in which it is possible to follow the mathematics very well. We have chosen to make an exact treatment of a problem which is something like the real problem rather than to do the usual thing, which is to make a rough treatment of the actual problem.

In order to apply our results to the actual case of the earth itself we must consider inhomogeneous masses. Jeans attacked the problem in two ways. His first method was to consider a model which consisted of a nucleus of finite density surrounded by an atmosphere of zero density. Clearly this is the limiting case of the kind of a two fluid system which Wiechert (1897) worked with. The problem is quite tractable mathematically once the study has been made on the homogeneous sphere. It is simply a matter of defining one of the geopotential surfaces above the nucleus as the true surface. The volume enclosed between this surface and the nucleus is called the atmosphere; it is referred to as V_a; compared with V_n of the nucleus. The results which have already been derived for the behavior of the homogeneous mass can now be applied at once to this theoretical inhomogeneous planet.

In particular, Jeans found that if the ratio of the volume of the atmosphere to the volume of the nucleus exceeded about 1/3, then it would turn out that fission would not take place along the sequence of the Jacobi ellipsoids. The rapidly rotating Maclaurin spheroid would develop a fissure around its equatorial zone through which matter would be ejected. This could also be expressed by saying that the contours of the geopotential no longer close around the earth.

He finds that there are two possible sequences of configurations: for a body in which the nucleus is small and very dense compared to the rest of its structure we have equatorial ejection of matter; on the other hand, if the nucleus is sufficiently large compared to the whole mass, then the behavior is qualitatively like that of a homogeneous mass, which we have been discussing.

It is true that the model does not really resemble the earth, but let us do the best we can to fit the earth to it. The polar moment of inertia C of the earth is known to be given by:

$$\frac{C}{Ma^2} = 0.3307 .$$

If the earth were homogeneous, we would have 0.4 instead of 0.3307. Thus, the earth has approximately 5/6 as much angular momentum as a homogeneous sphere of the same size. The question is, how big a homogeneous sphere would we need in order to have the same angular momentum as the earth, assuming that the total mass were the same? The answer is that the ratio of the radii should be the square root of 5/6 or 0.91. The ratio of the volumes is then just about 3/4. Hence, if we had an object consisting of the homogeneous sphere in the interior and a weightless shell outside so arranged that the space V_a between the shells was about 1/3 the volume of the inner shell, then this composite object would have approximately the same angular momentum and approximately the same value of C/Ma^2 as the earth. Jeans shows that this configuration is just on the borderline of the cases when fission takes place by the formation of a Jacobi ellipsoid. For more homogeneous bodies, fission is sure to take place by the development of the Jacobi ellipsoid; for less homogeneous bodies, that is bodies with a similar nucleus, break-up is sure to take place by the spreading away of a portion of the atmosphere around the equator. From this treatment it appears that the earth is near the limiting case.

Jeans' second, and more realistic model, involves the assumption of a polytrophic distribution of density. Polytropic density distributions have been extensively studied in the theory of the internal constitution of the stars, largely because R. Emden (1907) made a series of numerical integrations of them. The terminology of these spheres goes back toEmden's assumption that stars are in convective equilibrium. For convective equilibrium, the ratio γ of the specific heat at constant pressure to the specific heat at constant volume is of decisive importance. Emden took as his parameter the quantity n given by the equation

$$\gamma = 1 + \frac{1}{n}$$

The relation of n to any of the physically significant parameters of the distribution can only be reached through some detailed numerical integrations; as a consequence, n is for many purposes, and in particular for this one, merely a parameter which defines the density distribution. For $n = 0$, the density is uniform. For $n = 1$, it turns out that it is represented by the function $\frac{a}{r} \sin \frac{r}{a}$. For $n = 3$, we have the kind of distributions with a strong concentration to the center which are believed to be typical of stars like the sun. For $n = 5$, the star lacks an outer boundary, and for $n = \infty$ we have the distribution which would characterize an isothermal atmosphere and would extend to infinity. Jeans has calculated the behavior of polytropic gas spheres rotating with sufficient rapidity to break up. He finds that if the polytropic index is less than about 0.8 the star will be sufficiently homogeneous so that it will break up via the formation of Jacobi ellipsoids. If, however, the polytropic index exceeds this quantity, it would break up by the formation of an equatorial ring somewhat like Saturn's ring. Recently Roberts (1963) has restudied this problem; he finds that the critical value of the polytropic index is near 1.0.

A numerical integration of the Emden table for the polytrope n = 0.5 shows that the value of C/Ma^2 will be 0.32. For the earth the same ratio is 0.33; it follows that the earth is slightly more homogeneous than the Emden polytrope n - 0.5. Here we see strong evidence that the earth would tend to break up through the formation of a Jacobi ellipsoid rather than by the equatorial ejection of matter.

The actual situation inside the earth may well be intermediate between these two extreme models. Hence the actual earth would probably break up via the Jacobi ellipsoid.

A second point on which Jeans made important numerical investigations is the question of the effect of the internal density distribution on the limiting value of the angular momentum required for break up. For the case of the homogeneous ellipsoid and the somewhat similar case of nearly homogeneous ellipsoids, Jeans has sought the value of the angular velocity ω at which the transition would take place from a Maclaurin spheroid to a Jacobi ellipsoid. He finds the following general formula

$$\frac{\omega^2}{2\pi} = 0.18712\,\bar{\rho} + 0.06827\,(\rho_0 - \sigma) +$$

$$[0.01602 + 0.07098\,(\gamma - 2)]\left(\frac{\rho_0 - \sigma}{\rho_0}\right)^2 \qquad (499)$$

which is applicable really only to relatively small deviations from a homogeneous mass. In (499), $\bar{\rho}$ is the mean density, ρ_0 is the density at the center of the earth, and σ is the density at the boundary.

When this series is applied to the earth, we find that the critical period of rotation is $1^h\,58^m$. For a homogeneous body of the earth's mass, it is $2^h\,40^m$; and if a homogeneous body rotating at this speed is transformed, without change of angular momentum, into an inhomogeneous body for which

$$\frac{C}{Ma^2} = 0.33$$

the period of rotation is $2^h\,11^m$. It would seem to follow that the earth could not have broken up as a result of the formation of the core since it would still be rotating too slowly.

The result is, however, very doubtful, as Jeans would have been the first to say; the series does not converge well, and in fact the last term is larger than the one which precedes it, in the case of the earth. Jeans applied the series only to the case in which γ is near 2, which improves the convergence.

I have made some calculations based on later work by Roberts, which suggest that in fact the critical period for the earth is near $2^h\ 18^m$, so that the earth can in fact be destabilized by the formation of the core.

References

Emden, R., Gaskugeln, B.G. Teubner, Leipzig and Berlin, 1907.

Jeans, J. H., Problems of Cosmogony and Stellar Dynamics, Cambridge University Press, 1919.

Moulton, F. R., An Introduction to Celestial Mechanics, Macmillan, New York, 1916.

Roberts, P. H., On highly rotating polytropes, I, Astrophysical Journal 137, 1129-1141, 1963.

Thomson, W., and P. G. Tait, Principles of Mechanics and Dynamics, reprinted by Dover Press, New York, 1962, from Treatise on Natural Philosophy, Cambridge University Press, 1912.

Weichert, E., Ueber die Massenvertheilung im innern der Erde, Göttingen, Gesellschaft der Wissenschaften, Nachrichten, volume for 1897, 221-243, 1897.

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Motion in the Vicinity of the Triangular Libration Centers

I. **Introduction.** This paper concerns the restricted problem of three bodies: Two masses (P_1 and P_2) move in circular orbits under their mutual gravitation, while a third body (P, called the planetoid) of negligible mass (that is, it is acted upon by P_1 and P_2, but does not perturb their motion) moves in the same plane. We shall assume that the mass P_1 is greater than the mass P_2. We are interested in periodic librations of the planetoid around the triangular equilibrium point.

Periodic solutions of this type have been studied so extensively that we cannot attempt a thorough citation of the literature. However, the common interest was mainly in the first order analysis of these periodic solutions. Some authors, like E. W. Brown, H. R. Willard, and P. Pedersen, studied them up to the third order.

This is the first of a series of studies that will finally enable us to compute a very close numerical approximation of the short and long period orbits, both in the Sun-Jupiter and in the Earth-Moon systems, so that we shall be able, at reasonable expense, to extend these families by numerical integration. It is hoped to check E. W. Brown's conjecture that they go through orbits which are doubly asymptotic to the straight line equilibrium points L_3, L_2 and L_1 successively. For the long period orbits in the Sun-Jupiter system,

E. Rabe already extended the family through the orbit doubly asymptotic to L_3; but the Steffensen's algorithm he uses in order to integrate the differential equations proved to be so slow and heavy that he could not push further along. On the other hand, while a fourth order Runge-Kutta method is quite adequate for the purpose, the orbits depend so sensitively on the initial conditions that it is necessary to start the integrations with an approximation closer than the one given only by a variation orbit. We hope that our Fourier series will grant us these "good" initial values.

Meanwhile, the completely canonical transformation which we introduce at the first order suggests that we use the adelphic integral satisfied by the solutions in the vicinity of the triangular equilibrium point. Work along that line is now in progress; we already hold a *second* order approximation of the general librations in terms of four canonical variables, two action momenta and two angle coordinates.

Here we present a treatment of the first order general librations around the triangular equilibrium, and the Fourier series computed up to the third order for long and short period librations, in the Sun-Jupiter system as well as in the Earth-Moon system.

II. **The plane restricted problem of three bodies.** We shall begin with fixing the units of length, mass, and time. As the unit of length we choose the distance between the two finite masses, as the unit of mass the sum of the two finite masses, and, finally, the unit of time is determined by putting the angular velocity of the two finite masses equal to 1. With this choice of units, the gravitational constant reduces to 1.

We shall call the values of the two finite masses μ and $1 - \mu$, assuming $\mu \leqq 1/2$. The motion of the three particles (P_1, P_2 and P) will be referred to a rotating coordinate system, the so-called barycentric synodical system. Its origin is the center of mass of the two finite masses, the x-axis passes through the finite masses, the positive direction being that from the origin to P_2. The positive direction of revolution is chosen as the direction of the absolute motion of the two finite masses. In the barycentric synodical system, the coordinates of P_1 and P_2 are $(-\mu, 0)$ and $(1-\mu, 0)$.

The plane restricted problem of three bodies is described by the Lagrangian function

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - \dot{x}y) + \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{\rho_1} \quad \frac{\mu}{\rho_2},$$

to which are associated the canonical moments

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} - y, \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y} + x$$

and the Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2) - (xp_y - yp_x) - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}. \tag{1}$$

Here ρ_1 and ρ_2 are the distances of P to P_1 and P_2; they are thus defined by the relations

$$\rho_1^2 = (x + \mu)^2 + y^2, \quad \rho_2^2 = (x - 1 + \mu)^2 + y^2.$$

Since (1) is conservative, the canonical equations admit the integral

$$H = -\frac{1}{2}C$$

where the integration constant C is called the Jacobi constant.

III. **Motion in the neighborhood of the point** L_4. For reasons of convenience we put

$$\gamma = 1 - 2\mu;$$

in the phase space, the translation

$$x = \frac{1}{2}\gamma + X,$$

$$y = \frac{1}{2}\sqrt{3} + Y,$$

$$p_x = -\frac{1}{2}\sqrt{3} + p_X,$$

$$p_y = \frac{1}{2}\gamma + p_Y$$

is a conservative completely canonical homeomorphism; in the configuration space, that is to say in the (x, y)-plane, it translates the coordinate origin from the barycenter G to the point whose coordinates are $(1/2\gamma, 1/2\sqrt{3})$. This point is commonly denoted L_4. In the new coordinate system L_4XY, the Hamiltonian function (1) is transformed into the Hamiltonian function

$$H = \frac{1}{2}(p_X^2 + p_Y^2) - (Xp_Y - Xp_X) - \Omega \tag{2}$$

where the force function Ω is defined by the relation

$$2\Omega = \gamma X + \sqrt{3}\,Y + \frac{1+\gamma}{\rho_1} + \frac{1-\gamma}{\rho_2}. \tag{3}$$

In (2), the terms $-(3/8) - (1/8)\gamma^2$ were neglected, i.e., the Jacobi constant C is replaced by a modified Jacobi constant C' such that

$$C = \frac{3}{4} + \frac{1}{4}\gamma^2 + C'.$$

To analyze the motion of P in the neighborhood of L_4, it is required to expand (3) in a power series of X and Y. This is done by introducing the complex coordinates

$$R = X + iY, \qquad S = X - iY$$

so that

$$X = \frac{1}{2}(R + S), \qquad Y = -\frac{i}{2}(R - S).$$

At the same time, the complex number

$$q = \exp\left(i\frac{\pi}{6}\right)$$

is brought in so that the distance functions

$$\rho_1^2 = \left(\frac{1}{2} + X\right)^2 + \left(\frac{1}{2}\sqrt{3} + Y\right)^2,$$

$$\rho_2^2 = \left(\frac{1}{2} - X\right)^2 + \left(\frac{1}{2}\sqrt{3} + Y\right)^2$$

are given the symmetric form

$$\rho_1^2 = (q^2 + R)(q^{-2} + S), \tag{4a}$$

$$\rho_2^2 = (q^{-2} - R)(q^2 - S). \tag{4b}$$

To produce the power series expansion of (3), it is therefore enough to consider the binomial law

$$(a+z)^{-1/2} = \sum_{m=0}^{\infty} \left(-\frac{1}{2}\right)^m \frac{m!!}{m!} a^{-(2m+1)/2} z^m$$

where $m!$ represents the product of the m first natural integers, $m!!$ represents the product of the m first odd natural integers, $0! = 1$ and $0!! = 1$.

Applying this binomial law to (4a) and (4b), we obtain

$$\frac{1}{\rho_1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{m+n} \frac{m!!\, n!!}{m!\, n!} q^{-2(m-n)} R^m S^n,$$

$$\frac{1}{\rho_2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{m+n} \frac{m!!\, n!!}{m!\, n!} q^{2(m-n)} R^m S^n,$$

and we conclude that

$$\frac{1+\gamma}{\rho_1} + \frac{1-\gamma}{\rho_2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Q_{m,n} R^m S^n \tag{5}$$

where, for all $m \geqq 0$ and all $n \geqq 0$,

$$Q_{m,n} = \left(\frac{1}{2}\right)^{m+n} \frac{m!!\, n!!}{m!\, n!} \left[(1-\gamma)\, q^{2(m-n)} + (-1)^{m-n}(1+\gamma)\, q^{-2(m-n)}\right].$$

We neglect the constant term $Q_{0,0} = 2$, i.e., we define a new Jacobi constant Γ so that

$$C = \frac{11}{4} + \frac{1}{4}\gamma^2 + \Gamma.$$

For the terms of first degree, we notice on one side that

$$Q_{1,0} = -\frac{1}{2}\left[(1+\gamma)\, q^{-2} - (1-\gamma)\, q^2\right], \tag{6a}$$

$$Q_{0,1} = -\frac{1}{2}\left[(1+\gamma)\, q^{2} - (1-\gamma)\, q^{-2}\right] \tag{6b}$$

while, on the other side, we compute that

$$\gamma X + \sqrt{3}\, Y = \frac{1}{2}\,[(1+\gamma)\,q^{-2} - (1-\gamma)\,q^{2}]R$$
$$+ \frac{1}{2}\,[(1+\gamma)\,q^{2} - (1-\gamma)\,q^{-2}]S.$$

Consequently, the function 2Ω defined by (3) contains no term of the first degree in R and S. It means that, as it is well known, L_4 is an equilibrium point. For this equilibrium, whatever the mass ratio μ may be, the Jacobi constant Γ is equal to 0.

In view of the coefficients (6a) and (6b), we notice that

$$[(1+\gamma)\,q^{2} - (1-\gamma)\,q^{-2}][(1+\gamma)\,q^{-2} - (1-\gamma)\,q^{2}] = 3 + \gamma^{2}.$$

This identity in μ leads to defining a function $\zeta > 0$ of the mass ratio μ by means of the relation

$$\zeta^{2} = 3 + \gamma^{2}$$

and an angle α by means of the relations

$$\phi = \exp(i\alpha) = \frac{1}{\zeta}\,[(1+\gamma)\,q^{2} - (1-\gamma)\,q^{-2}],$$

$$\phi^{-1} = \exp(-\,i\alpha) = \frac{1}{\zeta}\,[(1+\gamma)\,q^{-2} - (1-\gamma)\,q^{2}].$$

In real terms

$$\cos\alpha = \frac{\gamma}{\zeta}, \qquad \sin\alpha = \frac{\sqrt{3}}{\zeta}.$$

Now for the coefficients of second degree in (5), i.e.,

$$Q_{2,0} = -\,\frac{3}{8}\,[(1+\gamma)\,q^{2} + (1-\gamma)\,q^{-2}],$$

$$Q_{1,1} = \frac{1}{2},$$

$$Q_{0,2} = -\,\frac{3}{8}\,[(1+\gamma)\,q^{-2} + (1-\gamma)\,q^{2}],$$

we observe that

$$[(1+\gamma)\,q^{2} + (1-\gamma)\,q^{-2}][(1+\gamma)\,q^{-2} + (1-\gamma)\,q^{-2}] = 1 + 3\gamma^{2}.$$

Consequently, we define a function $\delta > 0$ of the mass ratio μ by means of the relation

$$\delta^2 = 1 + 3\gamma^2$$

and an angle β by means of the relations

$$\theta^2 = \exp(2i\beta) = \frac{1}{\delta}\left[(1+\gamma)\,q^2 + (1-\gamma)\,q^{-2}\right],$$

$$\theta^{-2} = \exp(-2i\beta) = \frac{1}{\delta}\left[(1+\gamma)\,q^{-2} + (1-\gamma)\,q^2\right].$$

In real terms

$$\cos 2\beta = \frac{1}{\delta}, \qquad \sin 2\beta = \frac{\gamma}{\delta}\sqrt{3}.$$

Define now the rotation

$$X = \xi\cos\beta + \eta\sin\beta,$$

$$Y = -\xi\sin\beta + \eta\cos\beta$$

which turns the (X, Y) coordinate system at L_4 about an angle equal to $-\beta$. As it will be seen later, the new (ξ,η)-coordinate system is made of the principal axis of the first order periodic librations around L_4.

The orthogonal transformation in the configuration space is extended to a conservative completely canonical homeomorphism in the phase space if it is multiplied by the following orthogonal mapping in the moment space:

$$p_X = p_\xi\cos\beta + p_\eta\sin\beta,$$

$$p_Y = -p_\xi\sin\beta + p_\eta\cos\beta.$$

In the complex coordinates, the rotation results in substituting for the complex coordinates R and S the new complex coordinates

$$r = \xi + i\eta, \qquad s = \xi - i\eta$$

such that

$$r = \theta R, \qquad s = \theta^{-1}S.$$

Consequently, the force function Ω becomes the power series

$$\Omega = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Omega_{m,n}\,r^m s^n$$

where

$$\Omega_{0,0} = \Omega_{1,0} = \Omega_{0,1} = 0$$

and, for all m and all n such that $m \geqq 0$, $n \geqq 0$, and $m + n \geqq 2$,

$$\Omega_{m,n} = \frac{1}{2^{m+n+1}} \frac{m!!\,n!!}{m!\,n!} \left[(1-\gamma)\, q^{2(m-n)} \right.$$

$$\left. + (-1)^{m-n}(1-\gamma)\, q^{-2(m-n)}\right]\theta^{m-n}.$$

We list here these complex coefficients up to the fourth degree(1)

$$2^4\Omega_{2,0} = 2^4\Omega'_{0,2} = -3\delta,$$

$$2^4\Omega_{1,1} = 4;$$

$$2^5\Omega_{3,0} = 2^5\Omega'_{0,3} = 10\gamma\theta^{-3},$$

$$2^5\Omega_{2,1} = 2^5\Omega'_{1,2} = -3\zeta\phi^{-1}\theta^{-1};$$

$$2^8\Omega_{4,0} = 2^8\Omega'_{0,4} = -35\delta\theta^{-6},$$

$$2^8\Omega_{3,1} = 2^8\Omega'_{1,3} = -20\delta,$$

$$2^8\Omega_{2,2} = 36.$$

Having obtained the expansion of the force function Ω as a power series of r and s, it is a matter of easy, but somewhat tedious, algebra to express it as a power series of the Cartesian coordinates ξ and η:

$$\Omega = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \omega_{p,q}\xi^p\eta^q.$$

Here is the list of its coefficients up to the fourth degree

$$\omega_{0,0} = \omega_{1,0} = \omega_{1,0} = 0;$$

$$8\omega_{2,0} = 2 - 3\delta,$$

$$\omega_{1,1} = 0,$$

$$8\omega_{0,2} = 2 + 3\delta;$$

$$16\omega_{3,0} = 10\gamma\cos 3\beta - 3\zeta\cos(\alpha+\beta),$$

$$16\omega_{2,1} = 30\gamma\sin 3\beta - 3\zeta\sin(\alpha+\beta),$$

$$16\omega_{1,2} = -30\gamma\cos 3\beta - 3\zeta\cos(\alpha+\beta),$$

(1) In the subsequent formulae, A' denotes the conjugate of the complex number A.

$$16\omega_{0,3} = -10\gamma\sin 3\beta - 3\zeta\sin(\alpha+\beta);$$

$$128\omega_{4,0} = 18 - (20 + 35\cos 6\beta)\,\delta,$$

$$128\omega_{3,1} = -140\delta\sin 6\beta,$$

$$128\omega_{2,2} = 36 + 210\delta\cos 6\beta,$$

$$128\omega_{1,3} = 140\delta\sin 6\beta,$$

$$128\omega_{0,4} = 18 + (20 - 35\cos 6\beta)\,\delta.$$

IV. **First order librations around L_4.** Restricted to the terms of second degree in ξ and η, the Hamiltonian function describing the librations around L_4 is

$$H_2 = \frac{1}{2}\,(p_\xi^2 + p_\eta^2) - (\xi p_\eta - \eta p_\xi) - \frac{1}{4}\left(1 - \frac{3}{2}\,\delta\right)\xi^2 - \frac{1}{4}\left(1 + \frac{3}{2}\,\delta\right)\eta^2$$

and the canonical equations derived from it

$$\dot{\xi} = \frac{\partial H_2}{\partial p_\xi} = p_\xi + \eta, \qquad \dot{p}_\xi = -\frac{\partial H_2}{\partial \xi} = p_\eta + \frac{1}{2}\left(1 - \frac{3}{2}\,\delta\right)\xi,$$

$$\dot{\eta} = \frac{\partial H_2}{\partial p_\eta} = p_\eta - \xi, \qquad \dot{p}_\eta = -\frac{\partial H_2}{\partial \eta} = -p_\xi + \frac{1}{2}\left(1 + \frac{3}{2}\,\delta\right)\eta,$$

form a homogeneous system of four linear differential equations of the first order with constant coefficients.

We define the vector

$$\zeta = (\xi, \eta, p_\xi, p_\eta)$$

and the matrix

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \frac{1}{2}\left(1 - \frac{3}{2}\,\delta\right) & 0 & 0 & 1 \\ 0 & \frac{1}{2}\left(1 + \frac{3}{2}\,\delta\right) & -1 & 0 \end{pmatrix}$$

so that we can write the canonical equations in the simpler form

$$\dot{\zeta} = \mathscr{A}\,\zeta.$$

As is well known, to study the nature of their solutions, it is enough to analyze the intrinsic properties of matrix $\mathscr{A}$.

In order to obtain the proper values of matrix $\mathscr{A}$, i.e., the roots of the characteristic equation

$$\det(s\mathscr{E} - \mathscr{A}) = 0 \tag{6}$$

where $\mathscr{E}$ denotes the four dimensional unit matrix, we introduce the matrix

$$\mathscr{R} = \begin{bmatrix} s & -1 & 1 & 0 \\ 1 & s & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Since $\det \mathscr{R} = +1$, matrix $\mathscr{R}$ is inversible and

$$\mathscr{R}^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -s & 1 \\ 0 & 1 & -1 & -s \end{bmatrix}.$$

Then we compute the matrix product

$$\mathscr{B} = \mathscr{R}\mathscr{A}\mathscr{R}^{-1} = \begin{pmatrix} 0 & 0 & s^2 - \frac{3}{2}\left(1 - \frac{1}{2}\delta\right) & 2s \\ 0 & 0 & 2s & s^2 - \frac{3}{2}\left(1 + \frac{1}{2}\delta\right) \\ -1 & 0 & 2s & -2 \\ 0 & -1 & 2 & 2s \end{pmatrix}.$$

The polynomial equation

$$\det(s\mathscr{E} - \mathscr{B}) = 0$$

is, of course, equivalent to (6), but it is simpler to solve. Indeed, it is at once computed that

$$\det(s\mathscr{E} - \mathscr{B}) = s^4 + s^2 + \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right) = 0. \tag{7}$$

The four proper values of $\mathscr{A}$ are distinct if and only if any one of the following four conditions is satisfied:

$$\delta > \delta_1 = \frac{4}{3}\sqrt{2} \qquad = 1.885\ 618\ 085 \cdots,$$

$$\gamma > \gamma_1 = \frac{1}{9}\sqrt{69} \qquad = 0.922\ 958\ 208\cdots,$$

$$\mu < \mu_1 = \frac{1}{2}\left(1 - \frac{1}{9}\sqrt{69}\right) = 0.038\ 520\ 896\cdots,$$

$$\mu(1-\mu) < \frac{1}{27} \qquad = 0.037\ 037\ 037\cdots.$$

We propose to call any one of these special values a *critical value* at the first order. Under any of the above assumptions, the four proper values of $\mathscr{A}$ are purely imaginary, thus of the form $\pm\, in_s$ and $\pm\, in_l$, where n_s and n_l are strictly positive real numbers defined by the relations

$$n_s^2 = \frac{1}{2} + \frac{1}{4}\sqrt{(9\delta^2 - 32)},$$

$$n_l^2 = \frac{1}{2} - \frac{1}{4}\sqrt{(9\delta^2 - 32)}.$$

In view of expressing in a simple way the proper vectors of $\mathscr{A}$ corresponding to these four proper values, let us define the following functions of the mass ratio

$$A_s = \sqrt{\left(n_s^2 + \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\right)}, \qquad A_l = \sqrt{\left(n_l^2 + \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\right)},$$

$$B_s = \sqrt{\left(n_s^2 + \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\right)}, \qquad B_l = \sqrt{\left(n_l^2 + \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\right)}.$$

The form of the characteristic equation (7) clearly implies that

$$(8) \qquad A_s B_s = 2n_s, \qquad A_l B_l = 2n_l.$$

With the help of these two relations, it is easily seen that in the following table, each column represents a proper vector corresponding to the proper value written as a heading

	in_s	in_l	$-\, in_s$	$-\, in_l$
ξ	A_s	A_l	A_s	A_l
η	iB_s	iB_l	$-\, iB_s$	$-\, iB_l$
p_ξ	$i(n_s A_s - B_s)$	$i(n_l A_l - B_l)$	$-\, i(n_s A_s - B_s)$	$-\, i(n_l A_l - B_l)$
p_η	$A_s - n_s B_s$	$A_l - n_l B_l$	$A_s - n_s B_s$	$A_l - n_l B_l$

.

We denote by $\mathscr{C}$ this four-dimensional *complex* matrix; from linear algebra, we know that it is a regular matrix.

Before establishing the most essential property of $\mathscr{C}$, we should check the two relations

$$A_s(n_l A_l - B_l) = B_l(A_s - n_s B_s), \tag{9a}$$

$$A_l(n_s A_s - B_s) = B_s(A_l - n_l B_l). \tag{9b}$$

To this effect we compute

$$\begin{aligned}\Phi &= A_s B_l [A_s(n_l A_l - B_l) - B_l(A_s - n_s B_s)] \\ &= n_l A_s^2 A_l B_l + n_s A_s B_s B_l^2 - 2A_s^2 B_l^2,\end{aligned}$$

and we replace the quantities A_s^2, B_l^2 by their definition so that

$$\Phi = 2\left(n_s^2 n_l^2 - \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right)\right).$$

But, because n_s^2 and n_l^2 are the roots of the quadratic equation

$$m^2 - m + \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right) = 0,$$

we have that

$$n_s^2 n_l^2 = \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right),$$

this shows that $\Phi = 0$ and consequently it proves (9a). Then (9b) is proved from (9a) by permuting the indices s and l.

At this stage, we consider the symplectic matrix

$$\mathscr{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

A straightforward matricial computation gives the important relation([2])

$${}^t\mathscr{C}\,\mathscr{J}\mathscr{C} = +2i(n_s^2 - n_l^2)\begin{pmatrix} 0 & 0 & -n_s & 0 \\ 0 & 0 & 0 & n_l \\ n_s & 0 & 0 & 0 \\ 0 & -n_l & 0 & 0 \end{pmatrix}. \tag{10}$$

([2]) For a given square matrix $\mathscr{A}$, its transpose is denoted by ${}^t\mathscr{A}$.

As it is apparent from (10), several simple modifications will easily transform matrix $\mathscr{C}$ into a symplectic matrix $\mathscr{D}$. To this effect we define the strictly positive numbers ρ_s and ρ_l so that

$$\rho_s^2 = \frac{1}{2n_s(n_s^2 - n_l^2)}, \qquad \rho_l^2 = \frac{1}{2n_l(n_s^2 - n_l^2)},$$

and the complex matrix $\mathscr{R}$

$$\mathscr{R} = \begin{pmatrix} i\rho_s & 0 & 0 & 0 \\ 0 & -i\rho_l & 0 & 0 \\ 0 & 0 & \rho_s & 0 \\ 0 & 0 & 0 & \rho_l \end{pmatrix}.$$

Since det $\mathscr{R} = \rho_s^2\rho_l^2$, the matrix $\mathscr{R}$ is regular. Hence the matrix product

$$\mathscr{D} = \mathscr{C}\mathscr{R}$$

is also a regular complex matrix. But it is readily seen that

$$\mathscr{D} = \begin{pmatrix} ia_s & -ia_l & a_s & a_l \\ -b_s & b_l & -ib_s & -ib_l \\ -(n_sa_s - b_s) & n_la_l - b_l & -i(n_sa_s - b_s) & -i(n_la_l - b_l) \\ i(a_s - n_sb_s) & -i(a_l - n_lb_l) & a_s - n_sb_s & a_l - n_lb_l \end{pmatrix}$$

where the coefficients are defined by the relations

$$a_s = \rho_s A_s, \qquad b_s = \rho_s B_s,$$
$$a_l = \rho_l A_l, \qquad b_l = \rho_l B_l.$$

Now we are ready to check that

$${}^t\mathscr{D}\mathscr{J}\mathscr{D} = \mathscr{J},$$

and this means that $\mathscr{D}$ is symplectic. Consequently, the linear mapping

$$\zeta = \mathscr{D}w.$$

of the phase space $\zeta = (\xi, \eta, p_\xi, p_\eta)$ onto the *complex* phase space $w = (u_s, u_l, v_s, v_l)$ is a completely canonical mapping. The fact that ζ is real implies the reality conditions

$$v_s = -iu_s', \qquad v_l = iu_l'.$$

In order to transform H_2, we proceed in this way. First we observe that, by construction,

$$\mathscr{A}\mathscr{D} = \mathscr{D}\mathscr{N}$$

where $\mathscr{N}$ is a diagonal matrix whose nonzero coefficients are the proper values of $\mathscr{A}$ in this order: $(in_s, in_l, -in_s, -in_l)$. Then, we see that

$$^t\mathscr{D}\mathscr{J}\mathscr{A}\mathscr{D} = {}^t\mathscr{D}\mathscr{J}\mathscr{D}\mathscr{N} = \mathscr{J}\mathscr{N},$$

and we end up with

$$H_2 = +\frac{1}{2}(\zeta|\,\mathscr{J}\mathscr{A}\zeta) = +\frac{1}{2}(w|{}^t\mathscr{D}\mathscr{J}\mathscr{A}\mathscr{D}w) = +\frac{1}{2}(w|\,\mathscr{J}\mathscr{N}w),$$

which means that the Hamiltonian function H_2 is reduced to its complex "normal" form

$$H_2 = +in_s u_s v_s + in_l u_l v_l.$$

With E. T. Whittaker, we now consider the completely canonical mappings

$$\begin{aligned} u_s &= \sqrt{I_s}\, e^{\iota\phi_s}, & u_l &= \sqrt{I_l}\, e^{-\iota\phi_l}, \\ v_s &= -i\sqrt{I_s}\, e^{-\iota\phi_s}, & v_l &= i\sqrt{I_l}\, e^{\iota\phi_l}. \end{aligned}$$

In view of the reality conditions as stated above, the coordinates ϕ_s and ϕ_l, as well as the moments I_s and I_l are real. Whittaker's canonical mappings reduce H_2 to its real "normal" form

$$H_2 = +n_s I_s - n_l I_l.$$

We can summarize our first order study as follows: When $\mu(1-\mu) < 1/27$, the laws of motion around L_4 take the elementary form

$$\begin{aligned} \phi_s &= +n_s t + \epsilon_s, & I_s &= \text{const}, \\ \phi_l &= -n_l t + \epsilon_l, & I_l &= \text{const}. \end{aligned}$$

In Cartesian coordinates $(\xi, \eta, p_\xi, p_\eta)$, they are expressed by the formulae

$$\begin{aligned} \xi &= -2a_s I_s^{1/2}\sin\phi_s - 2a_l I_l^{1/2}\sin\phi_l, \\ \eta &= -2b_s I_s^{1/2}\cos\phi_s + 2b_l I_l^{1/2}\cos\phi_l, \\ p_\xi &= -2(n_s a_s - b_s)\, I_s^{1/2}\cos\phi_s + 2(n_l a_l - b_l)\, I_l^{1/2}\cos\phi_l, \\ p_\eta &= -2(a_s - n_s b_s)\, I_s^{1/2}\sin\phi_s - 2(a_l - n_l b_l)\, I_l^{1/2}\sin\phi_l. \end{aligned}$$

To $I_l = 0$ correspond the short period librations around L_4; to $I_s = 0$ correspond the long period librations.

V. **Critical mass ratios of order** $k \geqq 2$. Since $n_s > n_l$ for any $\mu < \mu_1$, on one hand there exists no integer k such that $n_l = kn_s$, but on the other hand there might exist an integer k such that $n_s = kn_l$.

In view of the fact that

$$n_s^2 + n_l^2 = 1 \quad \text{and} \quad n_s^2 n_l^2 = \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right),$$

the commensurability relation $n_s = kn_l$ turns out to be verified for a mass ratio such that

$$\frac{k^2}{(k^2+1)^2} = \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right).$$

This equation is explicitly solved with respect to μ to give a denumerable sequence of mass ratios

$$\mu_k = \frac{1}{2}\left(1 - \sqrt{\left(1 - \frac{16k^2}{27(k^2+1)^2}\right)}\right) \quad (k \geqq 1).$$

The notation is exempt from ambiguity because, for $k = 1$, the right-hand member restores the singular value of μ which has been denoted already by μ_1.

We propose to call μ_k the *critical mass ratio of order k.*

Indeed, according to a well-known theorem (Siegel, 1956), for any $\mu < \mu_1$, the restricted three-body problem admits a family of real periodic librations around L_4 that have the following properties:

(a) they can be expressed as series of powers of a real parameter ϵ;

(b) the periodic libration corresponding to $\epsilon = 0$ is the equilibrium configuration;

(c) the period $T_s(\epsilon)$ of these periodic librations is a series of powers of ϵ such that $T_s(0) = 2\pi/n_s = T_s$.

In other words, for any $\mu < \mu_1$, the infinitesimal short period librations around L_4 belong to a family of periodic librations around L_4 which, in Strömgren's terminology, terminates naturally at L_4.

But the same cannot be said for sure concerning the infinitesimal long period librations around L_4 unless there exists no commensurability of the type $n_s = kn_l$, that is to say, unless μ is not a critical mass ratio.

Actually, it will be seen later how this commensurability can be described as a *resonance* between the long period librations and the short period ones. Often this resonance has been overlooked, although it may account for peculiarities met by E. K. Rabe (1961 and 1962) and L. J. Wolaver (1963) in their computations of the long period librations around L_4.

For instance, Table I gives the first twenty critical mass ratios, and indicates that the Sun-Jupiter mass ratio stands between μ_{12} and μ_{13}. Therefore, a resonance that would amplify the coefficients of the twelfth and thirteenth harmonics in the coordinate series is to be expected. As long as the value of the orbital parameter that characterizes a periodic libration is kept small enough, coefficients of these two sensitive harmonics do not appear in the numerical computations; for too large values of the orbital parameter, the slow convergence of the Fourier series interferes with that resonance and usually damps it out. From the figures produced by E. K. Rabe (1962), we conjecture that $d_0 = 1.025$ is about the right value to be given to the orbital parameter d_0 used by this author so that the near resonance of the 13th order shows itself at its best. On his side, Rabe hints at a commensurability between the synodical period of the libration and the sidereal period of Jupiter around the Sun. However, a mathematical analysis of the orbits along Siegel's method does not show up at any stage a resonance of that type. Rabe's erroneous conjecture is based on a fortuitous approximation; as may be read from the table in the appendix, the first order long period for the Sun-Jupiter case is almost equal to 26π in our canonical units, that is to say, 13 times the sidereal period of Jupiter.

As another instance of the pervading influence of this resonance between long and short period librations, we may cite the difference in shape between long period orbits for the Earth-Moon and the Sun-Jupiter systems. For sufficiently small values of the orbital parameter, the orbits in the Earth-Moon case (E. K. Rabe and A. Schanzle, 1962, L. J. Wolaver, 1963) present the same form as the corresponding ones in the Sun-Jupiter case. When the orbital parameter increases, loops begin to occur. This dissemblance is caused by the near resonance of the short period on the third harmonic, since the Earth-Moon ratio is found between the critical mass ratios μ_3 and μ_4 and nearer to μ_3 than to μ_4.

TABLE I. Critical mass ratios

$\mu_1 = 0.038\ 520\ 896$	$\mu_{11} = 0.001\ 205\ 830$
$\mu_2 = 0.024\ 293\ 897$	$\mu_{12} = 0.001\ 015\ 697$
$\mu_3 = 0.013\ 516\ 016$	$\mu_{13} = 0.000\ 867\ 085$
$\mu_4 = 0.008\ 270\ 373$	$\mu_{14} = 0.000\ 748\ 764$
$\mu_5 = 0.005\ 509\ 203$	$\mu_{15} = 0.000\ 653\ 049$
$\mu_6 = 0.003\ 911\ 084$	$\mu_{16} = 0.000\ 574\ 539$
$\mu_7 = 0.002\ 912\ 185$	$\mu_{17} = 0.000\ 509\ 354$
$\mu_8 = 0.002\ 249\ 197$	$\mu_{18} = 0.000\ 454\ 645$
$\mu_9 = 0.001\ 787\ 848$	$\mu_{19} = 0.000\ 408\ 285$
$\mu_{10} = 0.001\ 454\ 406$	$\mu_{20} = 0.000\ 368\ 661$

VI. **Dalembert characteristic of the periodic librations.** Under the restrictions explained in the last paragraph, C. L. Siegel proves the existence of families of periodic librations around an equilibrium configuration by first building formally the solutions and then showing that they have existential meaning.

According to C. L. Siegel's algorithm, the variables u_s, u_l, v_s, v_l are to be expanded in power series of two complex valued functions ρ and σ of time t. At the same time, a complex valued function n of ρ and σ is built as a power series of the product $\rho\sigma$, while the functions ρ and σ are to be determined as solutions of the differential system

$$(11) \qquad \dot\rho = n\rho, \qquad \dot\sigma = -n\sigma.$$

In the case of short period librations, the power series u_s, u_l, v_s, v_l are of the form

$$u_s = \rho + U_s(\rho, \sigma), \qquad v_s = \sigma + V_s(\rho, \sigma),$$
$$u_l = U_l(\rho, \sigma), \qquad v_l = V_l(\rho, \sigma)$$

where U_s, U_l, V_s, and V_l all begin with quadratic terms in ρ and σ; moreover U_s should not contain terms of the form $\rho(\rho\sigma)^l$ and V_s should not contain terms of the form $\sigma(\rho\sigma)^l$.

Consequently, the differential equations (11) imply that the product $\rho\sigma$ is a constant and that, in its turn, implies that n is a constant. Therefore

$$\rho = \rho_0 e^{nt} \qquad \sigma = \sigma_0 e^{-nt}.$$

Now the reality conditions impose

$$\rho_0' = -i\sigma_0.$$

The Hamiltonian being conservative, the origin of time can be chosen in such a way that ρ_0 is real and positive, let us say $\rho_0 = \epsilon$. Hence, the complex valued "normal" coordinates (u_s, u_l, v_s, v_l) appear as *complex* Fourier series with multiples of nt as arguments and power series of ϵ as coefficients. The lowest power of ϵ occurring in the coefficient of $\exp(knt)$ equals the multiple k of nt in its argument; the power series from there onward progresses in powers of ϵ^2. In other words, the normal variables come out as complex Fourier series with the Dalembert characteristic (Brouwer and Clemence, 1961).

Going back to the real Cartesian coordinates ξ and η by means of the formulae

$$\xi = a_s(iu_s + v_s) + a_l(-iu_l + v_l),$$
$$\eta = -b_s(u_s + iv_s) + b_l(u_l - iv_l),$$

we thus find that ξ and η expand as real Fourier series with multiples of nt as arguments and power series of ϵ as coefficients and that both series exhibit the Dalembert characteristic.

Obviously all that has just been said about the family of short period librations can be applied with obvious modifications to the family of long period librations, provided μ is not one of the critical mass ratios.

Our purpose is to build a scheme that yields numerically the Fourier series ξ and η for both families of librations around L_4. As we shall become aware soon after the second order in ϵ, computations become quite cumbersome unless convenient notations are proposed at the very start. The following ones have proved to be quite adequate:

(a) The "mean motion" n will denote the series

$$n = \sum_{k=0}^{\infty} n_k \epsilon^{2k};$$

(b) The Jacobi constant Γ will denote the series

$$\Gamma = \sum_{k=0}^{\infty} \Gamma_k \epsilon^{2k};$$

(c) $C_{p,q,k,l}$ (resp. $S_{p,q,k,l}$) will denote the coefficient of ϵ^{k+2l} in the power series that is the coefficient of $\cos knt$ (resp. of $\sin knt$) in the product $\xi^p \eta^q$;

(d) $\dot{C}_{1,0,k,l}$ (resp. $\dot{S}_{1,0,k,l}$) will denote the coefficient of ϵ^{k+2l} in the power series which is the coefficient of $\cos knt$ (resp. of $\sin knt$) in the derivative $\dot{\xi}$; the symbols $\dot{C}_{1,0,k,l}$ and $\dot{S}_{0,1,k,l}$ have a similar meaning with respect to $\dot{\eta}$. Corresponding symbols: $\dot{C}_{2,0,k,l}$, $\dot{S}_{2,0,k,l}$, $\dot{C}_{0,2,k,l}$, and $\dot{S}_{0,2,k,l}$ are introduced for the second powers $\dot{\xi}^2$ and $\dot{\eta}^2$.

All those coefficients will be determined by induction from the differential equations

$$\ddot{\xi} - 2\dot{\eta} - \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\xi = \frac{\partial}{\partial \xi}\Omega^*,$$

$$\ddot{\eta} + 2\dot{\xi} - \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\eta = \frac{\partial}{\partial \eta}\Omega^*$$

that are deduced from the Lagrangian function

$$L = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2) + (\xi\dot{\eta} - \dot{\xi}\eta) + \frac{1}{2}(\xi^2 + \eta^2) + \Omega$$

canonically associated with the Hamiltonian function H; here Ω^* represents the force function Ω stripped of its second degree terms. The Jacobi constant Γ will be computed from the Jacobi integral which is now written

$$\Gamma = 2\Omega + (\xi^2 + \eta^2) - (\dot{\xi}^2 + \dot{\eta}^2).$$

To start the induction we use the first order normal librations as they were found above. For the phase constants, we choose the values

$$\epsilon_s = \pi \qquad \epsilon_l = 0$$

so that, for short period librations as well as for the long period librations, on the first order ellipse, the planetoid starts its fundamental oscillation from a point on the positive side of the η-axis.

TABLE II. Initial coefficients

	Short period	Long period
$C_{1,0,0,0}$	0	0
$C_{1,0,1,0}$	0	0
$S_{1,0,1,0}$	$2a_s$	$2a_l$
$C_{0,1,0,0}$	0	0
$C_{0,1,1,0}$	$2b_s$	$2b_l$
$S_{0,1,1,0}$	0	0
n_0	n_s	n_l
Γ_0	0	0
Γ_2	$-2n_s$	$2n_l$

VII. **Second order librations around L_4.** To the second order in ϵ, the Lagrangian equations of motion are

$$\ddot{\xi} - 2\dot{\eta} - \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\xi = 3\omega_{3,0}\xi^2 + 2\omega_{2,1}\xi\eta + \omega_{1,2}\eta^2,$$

$$\ddot{\eta} + 2\dot{\xi} - \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\eta = \omega_{2,1}\xi^2 + 2\omega_{1,2}\xi\eta + 3\omega_{0,3}\eta^2.$$

It is proposed to determine the six coefficients $C_{1,0,0,1}$, $C_{1,0,2,0}$, $S_{1,0,2,0}$, $C_{0,1,0,1}$, $C_{0,1,2,0}$, and $S_{0,1,2,0}$ so that

$$\begin{aligned}\xi &= C_{1,0,0,1}\epsilon^2 + C_{1,0,1,0}\epsilon\cos n_0 t + S_{1,0,1,0}\epsilon\sin n_0 t\\ &\quad + C_{1,0,2,0}\epsilon^2\cos 2n_0 t + S_{1,0,2,0}\epsilon^2\sin 2n_0 t,\\ \eta &= C_{0,1,0,1}\epsilon^2 + C_{0,1,1,0}\epsilon\cos n_0 t + S_{0,1,1,0}\epsilon\sin n_0 t\\ &\quad + C_{0,1,2,0}\epsilon^2\cos 2n_0 t + S_{0,1,2,0}\epsilon^2\sin 2n_0 t\end{aligned}$$

will be a solution of these equations up to second order in ϵ.

In order to compute the second order terms in the right-hand members of the Lagrangian equations, we need the coefficients of that order in the three functions ξ^2, $\xi\eta$ and η^2; in our notation they are

$$C_{2,0,0,1} = 2a_s^2, \qquad C_{1,1,0,1} = 0, \qquad C_{0,2,0,1} = 2b_s^2,$$
$$C_{2,0,2,0} = -2a_s^2, \qquad C_{1,1,2,0} = 0, \qquad C_{0,2,2,0} = 2b_s^2,$$
$$S_{2,0,2,0} = 0, \qquad S_{1,1,2,0} = 2a_s b_s, \qquad S_{0,2,2,0} = 0$$

for the short period librations. Corresponding coefficients for the long period librations are derived from those merely by changing the subscript.

Then, if the right-hand members of the Lagrangian equations are decomposed into the sums

$$X_2 = X_{0,1}\epsilon^2 + CX_{2,0}\epsilon^2 \cos 2n_0t + SX_{2,0}\epsilon^2 \sin 2n_0t,$$
$$Y_2 = Y_{0,1}\epsilon^2 + CY_{2,0}\epsilon^2 \cos 2n_0t + SY_{2,0}\epsilon^2 \sin 2n_0t,$$

we can evaluate the second order coefficients

$$X_{0,1} = 3\omega_{3,0}\, C_{2,0,0,1} + 2\omega_{2,1}\, C_{1,1,0,1} + \omega_{1,2}\, C_{0,2,0,1},$$
$$CX_{2,0} = 3\omega_{3,0}\, C_{2,0,2,0} + 2\omega_{2,1}\, C_{1,1,2,0} + \omega_{1,2}\, C_{0,2,2,0},$$
$$SX_{2,0} = 3\omega_{3,0} S_{2,0,2,0} + 2\omega_{2,1} S_{1,1,2,0} + \omega_{1,2} S_{0,2,2,0},$$
$$Y_{0,1} = \omega_{2,1}\, C_{2,0,0,1} + 2\omega_{1,2}\, C_{1,1,0,1} + 3\omega_{0,3}\, C_{0,2,0,1},$$
$$CY_{2,0} = \omega_{2,1}\, C_{2,0,2,0} + 2\omega_{1,2}\, C_{1,1,2,0} + 3\omega_{0,3}\, C_{0,2,2,0},$$
$$SY_{2,0} = \omega_{2,1} S_{2,0,2,0} + 2\omega_{1,2} S_{1,1,2,0} + 3\omega_{0,3} S_{0,2,2,0}.$$

From those preliminaries, the unknown coefficients appear as solutions of the linear equations

$$-\frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right) C_{1,0,0,1} = X_{0,1},$$

$$-\frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right) C_{0,1,0,1} = Y_{0,1},$$

$$-\left[4n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right)\right] C_{1,0,2,0} - 4n_0 S_{0,1,2,0} = CX_{2,0},$$

$$-4n_0\, C_{1,0,2,0} - \left[4n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right)\right] S_{0,1,2,0} = SY_{2,0},$$

$$-\left[4n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right)\right] S_{1,0,2,0} - 4n_0\, C_{0,1,2,0} = SX_{2,0},$$

$$4n_0 S_{1,0,2,0} - \left[4n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right)\right] C_{0,1,2,0} = CY_{2,0}.$$

Of the last two systems, the determinant is

$$\Delta_2 = 16n_0^4 - 4n_0^2 + \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right).$$

But from the definition of the mean motion n_0, we have the relation

$$n_0^4 - n_0^2 + \frac{9}{4}\left(1 - \frac{1}{4}\delta^2\right) = 0,$$

so that, by elimination of δ, we obtain

$$\Delta_2 = 3n_0^2(5n_0^2 - 1).$$

Δ_2 is equal to zero if and only if $n_0^2 = 1/5$, which means that the long and short mean motions are bound by the commensurability relation $n_s = 2n_l$. Thus we find again the critical case of order 2 singled out by Siegel's theorem. In our context, it suggests that, for the critical mass ratio μ_2, coefficients $C_{1,0,2,0}$, $S_{1,0,2,0}$, $C_{0,1,2,0}$, and $S_{0,1,2,0}$ for the long period librations are wrongly assumed to be of second order in ϵ and that the long period librations no longer exhibit the Dalembert characteristic. To say it in other words, the commensurability ratio $n_s = 2n_l$ may be described as a resonance of the short period librations on the long period ones; this resonance amplifies so much the second harmonics that their coefficients can no longer be assumed to be of an order of magnitude less than the coefficients of the first harmonics.

Reserving this singular case for closer scrutiny elsewhere, we assume that μ is sufficiently far away from the critical mass ratio μ_2 so that no substantial difficulty is met while solving the last two systems.

At this stage, the Jacobi integral

$$\begin{aligned}\Gamma = {} & (2\omega_{2,0} + 1)\,\xi^2 + (2\omega_{0,2} + 1)\,\eta^2 \\ & + 2(\omega_{3,0}\,\xi^3 + \omega_{2,1}\,\xi^2\eta + \omega_{1,2}\,\xi\eta^2 + \omega_{0,3}\eta^3) \\ & - (\dot{\xi}^2 + \dot{\eta}^2)\end{aligned}$$

should be used to check the number of significant figures present in the second order coefficients, for these are obtained by dividing by Δ_2 which, in general, is rather small. To do so, we need the third order coefficients in the functions ξ^2, η^2, ξ^3, $\xi^2\eta$, $\xi\eta^2$, η^3; in view of its use in the third order analysis, we add to them the coefficients in the function $\xi\eta$.

$$C_{2,0,1,1} = S_{1,0,1,0} S_{1,0,2,0},$$
$$C_{2,0,3,0} = -S_{1,0,1,0} S_{1,0,2,0},$$
$$S_{2,0,1,1} = -S_{1,0,1,0} C_{1,0,2,0} + 2C_{1,0,0,1} S_{1,0,1,0},$$
$$S_{2,0,3,0} = S_{1,0,1,0} C_{1,0,2,0},$$

$$C_{0,2,1,1} = 2C_{0,1,1,0} C_{0,1,0,1} + C_{0,1,1,0} C_{0,1,2,0},$$
$$C_{0,2,3,0} = C_{0,1,1,0} C_{0,1,2,0},$$
$$S_{0,2,1,1} = C_{0,1,1,0} S_{0,1,2,0},$$
$$S_{0,2,3,0} = C_{0,1,1,0} S_{0,1,2,0},$$

$$C_{1,1,1,1} = C_{1,0,0,1} C_{0,1,1,0} + \frac{1}{2} C_{1,0,2,0} C_{0,1,1,0} + \frac{1}{2} S_{1,0,1,0} S_{0,1,2,0},$$

$$C_{1,1,3,0} = \frac{1}{2} C_{1,0,2,0} C_{0,1,1,0} - \frac{1}{2} S_{1,0,1,0} S_{0,1,2,0},$$

$$S_{1,1,1,1} = S_{1,0,1,0} C_{0,1,0,1} + \frac{1}{2} S_{1,0,2,0} C_{0,1,1,0} - \frac{1}{2} S_{1,0,1,0} C_{0,1,2,0},$$

$$S_{1,1,3,0} = \frac{1}{2} S_{1,0,2,0} C_{0,1,1,0} + \frac{1}{2} S_{1,0,1,0} C_{0,1,2,0},$$

$$C_{3,0,1,1} = 0,$$
$$C_{3,0,3,0} = 0,$$

$$S_{3,0,1,1} = C_{2,0,0,1} S_{1,0,1,0} - \frac{1}{2} C_{2,0,2,0} S_{1,0,1,0},$$

$$S_{3,0,3,0} = \frac{1}{2} S_{1,0,1,0} C_{2,0,2,0},$$

$$C_{2,1,1,1} = C_{2,0,0,1} C_{0,1,1,0} + \frac{1}{2} C_{2,0,2,0} C_{0,1,1,0},$$

$$C_{2,1,3,0} = \frac{1}{2} C_{2,0,2,0} C_{0,1,1,0},$$

$$S_{2,1,1,1} = 0,$$
$$S_{2,1,3,0} = 0,$$

$$C_{1,2,1,1} = 0,$$
$$C_{1,2,3,0} = 0,$$

$$S_{1,2,1,1} = C_{0,2,0,1} S_{1,0,1,0} - \frac{1}{2} C_{0,2,2,0} S_{1,0,1,0},$$

$$S_{1,2,3,0} = \frac{1}{2}\, C_{0,2,2,0} S_{1,0,1,0},$$

$$C_{0,3,1,1} = C_{0,1,1,0}\, C_{0,2,0,1} + \frac{1}{2}\, C_{0,1,1,0}\, C_{0,2,2,0},$$

$$C_{0,3,3,0} = \frac{1}{2}\, C_{0,1,1,0}\, C_{0,2,2,0},$$

$$S_{0,3,1,1} = 0,$$

$$S_{0,3,3,0} = 0.$$

We conclude this list of formulae with the corresponding ones for the derivatives $\dot\xi$, $\dot\eta$, $\dot\xi^2$, $\dot\eta^2$:

$$\dot C_{1,0,2,0} = 2n_0 S_{1,0,2,0}, \qquad \dot C_{0,1,2,0} = 2n_0 S_{0,1,2,0},$$

$$\dot S_{1,0,2,0} = -\, 2n_0\, C_{1,0,2,0}, \qquad \dot S_{0,1,2,0} = -\, 2n_0\, C_{0,1,2,0},$$

$$\dot C_{2,0,1,1} = \dot C_{2,0,3,0} = \dot C_{1,0,1,0}\, \dot C_{1,0,2,0},$$

$$\dot S_{2,0,1,1} = \dot S_{2,0,3,0} = \dot C_{1,0,1,0}\, \dot S_{1,0,2,0},$$

$$\dot C_{0,2,1,1} = -\, \dot C_{0,2,3,0} = \dot S_{0,1,1,0}\, \dot S_{0,1,2,0},$$

$$\dot S_{0,2,1,1} = -\, \dot S_{0,2,3,0} = -\, \dot S_{0,1,1,0}\, \dot C_{0,1,2,0}.$$

Now the Jacobi integral is expanded in a Fourier series and we extract its third order part

$$C\,\Gamma_{1,1} \cos n_0 t + S\,\Gamma_{1,1} \sin n_0 t + C\,\Gamma_{3,0} \cos 3n_0 t + S\,\Gamma_{3,0} \sin 3n_0 t$$

where

$$\begin{aligned} C\,\Gamma_{1,1} = {} & (2\omega_{2,0} + 1)\, C_{2,0,1,1} + (2\omega_{0,2} + 1)\, C_{0,2,1,1} \\ & + 2(\omega_{3,0}\, C_{3,0,1,1} + \omega_{2,1}\, C_{2,1,1,1} + \omega_{1,2}\, C_{1,2,1,1} + \omega_{0,3}\, C_{0,3,1,1}) \\ & - \dot C_{2,0,1,1} - \dot C_{0,2,1,1}, \end{aligned}$$

$$\begin{aligned} S\,\Gamma_{1,1} = {} & (2\omega_{2,0} + 1)\, S_{2,0,1,1} + (2\omega_{0,2} + 1)\, S_{0,2,1,1} \\ & + 2(\omega_{3,0,}\, S_{3,0,1,1} + \omega_{2,1} S_{2,1,1,1} + \omega_{1,2} S_{1,2,1,1} + \omega_{0,3} S_{0,3,1,1}) \\ & - \dot S_{2,0,1,1} - \dot S_{0,2,1,1}, \end{aligned}$$

$$\begin{aligned} C\,\Gamma_{3,0} = {} & (2\omega_{2,0} + 1)\, C_{2,0,3,0} + (2\omega_{0,2} + 1)\, C_{0,2,3,0} \\ & + 2(\omega_{3,0}\, C_{3,0,3,0} + \omega_{2,1}\, C_{2,1,3,0} + \omega_{1,2}\, C_{1,2,3,0} + \omega_{0,3}\, C_{0,3,3,0}) \\ & - \dot C_{2,0,3,0} - \dot C_{0,2,3,0}, \end{aligned}$$

$$\begin{aligned} S\,\Gamma_{3,0} = {} & (2\omega_{2,0} + 1)\, S_{2,0,3,0} + (2\omega_{0,2} + 1)\, S_{0,2,3,0} \\ & + 2(\omega_{3,0} S_{3,0,3,0} + \omega_{2,1} S_{2,1,3,0} + \omega_{1,2} S_{1,2,3,0} + \omega_{0,3} S_{0,3,3,0}) \\ & - \dot S_{2,0,3,0} - \dot S_{0,2,3,0}. \end{aligned}$$

Third order coefficients in the Jacobi integral should be equal to zero; to ascertain the accuracy of our numerical evaluations, we check how close to zero they will remain.

VIII. **Third order librations around** L_4. In the right-hand members of the third order Lagrangian equations

$$\ddot{\xi} - 2\dot{\eta} - \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\xi = 3\omega_{3,0}\xi^2 + 2\omega_{2,1}\xi\eta + \omega_{1,2}\eta^2 + 4\omega_{4,0}\xi^3 + 3\omega_{3,1}\xi^2\eta + 2\omega_{2,2}\xi\eta^2 + \omega_{1,3}\eta^3,$$

$$\ddot{\eta} + 2\dot{\xi} - \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\eta = \omega_{2,1}\xi^2 + 2\omega_{1,2}\xi\eta + 3\omega_{0,3}\eta^2 + \omega_{3,1}\xi^3 + 2\omega_{2,2}\xi^2\eta + 3\omega_{1,3}\xi\eta^2 + 4\omega_{0,4}\eta^3,$$

we replace the different functions $\xi^2, \cdots, \eta^3$ by their third order Fourier series, and we collect the various coefficients of the third harmonics. Assuming that these expressions are of the form

$$CX_{1,1}\cos nt + CX_{3,0}\cos 3nt + SX_{1,1}\sin nt + SX_{3,0}\sin 3nt$$

for the right-hand member of the first equation, and of the form

$$CY_{1,1}\cos nt + CY_{3,0}\cos 3nt + SY_{1,1}\sin nt + SY_{3,0}\sin 3nt$$

for the right-hand member of the second equation, we readily obtain that

$$CX_{1,1} = 3\omega_{3,0}C_{2,0,1,1} + 2\omega_{2,1}C_{1,1,1,1} + \omega_{1,2}C_{0,2,1,1} + 4\omega_{4,0}C_{3,0,1,1} + 3\omega_{3,1}C_{2,1,1,1} + 2\omega_{2,2}C_{1,2,1,1} + \omega_{1,3}C_{0,3,1,1},$$

$$CX_{3,0} = 3\omega_{3,0}C_{2,0,3,0} + 2\omega_{2,1}C_{1,1,3,0} + \omega_{1,2}C_{0,2,3,0} + 4\omega_{4,0}C_{3,0,3,0} + 3\omega_{3,1}C_{2,1,3,0} + 2\omega_{2,2}C_{1,2,3,0} + \omega_{1,3}C_{0,3,3,0},$$

$$SX_{1,1} = 3\omega_{3,0}S_{2,0,1,1} + 2\omega_{2,1}S_{1,1,1,1} + \omega_{1,2}S_{0,2,1,1} + 4\omega_{4,0}S_{3,0,1,1} + 3\omega_{3,1}S_{2,1,1,1} + 2\omega_{2,2}S_{1,2,1,1} + \omega_{1,3}S_{0,3,1,1},$$

$$SX_{3,0} = 3\omega_{3,0}S_{2,0,3,0} + 2\omega_{2,1}S_{1,1,3,0} + \omega_{1,2}S_{0,2,3,0} + 4\omega_{4,0}S_{3,0,3,0} + 3\omega_{3,1}S_{2,1,3,0} + 2\omega_{2,2}S_{1,2,3,0} + \omega_{1,3}S_{0,3,3,0},$$

$$CY_{1,1} = \omega_{2,1}C_{2,0,1,1} + 2\omega_{1,2}C_{1,1,1,1} + 3\omega_{0,3}C_{0,2,1,1} + \omega_{3,1}C_{3,0,1,1} + 2\omega_{2,2}C_{2,1,1,1} + 3\omega_{1,3}C_{1,2,1,1} + 4\omega_{0,4}C_{0,3,1,1},$$

$$CY_{3,0} = \omega_{2,1}C_{2,0,3,0} + 2\omega_{1,2}C_{1,1,3,0} + 3\omega_{0,3}C_{0,2,3,0} + \omega_{3,1}C_{3,0,3,0} + 2\omega_{2,2}C_{2,1,3,0} + 3\omega_{1,3}C_{1,2,3,0} + 4\omega_{0,4}C_{0,3,3,0},$$

$$SY_{1,1} = \omega_{2,1} S_{2,0,1,1} + 2\omega_{1,2} S_{1,1,1,1} + 3\omega_{0,3} S_{0,2,1,1} + \omega_{3,1} S_{3,0,1,1}$$
$$+ 2\omega_{2,2} S_{2,1,1,1} + 3\omega_{1,3} S_{1,2,1,1} + 4\omega_{0,4} S_{0,3,1,1},$$
$$SY_{3,0} = \omega_{2,1} S_{2,0,3,0} + 2\omega_{1,2} S_{1,1,3,0} + 3\omega_{0,3} S_{0,2,3,0} + \omega_{3,1} S_{3,0,3,0}$$
$$+ 2\omega_{2,2} S_{2,1,3,0} + 3\omega_{1,3} S_{1,2,3,0} + 4\omega_{0,4} S_{0,3,3,0}.$$

Once the right-hand members are known, we are able to formulate the linear equations that will produce the second order coefficient n_1 in the mean motion n and the third order coefficients $C_{1,0,1,1}$, $S_{1,0,1,1}$, $C_{1,0,3,0}$, $S_{1,0,3,0}$, $C_{0,1,1,1}$, $S_{0,1,1,1}$, $C_{0,1,3,0}$, $S_{0,1,3,0}$ in the Fourier series of the periodic librations. Focusing first on the first harmonic, we obtain the two systems

$$-\left[n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\right] C_{1,0,1,1} \quad - 2n_0 S_{0,1,1,1} = CX_{1,1},$$
$$- 2n_0 C_{1,0,1,1} - \left[n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\right] S_{0,1,1,1} = SY_{1,1},$$
$$-\left[n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\delta\right)\right] S_{1,0,1,1} \quad + 2n_0 C_{0,1,1,1}$$
$$+ 2(C_{0,1,1,0} - n_0 S_{1,0,1,0}) n_1 = SX_{1,1},$$
$$2n_0 S_{1,0,1,1} - \left[n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\delta\right)\right] C_{0,1,1,1}$$
$$+ 2(S_{1,0,1,0} - n_0 C_{0,1,1,0}) n_1 = CY_{1,1}.$$

Since, by definition of n_0, the determinant of the first system is equal to zero, its right-hand members are bound by the relation

$$a_s CX_{1,1} = b_s SY_{1,1}.$$

We are at leisure to give an arbitrary value to one of the two unknown coefficients, and we choose to put

$$S_{0,1,1,1} = 0.$$

Any other choice would do as well and could be reduced to ours by a proper modification of the orbital parameter ϵ. From this choice, it follows that

$$C_{1,0,1,1} = -\frac{SY_{1,1}}{2n_0} = -\frac{a_s CX_{1,1} + b_s SY_{1,1}}{2n_0 b_s}.$$

In the second system, we choose to put

$$C_{0,1,1,1} = 0,$$

so that it constitutes now a system of two linear equations in the two unknowns $S_{1,0,1,1}$ and n_1. Through easy algebraic manipulations, its determinant is shown to be equal to $2b_s/\rho_s^2$. Hence

$$S_{1,0,1,1} = \frac{2\rho_s^2}{b_s}\left[(a_s - n_s b_s)\, SX_{1,1} - (b_s - n_s a_s)\, CY_{1,1}\right]$$

$$n_1 = -\frac{1}{2}\,(a_s SX_{1,1} + b_s CY_{1,1}).$$

After the first harmonic, we look for the coefficients of the third harmonic; they are easily seen to be solutions of the linear equations

$$-\left[9n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right)\right] C_{1,0,3,0} \quad - 6n_0 S_{0,1,3,0} = CX_{3,0},$$

$$- 6n_0\, C_{1,0,3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right)\right] S_{0,1,3,0} = SY_{3,0},$$

$$-\left[9n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right)\right] S_{1,0,3,0} \quad + 6n_0\, C_{0,1,3,0} = SX_{3,0},$$

$$6n_0\, S_{1,0,3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right)\right] C_{0,1,3,0} = CY_{3,0},$$

For both systems, the determinant is

$$\Delta_3 = \left[9n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right)\right]\left[9n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right)\right] - 36n_0^2.$$

By eliminating δ, in the same way as we did for Δ_2, we obtain

$$\Delta_3 = 8n_0^2(10n_0^2 - 1).$$

Thus Δ_3 is zero if and only if either $n_0 = 0$ or $n_0^2 = 1/10$. The first case is evidently to be excluded since we assume $\mu \neq 0$; the second case means that $n_s^2 = 9/10$ and $n_l^2 = 1/10$. In other words, this is the case where long and short periods of the first order would be bound by the commensurability ratio $n_s = 3n_l$. Here again appears a critical mass ratio, namely μ_3, and we see here that this mass ratio holds its exceptional significance from the fact that the resonance between long and short period librations amplifies so much the amplitudes of the third harmonics that their coefficients can no longer be assumed to be of an order of magnitude less than

TABLE III. Third Order Librations
(Earth-Moon System)
$\mu = 0.012\ 139\ 605$

	Short period	Long period
n_0	0.954 546 929 040	0.298 060 665 403
$C_{1,0,1,0}$	0	0
$S_{1,0,1,0}$	3.145 804 428 481	4.998 242 126 170
$C_{0,1,1,0}$	1.546 278 039 770	0.973 199 813 531
$S_{0,1,1,0}$	0	0
Γ_1	− 1.909 093 858 080	0.596 121 330 807
$C_{1,0,0,1}$	− 5.196 143 148 983	− 15.681 265 476 799
$C_{1,0,2,0}$	0.088 564 803 926	− 9.103 149 063 171
$S_{1,0,2,0}$	− 0.608 280 006 459	1.513 029 566 916
$C_{0,1,0,1}$	− 1.238 971 241 052	− 5.650 509 279 875
$C_{0,1,2,0}$	1.270 580 758 217	6.427 979 036 353
$S_{0,1,2,0}$	− 0.030 933 383 091	3.301 138 379 583
$C\,\Gamma_{1,1}$	3×10^{-14}	1.25×10^{-13}
$S\,\Gamma_{1,1}$	-1.6×10^{-14}	-0.77×10^{-13}
$C\,\Gamma_{3,0}$	-9×10^{-15}	1.57×10^{-13}
$S\,\Gamma_{3,0}$	-0.9×10^{-15}	-0.19×10^{-13}
n_1	0.231 038 268 650	− 0.678 344 111 000
$C_{1,0,1,1}$	25.537 262 586 851	294.136 073 022 983
$S_{1,0,1,1}$	− 4.811 517 827 891	− 1.669 982 829 189
$C_{1,0,3,0}$	− 0.094 139 704 578	− 42.497 292 211 543
$S_{1,0,3,0}$	1.185 743 769 640	− 106.219 595 043 159
$C_{0,1,1,1}$	0	0
$S_{0,1,1,1}$	0	0
$C_{0,1,3,0}$	− 1.460 803 788 615	− 52.443 419 368 986
$S_{0,1,3,0}$	− 0.076 054 033 011	39.833 133 928 733

TABLE IV. Third Order Librations
(Sun-Jupiter System)
$\mu = 0.000\ 953\ 875\ 35$

	Short period	Long period
n_0	0.996 757 525 556	0.080 463 875 413
$C_{1,0,1,0}$	0	0
$S_{1,0,1,0}$	2.848 471 915 939	8.697 980 412 788
$C_{0,1,1,0}$	1.422 683 843 549	0.465 909 875 774
$S_{0,1,1,0}$	0	0
Γ_1	− 1.993 515 051 113	0.160 927 750 827
$C_{1,0,0,1}$	− 4.246 089 894 994	− 49.013 967 358 655
$C_{1,0,2,0}$	0.004 996 362 605	− 16.845 454 779 570
$S_{1,0,2,0}$	− 0.506 418 069 883	0.699 464 015 741
$C_{0,1,0,1}$	− 1.013 949 242 336	− 18.775 890 330 070
$C_{0,1,2,0}$	1.016 117 577 440	18.904 603 278 380
$S_{0,1,2,0}$	− 0.001 597 387 632	1.795 975 982 339
$C\,\Gamma_{1,1}$	2×10^{-15}	3×10^{-13}
$S\,\Gamma_{1,1}$	-1.2×10^{-14}	-4×10^{-13}
$C\,\Gamma_{3,0}$	-1.7×10^{-14}	8×10^{-13}
$S\,\Gamma_{3,0}$	-4×10^{-18}	-7×10^{-14}
n_1	0.011 354 365 351	− 1.119 733 339 475
$C_{1,0,1,1}$	18,192 503 417 118	6592.373 487 503 260
$S_{1,0,1,1}$	− 3.266 907 250 147	13.196 039 832 980
$C_{1,0,3,0}$	− 0.005 046 838 573	− 4.188 635 838 735
$S_{1,0,3,0}$	0.845 304 608 549	− 28.375 986 177 146
$C_{0,1,1,1}$	0	0
$S_{0,1,1,1}$	0	0
$C_{0,1,3,0}$	− 1.083 494 388 458	− 10.192 147 632 790
$S_{0,1,3,0}$	− 0.004 018 376 132	73.056 804 001 727

the coefficients of the second harmonics. The periodic librations do not exhibit the Dalembert characteristic. Reserving this case for attention elsewhere, we assume that μ is different from μ_3. Now both systems are easily solved, and their solutions are given by the following relations:

$$\Delta_3 \cdot C_{1,0,3,0} = 6n_0 SY_{3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right) \right] CX_{3,0},$$

$$\Delta_3 \cdot S_{1,0,3,0} = -\, 6n_0\, CY_{3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 + \frac{1}{2}\,\delta\right) \right] SX_{3,0},$$

$$\Delta_3 \cdot C_{0,1,3,0} = -\, 6n_0 SX_{3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right) \right] CY_{3,0},$$

$$\Delta_3 \cdot S_{0,1,3,0} = 6n_0 CX_{3,0} - \left[9n_0^2 + \frac{3}{2}\left(1 - \frac{1}{2}\,\delta\right) \right] SY_{3,0}.$$

IX. **Application to the Sun-Jupiter and the Earth-Moon systems.** The computation schedules that we just explained have been applied to the two most important instances in the plane restricted problem of three bodies:

(a) the Earth-Moon system;
(b) the Sun-Jupiter system.

The values to be found in Tables III and IV agree with the general law about the periods, as it has been enunciated by P. Pedersen, up to the third order, the short period is a descreasing function of the orbital parameter ϵ, while the long period increases with ϵ.

References

1. P. Pedersen, *On the periodic orbits in the neighborhood of the triangular equilibrium points in the restricted problem of three bodies,* Monthly Notes **94** (1933), 167-185.

2. ———, *Fourier series for the periodic orbits around the triangular libration points,* Monthly Notes **95** (1934), 482-495.

3. E. K. Rabe, *Determination and survey of periodic Trojan orbits in the restricted problem of three bodies,* Astronom. J. **66** (1961), 500-513.

4. ———, *Additional periodic Trojan orbits and further studies of their stability features,* Astronom. J. **67** (1962), 382-390.

5. C. L. Siegel, *Vorlesungen über Himmelsmechanik,* Springer, Berlin, 1956.

6. L. J. Wolaver, Private communication, 1963.

7. D. Brouwer and G. M. Clemence, *Methods of celestial mechanics,* Academic Press, New York, 1961.

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Motion of a Particle in the Vicinity of a Triangular Libration Point in the Earth-Moon System

I. **Introduction.** Let m_1, m_2 and m_3 denote three point masses such that $m_1 > m_2 > m_3$. The masses move under the influence of their mutual gravitational attractions; the force between any two masses is inversely proportional to the square of their distance and proportional to the product of their masses. It is well known since Lagrange's work in 1772 (see [1]) that there are in this "three body problem" five exact solutions in which the three masses maintain a constant configuration which revolves with constant angular velocity. An important specialization of the three body problem is the restricted three body problem in which m_3 is infinitesimal and m_1 and m_2 move in circular orbits around their barycenter. The smallness of m_3 means that it does not influence the motion of m_1 and m_2. For many purposes it is convenient to describe the motion of m_3 in a coordinate system which is attached to m_1 and m_2. In this rotating coordinate system the five Lagrange solutions show up as five fixed points at which m_3 would be stationary if placed there with zero velocity (i.e., zero velocity in the rotating coordinates). It is further known that, in this rotating coordinate system, m_3 may describe small periodic orbits about the Lagrange solutions. Gyldén therefore called the points which correspond to the Lagrange solutions "centres of libration"; they are also often referred to as "libration points" or "Lagrange points."

The libration points are singular points of the differential equations of motion in the restricted problem of three bodies; they are also equilibrium points since the gravitational forces on a mass placed in such a point are balanced by the centrifugal force. Three libration points, the collinear points, are found on the line connecting the two large masses; the other two, the triangular points, form equilateral triangles with the two large masses. By linearizing the equations of motion Charlier in [**2**] showed that there are two classes of periodic infinitesimal orbits around the triangular libration points, namely those with short periods (periods very nearly equal to the period of the two large masses) and those with long periods (the periods depending on the mass ratio of the two large masses). Each of these classes consists of concentric, coaxial and similar ellipses with semi-major and minor axes in the ratio 2 : 1 for the short period orbits and a larger ratio, again depending on the mass ratio, for the long period orbits. Plummer in [**3**] considered Charlier's problem in a more general format and from his results some additional conclusions can be drawn (although they were not explicitly mentioned in his paper). For a mass ratio of the two large masses smaller than 1/27 both classes of orbits around the triangular points can be expressed with trigonometric functions; these points are therefore called stable libration points. Furthermore, only one of the classes of orbits around the collinear libration points can be expressed in trigonometric functions, the other requiring hyperbolic functions; the collinear points are therefore called unstable libration points.

The discovery in 1906 of the first of a group of asteroids which appear to oscillate (or, in astronomical terms, librate) around the Sun-Jupiter triangular libration points, gave further impetus to the study of these motions. This first asteroid discovered was called Achilles, and since subsequent asteroids of the group were also called after heroes from the Trojan war these asteroids are commonly referred to as the Trojan group. Brown in [**4**] considered the long period orbits around the triangular libration points by supposing finite amplitudes of libration and discussed in some detail the dependence of period and orbit shape on amplitude. In [**5**] he discussed libration orbits for a mass ratio of the two large masses greater than 1/27. Willard in [**6**] discussed the short period orbits, again of finite amplitude and computed a number of possible orbits. Whereas all this work was based on the restricted problem of three bodies, with the discovery of more Trojans attempts were made to take into consideration the actual physical circumstances. Among the first contributions were those by Linders in [7] and Smart in [8]; Brown in [9] published the explanation for his theory (published in its entirety in 1926 in [10]) which was accurate enough to compute the position of a Trojan asteroid within a few seconds of arc. This theory was applied numerically to Achilles by Brouwer in [11] and by Eckert in [12] to Hector, which has a particularly large libration amplitude. Since this theory was numerical it had to be set up separately for each asteroid. A group theory was outlined by Brown and Shook in [13] in which the interesting direct and indirect effects by Saturn were also discussed. Herz in [14] carried out some of the details of Brown and Shook's plan. Further work concerning the motion of the Trojans was accomplished by Wilkins and reported in [15], [16], [17], and [18].

Thüring in [19] and [20] considered again the problem of the long period motions, in particular the dependence of the period on amplitude. His subsequent contributions in [21] and [22] were largely based on numerical work and [22] was of particular interest because of the application of an electronic digital computer. Thüring's claim of the nonexistence of long period orbits through any arbitrary point was refuted by Rabe in [23] who made a survey of numerically computed long period libration orbits, expressed in Fourier series expansions. Rabe also discussed some aspects of the stability of such periodic orbits and extended these studies in [24]. Similar work was done by him and Schanzle in [25] on libration orbits for the earth-moon system. His most recent work, as discussed in [43] and [44] develops the idea that such periodic orbits should be used as intermediate orbits for the computation of real, nonperiodic orbits. Stumpff in [26] reconsidered and refined Thüring's theory, in particular with respect to the relations between long period orbits with very large amplitudes around the triangular libration points and the nonperiodic orbits in the neighborhood of the collinear libration points.

The study of libration points in the earth-moon system was initiated by Klemperer and Benedikt in [27]. They argued that, in analogy with the Trojan asteroids, there are to be found in the combined gravitational field of the earth and the moon two areas in which natural or artificial bodies would move while maintaining a more or less constant configuration (that of the equilateral triangle) with the earth and the moon. Again, as was the case with the Trojans, there was a subsequent discovery of two "faint cloudlike satellites" (anonymously reported by Kordylewski in [**28**]) in the neighborhood of L_5, the libration point 60° behind the moon. Later, the discovery of such a "cloud" near L_4 (the libration point 60° ahead of the moon) was also reported. Among some possible applications of the triangular earth-moon libration points suggested by Benedikt in [**29**] was the determination of the lunar mass; the supporting argument was the well-known relation between the libration period and the earth-moon mass ratio. This was validly refuted by Colombo in a letter to *Nature* (see [**30**]) by the argument that such periods would be difficult to observe because of perturbations by the sun. Colombo quoted there the work by the present author on the effect of solar perturbations, but put a little too much emphasis on the instability which seemed to be indicated by that work. Thus, Benedikt in turn refuted Colombo, in the same issue of *Nature* (see [**30**]) and equally validly, by supposing that there would be "sufficient permanency to carry out the required measurements." It is unfortunate that he quoted results of Sehnal in [**31**], because of Sehnal's inadmissable assumption that the sun stays permanently on the earth-moon axis. Colombo followed up on his first investigations with [**32**] in which he considered the motion near L_4 or L_5 under the influence of the sun, and the possibility of stabilizing it with a solar sail; in [**33**] he gave a numerical analysis of the influence of the moon orbital eccentricity.

Two reports by Ellis and Diana served as introduction to a study to be performed by this author for the U. S. Air Force RADC. The first by Ellis presented a review of Pederson's work in [**35**] on the critical mass ratio (1/27 for infinitesimal orbits) for noninfinitesimal orbits; the second [**36**], by Ellis and Diana, presented some numerically computed libration orbits in the restricted problem and also discussed the booster requirements for earth-based launch into a trajectory which would intercept a triangular libration point. This author in [**37**] extended this work by adding to the linearized equations of motion relative to a stable libration point in the restricted problem the principal effects of a fourth body representing the sun as it is related to the earth and moon. Two linear, second order differential equations with time varying coefficients, were thus obtained which in principle could be solved in powers of the small parameter (mass of sun divided by the cube of the earth-sun distance). The first order solution and the most significant parts of the second order solution were obtained and for a number of different initial conditions this presented a reasonably close agreement with numerically integrated orbits. It did appear that any so called "stability" was strongly influenced by the sun but it also appeared possible to choose the initial configuration of earth-moon-sun and initial conditions of the small particle such that this influence was small enough for a usefully long "libration life" to be possible. In a subsequent paper [38] the influence of the moon's orbital eccentricity was discussed and it was found that, if the sun was introduced in the consideration of motion near earth-moon libration points, the moon's orbital eccentricity would have to be considered also. Because of this it did not seem entirely practical to continue the work in terms of rectangular coordinates if greater accuracy were required. Because of the apparently great importance of these perturbations, it seemed worthwhile to develop a theory in terms of orbital elements; this theory is sketched in the present chapter. Before some additional theoretical matters are touched upon, mention must still be made of the work by Michael in [39] which discusses orbit envelopes as depending on initial conditions, based on a linearized analysis of the restricted problem.

It appears that the most fundamental questions about motion near libration points are those about the existence of periodic orbits and the stability of such orbits. Most generally, it concerns the behavior of solutions of differential equations at or near conditions of commensurability. If stable periodic solutions exist, such solutions may be used as intermediate orbits for the computation of nonperiodic orbits by perturbation analysis. According to the remarks made before, it appears that in the restricted problem of three bodies the existence of periodic orbits about the triangular libration points is well established. Actually, this result followed from the analysis of the linearized equations of motion. Accordingly, it served to exhibit the stability of the triangular configuration, as one of Lagrange's exact solutions of the restricted problem, only in so far as the linearization is valid; that is, only for infinitesimal disturbances. The apparent existence of noninfinitesimal periodic orbits (Brown, Thüring, Rabe) followed either from the analysis of higher order approximations of the differential equations (but still not exact) or from numerical work. The stability of such orbits, if studied at all, has been investigated only numerically. Only as late as 1959 it was shown by Littlewood in [**40**], which is an extremely difficult paper, that the triangular configuration itself (still in the restricted problem) is stable in the sense that for an initial disturbance of order ϵ the disturbance will remain of order ϵ for as long a time as $\exp(A\epsilon^{-1/2}|\log\epsilon|^{-3/4})$, where A depends only on the mass ratio. In a second, equally difficult, paper he improved his results somewhat (see [**41**]). A possibly stronger result was obtained by Leontovic in [**42**], where he states that in the restricted problem the triangular configuration is stable for mass ratios smaller than 1/27, possibly excluding a set of mass ratios of Lebesgue measure zero. All this says precious little about noninfinitesimal libration orbits and their stability. It appears thus to be very difficult to derive meaningful results by qualitative methods, and with the problem of libration orbits we may still be in the position of trying to come to general results by the study of particular analytical or numerical solutions. This seems to be typical for the development of nonlinear mechanics. There is, of course, an extensive literature concerning questions of this general kind. Still, the three papers by Leontovic and Littlewood are the only contributions specifically concerned with the triangular libration points which have come to the attention of this author.

Considering the modern trends in the study of nonlinear mechanics toward qualitative methods one may expect that any new work on triangular libration points, whether it be in the earth-moon or the sun-Jupiter case, should concentrate on the establishment of a proof of stability for libration orbits. If then a solution in the form of analytical expressions of the coordinates as functions of time with an exhibition of integration constants would be at all required, one should use periodic orbits (whose existence would first be proved) as intermediate orbits for the perturbation analysis. Two reasons discourage one from following this approach. First of all, even though the past few decades have seen a significant development of methods and theorems in nonlinear mechanics there is still very little known about systems of higher than second order. The methods of the phase plane, so convenient and easily visualized for second order systems, must be transferred to multidimensional phase space which introduces some formidable complications. Secondly, the few qualitative results which are known about the triangular libration points specifically have been derived only for the restricted three body problem which is really very special since its Hamiltonian does not contain the independent variable explicitly. On the other hand, preliminary studies have shown clearly that in the case of the earth-moon libration points the influences of the sun as the fourth body and of the moon's orbital eccentricity are quite important. The Hamiltonian of such a problem contains the independent variable in periodic terms of short and long periods, and especially (in the present problem) with periods commensurable, or nearly so, with the principal periods of the problem. Very little is known at all about how certain qualitative results derived for constant Hamiltonian could be transferred to a similar problem with time-varying Hamiltonian.

Thus, the purpose of the present chapter is to develop a solution in the form of analytical expressions for coordinates (strictly speaking, elements) as functions of time, containing integration constants which in some way can be related to initial conditions of position and velocity. This goal is slightly more general than is usually the case in the development of such a "theory" for astronomical purposes. In that case a theory is developed for a particular celestial body, even though there is considerable flexibility in choosing the stage of the work where this particular body is introduced. In some way the integration constants are numerically evaluated by relating the solution to observations and they need not be related to arbitrary initial conditions of position and velocity. In the present case a theory is developed for the more general purpose of precomputing the orbit of a not yet existing space vehicle, or of not yet observed natural bodies. Even more important, it is hoped that this theory will enable one to determine the initial conditions for a space vehicle to complete its mission most successfully, or to describe a libration orbit of greatest "stability." Of necessity, the integration constants will thus appear as symbols throughout the entire development. Probably the best that can be obtained is that the initial values of the elements are functions of the integration constants; the elements themselves are related to the position and velocity components through the equations of elliptic motion. Some intricate inversions will therefore be necessary before the theory can be used to compute an orbit from specific initial conditions. Still, many important features of the motion will emerge from the analytical expressions even without computation of specific orbits, more than could possibly be obtained by the numerical integration of many orbits. According to a statement made by Brown in 1923 concerning the libration orbits of Trojans, but still applicable today to libration orbits in the earth-moon system, the problem "presents so many points of mathematical and mechanical interest, that a general explanation of certain features of the motion and of the methods adapted to obtain a solution of the problem may not be out of place."

II. **Problem statement and outline of method of solution.** The problem is that of motion of a particle in the vicinity of the earth-moon triangular libration points. A representation of that motion is to be developed in the form of analytical expressions for certain variables as functions of the independent variable time. In this development the gravitational force of the sun and the moon's orbital eccentricity shall be considered. The analytical expressions shall serve the purpose of (1) pointing out certain general features of the motion, as for instance the effects of the sun, the moon's orbital eccentricity and the initial configuration on the appearance of various periodic terms and their amplitudes, (2) providing a means of orbit prediction (or ephemeris computation), after substitution of initial conditions and evaluation of the integration constants, (3) providing a theory for the determination of an orbit from observations, (4) providing a tool for the simulation of libration orbits from which insertion conditions may be determined for smallest libration amplitudes during a desired "libration lifetime," (5) providing the basis for the determination of station keeping requirements.

The method of solution is explained in the following sections. At this time the purpose is not to present a complete solution, but rather to highlight the essential features and difficulties of the problem and to present a plan according to which the details of the solution are to be carried out. The equations of motion are derived from a formulation of the problem in Jacobi coordinates. After the identification of the Main Problem and the problems due to the Direct Effect of the Sun and the Indirect Effect, a slight digression is made by writing the equations for the main problem in rotating rectangular coordinates attached to L_4. It is shown how the two fundamental frequencies are derived and how these are changed by the introduction of the sun's attraction. This provides a link with earlier work by this author and others. The method of solution adapted here begins with the introduction of Delaunay elements as variables. An explanation follows of the development in terms of these elements of the disturbing function for the main problem, the direct effect of the sun and the indirect effect. The appearance of short period terms, long period terms and libration terms is identified. The procedure by which the short period terms are to be eliminated according to the von Zeipel method is outlined and some remarks are made about the work which will be required in order to eliminate the long period terms. The libration terms are discussed in some greater detail because they embody the essential difficulties of the problem: the libration, or motion in mean longitude, and the variation of the semi-major axis. Especially of interest is the demonstration of the dependence of the libration frequency on amplitude.

III. **Equations of motion and disturbing function.** Let m_0, m_1, m_2, m_3 be the masses of earth, particle, moon and sun respectively. The equations of motion according to the inverse square law of gravitation can be formulated conveniently in terms of Jacobi coordinates, defined in Figure 1.

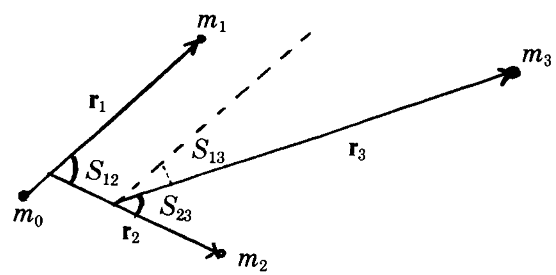


FIGURE 1. Jacobi Coordinates

The position of m_1 is given with respect to m_0 by the vector $\mathbf{r}_1$; the position of m_2 by the vector $\mathbf{r}_2$, beginning at the barycenter of m_0 and m_1; the position of m_3 by the vector $\mathbf{r}_3$, beginning at the barycenter of m_0, m_1 and m_2. To take full advantage of the generality of this formulation, the substitution $m_1 = 0$ is left until later. Since the motions of the moon and the sun with respect to the earth are known, only the equations of motion for m_1 are needed. They are in vector form

$$(1)\quad \ddot{\mathbf{r}}_1 = \frac{m_0 + m_1}{m_0 m_1}\frac{\partial F}{\partial \mathbf{r}_1}, \quad F = k^2 \sum_{i,j} \frac{m_i m_j}{|\mathbf{r}_{ij}|}, \quad i \neq j, \quad i,j = 0,1,2,3,$$

where k^2 is the gravitational constant and $\mathbf{r}_{ij}$ is the vector from m_i to m_j. To express the force function F in the Jacobi coordinates, we observe from Figure 1,

$$\begin{aligned}
\mathbf{r}_{01} &= \mathbf{r}_1 \\
\mathbf{r}_{02} &= \mathbf{r}_2 + k_1\mathbf{r}_1, \quad k_1 = \frac{m_1}{m_0 + m_1} \\
\mathbf{r}_{23} &= \mathbf{r}_3 - (1 - k_2)\,\mathbf{r}_2, \quad k_2 = \frac{m_2}{m_0 + m_1 + m_2} \\
\mathbf{r}_{03} &= k_1\mathbf{r}_1 + k_2\mathbf{r}_2 + \mathbf{r}_3 \\
\mathbf{r}_{13} &= \mathbf{r}_3 - (1 - k_1)\,\mathbf{r}_1 + k_2\mathbf{r}_2 \\
\mathbf{r}_{12} &= \mathbf{r}_2 - (1 - k_1)\,\mathbf{r}_1.
\end{aligned}$$

To get the inverses, $1/r_{ij}$, Legendre polynomials may be used without any difficulty for the first three expressions; for instance,

$$(2)\quad \begin{aligned}
r_{02}^{-1} &= r_2^{-1}\left(1 + 2k_1\frac{r_1}{r_2}\cos S_{12} + \frac{r_1}{r_2}\right)^{-1/2} \\
&= r_2^{-1}\left[1 - k_1\frac{r_1}{r_2}\cos S_{12} + k_1^2\frac{r_1^2}{r_2^2}\left(\frac{3}{2}\cos^2 S_{12} - \frac{1}{2}\right) + \cdots\right].
\end{aligned}$$

A little more difficult are the expressions for $1/r_{03}$ and $1/r_{13}$; they may be expanded according to the binomial theorem and it is then seen that two series of Legendre polynomials result and a series of "coupled terms." For instance, for $1/r_{03}$,

$$(3)\quad \begin{aligned}
r_{03}^{-1} = r_3^{-1}\Bigg[1 &- k_2\frac{r_2}{r_3}\cos S_{23} + \left(k_2\frac{r_2}{r_3}\right)^2\left(\frac{3}{2}\cos^2 S_{23} - \frac{1}{2}\right) - \cdots \\
&- k_1\frac{r_1}{r_3}\cos S_{13} + \left(k_1\frac{r_1}{r_3}\right)^2\left(\frac{3}{2}\cos^2 S_{13} - \frac{1}{2}\right) - \cdots \\
&+ k_1k_2\frac{r_1r_2}{r_3^2}\left(3\cos S_{13}\cos S_{23} - \cos S_{12}\right) + \cdots\Bigg],
\end{aligned}$$

and, similarly for $1/r_{13}$.

Finally, no such expansions can be used directly for r_{12}^{-1}, since both parts of $\mathbf{r}_{12}$ are of equal order of magnitude. Instead, we define

$$\Delta^2 = r_1^2 - 2r_1r_2\cos S_{12} + r_2^2 \tag{4}$$

and write

$$r_{12}^{-1} = \Delta^{-1}\left[1 + 2k_1\frac{r_1r_2}{\Delta^2}\cos S_{12} - (2-k_1)\,k_1\frac{r_1^2}{\Delta^2}\right]^{-1/2} \tag{5}$$

and the binomial theorem may again be used for the part in the brackets.

Now, all the expressions for the r_{ij}^{-1} are substituted in the disturbing function, equation (1), and finally the smallness of m_1 is taken into account by putting $m_1 = k_1 = 0$; in particular it is noted that the expansion of the factor in brackets, equation (5), is not needed since all the terms, except unity, vanish. The result is

$$\begin{aligned} F &= k^2\left[\frac{m_0+m_1}{r_1} + m_2\left(\frac{1}{\Delta} - \frac{r_1\cos S_{12}}{r_2^2}\right)\right] \\ &\quad + m_3k^2\left[\frac{r_1^2}{r_3^3}\left(\frac{3}{2}\cos^2 S_{13} - \frac{1}{2}\right) + \frac{r_1^3}{r_3^4}\left(\frac{5}{2}\cos^3 S_{13} - \frac{3}{2}\cos S_{13}\right)\right. \\ &\quad \left. + \frac{1}{2}\,k_2\frac{r_1^2r_2}{r_3^4}\left(3\cos S_{23} + 6\cos S_{13}\cos S_{12} - 15\cos^2 S_{13}\cos S_{23}\right)\right]. \end{aligned} \tag{6}$$

With the Main Problem is meant the problem of which the disturbing function is given by the first bracket in (6) and in which also the motion of the moon follows the ellipse which results from taking the elements of the moon's orbit to be constant. The second bracket in (6) is the disturbing function for the Direct Effect of the Sun; the Indirect Effect of the Sun is considered by using in the disturbing function the motion of the moon as it is perturbed by the sun. If units are chosen such that the average earth-moon distance, the sum of the masses of earth and moon, and the gravitational constant are all unity, the coefficient m_2k^2 is about .012 and the coefficient of the sun's force function, m_3k^2/r_3^3 is about .0052. This is a first indication that the sun, as the fourth body, plays an important role in this problem. The second term of the second bracket in (6) is about 400 times smaller than the first, but there are indications (see [**37**] and [**38**]) that it is important, because it introduces a nearly resonant frequency. The coupling term (the third term of the second bracket in (6)) has a coefficient of about 8×10^{-8} and can probably be neglected.

IV. **Relative rectangular coordinates.** Some of the important aspects of the problem are revealed by the linearization of the equations of motion. For this, consider the three bodies m_0, m_2 and m_1; also, confine the motion of m_1 to be in the plane of the motions of m_0 and m_2. Let (X, Y) be the rectangular coordinate system in this plane, centered at m_0, the X-axis going through m_2. This coordinate system rotates with the angular velocity of m_2, $\dot{v}$. The equations of motion of m_1 are then

$$\ddot{X} - 2\dot{Y}\dot{v} - \dot{v}^2 X - \ddot{v}Y = \frac{\partial F}{\partial X}$$

$$\ddot{Y} + 2\dot{X}\dot{v} - \dot{v}^2 Y + \ddot{v}X = \frac{\partial F}{\partial Y},$$

where $\cdot$ denotes d/dt, and F is the force function defined in equation (1), with $i, j = 0, 1, 2$. The units have been chosen such that $(m_0 + m_2) = 1$ and $k^2 = 1$. For motion of m_1 in an infinitesimal neighborhood of the leading triangular libration point L_4 these differential equations may be linearized by developing $\partial F/\partial X$ and $\partial F/\partial Y$ in a Taylor series about the point $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ and retaining only the constant terms and the terms which are linear in x and y, where $x = X - \frac{1}{2}$, $y = Y - \frac{1}{2}\sqrt{3}$. If the further simplification is made that m_2 moves in a circular orbit about m_0, so that $\dot{v} = 1$, the equations of motion of m_1 are

$$\begin{aligned} \ddot{x} - 2\dot{y} - c_1 x - c_2 y &= 0 \\ \ddot{y} + 2\dot{x} - c_2 x - c_3 y &= 0, \end{aligned} \tag{7}$$

with $c_1 = 3/4$, $c_2 = 3\sqrt{3}\,(1 - 2\mu)/4$, $c_3 = 9/4$, $\mu = m_2/(m_0 + m_2)$. The frequency equation of this fourth order system has only imaginary roots so that the solution can be expressed in terms of trigonometric functions. The frequencies are .95459 and .29792 (if the mass ratio μ is taken to be .01213) which corresponds to periods of 28.62 and 91.7 days.

An approximation to the perturbation by m_3 (the sun) can be found by subjecting the second line of equation (6) to the same expansion in a Taylor series. The equations of motion are then

$$\begin{aligned} &\ddot{x} - 2\dot{y} - c_1' x - c_2 y = \nu(1/4 + \text{ terms with } \cos\phi t, \cos 2\phi t) \\ &\ddot{y} + 2\dot{x} - c_2 x - c_3' y = \nu(\sqrt{3}/4 + \text{ terms with } \cos\phi t, \cos 2\phi t), \end{aligned} \tag{8}$$

where $c_1' = 3/4 + \nu/2$, $c_2 = 3\sqrt{3}\,(1 - 2\mu)/4$, $c_3 = 9/4 + \nu/2$,

$$\nu = m_2/r_3^3 = .00567, \quad \phi = .92520.$$

The terms in the right-hand sides with frequency 2ϕ come from the first term of the second line of equation (6), those with frequency ϕ come from the second term. The new fundamental frequencies are .9457 and .3161, corresponding to periods of 28.91 and 86.8 days; the shorter period is rather close to the length of the synodic month, 29.53 days. Also, the two fundamental frequencies are nearly commensurable (with ratio 2.99, versus 3.20 in the three body case). A solution of equation (8) could be attempted in powers of the "small parameter" ν, but the commensurabilities will unavoidably lead to trouble with small divisors.

If the orbital eccentricity of the moon is taken into account, there will be additional terms on the right-hand side with the small parameter $e = .056$ and frequency equal to unity; this new frequency is also close to one of the fundamental frequencies, again causing trouble with small divisors.

A solution of equation (8) was carried out, in terms of arbitrary initial position and velocity components, to include all terms with the first power of ν and the most important of the terms with the second power of ν. This solution is interesting for "engineering" purposes but it lacks accuracy. Most importantly, it was found that the realistic problem cannot be discussed reasonably by considering either the sun's effect or the moon's orbital eccentricity separately; both perturbations must be considered together. Because of the almost immediate difficulty with small divisors it appears quite impractical to formulate a solution of any reasonable accuracy in the relative rectangular coordinates. Of course, inasmuch as one may argue that these small divisors really reflect the typical physical behavior of the dynamical system and are therefore more or less independent of the particular mathematical formulation, one may expect that similar difficulties will be encountered with any other set of coordinates. But one may also hope to formulate the problem in such variables that the difficulty is postponed, or such that some of the equations are decoupled so that the difficulty appears in a lower order system. An outline of such a solution is the subject of the following sections.

V. **Formulation in orbital elements.** Just as the Lagrange solution (in which m_1 maintains the equilaterial triangle configuration with m_0 and m_2) was used as the basis for a perturbation analysis in rectangular coordinates, it will also serve as the basis for a formulation in terms of orbital elements. A basic difference between the two formulations is that in one the motion is given in the rotating coordinate system, whereas in the other the motion is given in inertial space. It must be noted that the Lagrange solutions hold not only when m_2 follows a circular orbit around m_0 but also for any elliptic orbital motion of m_2. The elements will thus be chosen such that for constant values of the elements of m_1 an elliptical orbit follows with the same eccentricity as m_2, according to the Lagrange solution. For this purpose the force function F is put in the form

$$F = \frac{m_0 + m_1 + m_2}{r_1} + m_2\left(\frac{1}{\Delta} - \frac{1}{r_1} - \frac{r_1 \cos S_{12}}{r_2^2}\right) + m_3[\ ], \tag{9}$$

where, in comparison with equation (6), m_2/r_1 is added to the first term and subtracted from the second. The brackets with the coefficient m_3 contain the sun's contribution, as in equation (6). Units have been chosen such that $k^2 = 1$.

Now, with $\mu = m_0 + m_1 + m_2$ and

$$m_2 R = m_2\left(\frac{1}{\Delta} - \frac{1}{r_1} - \frac{r_1 \cos S_{12}}{r_2^2}\right) + m_3[\ ], \tag{10}$$

the equations of motion are

$$\ddot{x} + \mu\frac{x}{r_1^3} = m_2\frac{\partial R}{\partial x}, \tag{11}$$

and similarly for y and z. If $m_3 = 0$ (neglect the sun), it is easily seen that at the triangular libration points R and $\operatorname{grad} R$ vanish, as they should according to the Lagrange solution. Let the elliptic orbit which then results from equation (11) be characterized by the elements

a, semimajor axis

e, eccentricity

i, inclination

l, mean anomaly

ϖ, longitude of perigee

h, angle from vernal equinox to line of nodes,

then, according to page 539 of [45], equation (11) may be replaced by a set of canonical equations

$$\dot{c}_i = \frac{\partial \widetilde{H}}{\partial w_i}, \quad \dot{w}_i = -\frac{\partial \widetilde{H}}{\partial c_i}, \quad i = 1,2,3 \tag{12}$$

with the Hamiltonian

$$\widetilde{H} = \frac{\mu^2}{2c_1^2} + m_2 R \tag{13}$$

and the variables (sometimes called modified Delaunay variables)

$$\begin{aligned} c_1 &= \sqrt{(\mu a)}, \quad w_1 = l + g + h, \text{ mean longitude} \\ c_2 &= \sqrt{(\mu a)}(\sqrt{(1-e^2)} - 1), \ w_2 = g + h = \varpi, \text{ longitude of perigee} \\ c_3 &= \sqrt{((\mu a)(1-e^2))}(\cos i - 1), \ w_3 = h, \text{ longitude of node.} \end{aligned} \tag{14}$$

A slightly more convenient form is obtained by dividing c_1, c_2, c_3, $\widetilde{H}$ and $\dot{c}_i$ by $\bar{c}_1 = \sqrt{(\mu \bar{a})}$ and by using

$$\mu = n^2 a^3 = \bar{n}^2 \bar{a}^3 \quad (n \text{ is "mean motion"}), \tag{15}$$

where the bar refers the symbol to the moon's orbit. The problem is then stated in the form of the canonical equations

$$\dot{c}_i = \frac{\partial H}{\partial w_i}, \quad \dot{w}_i = -\frac{\partial H}{\partial c_i}, \quad i = 1,2,3 \tag{16}$$

with the Hamiltonian

$$H = \bar{n}\left[\frac{1}{2c_1^2} + m\bar{a}R\right], \tag{17}$$

and the disturbing function

$$\begin{aligned} m\bar{a}R = m &\left(\frac{\bar{a}}{\Delta} - \frac{\bar{a}}{r_1} - \frac{\bar{a} r_1 \cos S_{12}}{r_2^2}\right) \\ &+ \frac{m_3}{m_0 + m_1} \frac{\bar{a} r_1^2}{r_3^3}\left[\left(\frac{3}{2}\cos^2 S_{13} - \frac{1}{2}\right) + \cdots\right] \end{aligned} \tag{18}$$

where $m = m_2/(m_0 + m_1)$.

This sixth order system is to be solved with six integration constants (or "arbitraries"); ideally these six integration constants should be initial conditions of position and velocity. The introduction of orbital elements has of course precluded any convenient reintroduction of rectangular components of position and velocity. Even stronger, because of the complications inherent in the formulation of a solution, it will also be impractical to let the integration constants be the initial values of the osculating elements.

VI. **Development of the disturbing function.**

a. *Main Problem.* Using equation (4), the disturbing function for the main problem (namely the first term in equation (18)) can be written as

$$\bar{a}R = \frac{\bar{a}}{\Delta} - \left(\frac{\bar{a}}{a}\right)\left(\frac{a}{r_1}\right) + \frac{1}{2}\left(\frac{\bar{a}}{r_2}\right)^3\left(\frac{\Delta}{a}\right)^2 - \frac{1}{2}\left(\frac{a}{\bar{a}}\right)^2\left(\frac{r_1}{a}\right)^2\left(\frac{\bar{a}}{r_2}\right)^3. \tag{19}$$

For convenience in using expansions for elliptic motion, wherever a radius appears it has been combined with its corresponding semimajor axis to form a dimensionless quantity. This disturbing function is to be expressed as a trigonometric series in terms of the canonical variables defined in equation (14). Actually, a development in terms of the elliptic elements a, e, i, l, ϖ, h is a little more convenient; the transformation to canonical variables is then performed whenever it is required, through the relations in equation (14). The procedure which is to be followed to achieve this development will now be outlined; the complete development will not be given here.

First of all, the expressions (a/r_1), $(a/r_2)^3$ and $(r_1/a)^2$ in equation (19) are easily expressed in trigonometric series in multiples of the anomaly with powers of the eccentricity as coefficients, by referring to Cayley's tables (Mem. Roy. Astronom. Soc. **29** (1861), 191-306). The expansion of the first term, $\bar{a}/\Delta$, is best obtained by using the binomial theorem on the expansion of $\Delta^2/\bar{a}^2$. To obtain the expansion of $\Delta^2/\bar{a}^2$, the expression $\cos S_{12}$ which appears in equation (4) has to be expressed in the true longitudes ψ and $\bar{\psi}$ of m_1 and m_2 respectively, the inclinations and the nodal longitudes. This expression is given in [**13**, p. 34].

With the following definitions

$$\Delta_0^2 = 1 + \left(\frac{a}{\bar{a}}\right)^2 - 2\frac{a}{\bar{a}}\cos\tau, \tag{20}$$

the difference in mean longitudes of m_1 and m_2

$$\tau = w_1 - \overline{w}_1,$$

(21)

$$\gamma = \sin\frac{i}{2}, \quad \overline{\gamma} = \sin\frac{\overline{i}}{2},$$

and

$$Q = \frac{1}{2}\sin i \sin \overline{i},$$

the expression $\Delta^2/\overline{a}^2$ becomes

(22)

$$\begin{aligned}\frac{\Delta^2}{\overline{a}^2} = \Delta_0^2 &+ \left(\frac{r_2^2}{\overline{a}^2} - 1\right) + \left(\frac{a}{\overline{a}}\right)^2\left(\frac{r_1^2}{a^2} - 1\right)\\ &- 2\frac{a}{\overline{a}}\left[\frac{r_1}{a}\frac{r_2}{\overline{a}}\cos(\psi - \overline{\psi}) - \cos\tau\right]\\ &+ 2\frac{a}{\overline{a}}\frac{r_1}{a}\frac{r_2}{\overline{a}}[(\gamma^2 + \overline{\gamma}^2 - \gamma^2\overline{\gamma}^2)\cos(\psi - \overline{\psi})\\ &\qquad - (1 - \gamma^2)\overline{\gamma}^2\cos(\psi + \overline{\psi} - 2\overline{h})\\ &\qquad - \gamma^2(1 - \overline{\gamma}^2)\cos(\psi + \overline{\psi} - 2h)\\ &\qquad - \gamma^2\overline{\gamma}^2\cos(\psi - \overline{\psi} - 2h + 2\overline{h})\\ &\qquad - Q\cos(\psi - \overline{\psi} - h + \overline{h})\\ &\qquad + Q\cos(\psi + \overline{\psi} - h - \overline{h})].\end{aligned}$$

The expansion can then be completed except for Δ_0^2, without great difficulties (but with a great amount of labor!) by substituting the elliptic expansions for r_1/a and $r_2/\overline{a}$. $\Delta^2/\overline{a}^2$ is of course of the form $(1 +$ terms of order e, e^2, etc.); the first term of equation (19), $\overline{a}/\Delta$, can therefore be obtained by applying the binomial theorem to equation (22). But first the term Δ_0^2 must be expanded; use is now made of the fact that a theory is to be developed for motion near the libration point. The difference in mean longitudes of m_0 and m_2 is therefore written as

(23)

$$\tau = w_1 - \overline{w}_1 = \tau_0 + \delta\tau, \quad \tau_0 = +60^\circ \text{ for } L_4,$$
$$\tau_0 = -60^\circ \text{ for } L_5.$$

(For convenience the following shall be specialized for L_4, so that $\tau = 60^\circ$.) The expansion of Δ_0^2 in a power series of $\delta\tau$ will then complete the expansion of $\Delta^2/\bar{a}^2$, whereupon $\bar{a}/\Delta$ can be obtained by the binomial theorem.

For the purpose of outlining the expansion of $\bar{a}R$ in a little more detail, let

$$\frac{\Delta^2}{\bar{a}^2} = 1 + O_1 + O_2 + O_3 + O_4 + \cdots$$

and

$$\left(\frac{\bar{a}}{r_2}\right)^3 = 1 + E_1 + E_2 + E_3 + E_4 + \cdots,$$

where O_1 and E_1 indicate all the terms with the first power of the eccentricities and inclinations as coefficient; O_2 and E_2 indicate the second order terms, and so on. Then

$$\frac{\bar{a}}{\Delta} = 1 - \frac{1}{2}(O_1 + O_2 + O_3 + O_4) + \frac{3}{8}(O_1^2 + 2O_1O_2 + \cdots) + \cdots.$$

Let also

$$\frac{a}{\bar{a}} = (1 + x_1)^2 \tag{24}$$

then after some amount of algebra it is seen that

$$\begin{aligned} \bar{a}R = & -\frac{3}{4}e^2 - \frac{3}{4}e^2\cos l + \frac{3}{2}e\bar{e}\cos(l + \bar{l}) + \frac{3}{2}e\bar{e}\cos(l - \bar{l}) \\ & + \frac{3}{8}O_1^2 + \frac{1}{2}E_1O_1 + \frac{3}{4}O_1O_2 - \frac{5}{16}O_1^3 \qquad (25) \\ & + \frac{1}{2}E_1O_2 + \frac{1}{2}E_2O_1 + \cdots. \end{aligned}$$

Therefore, the disturbing function contains no zeroth and first order terms; also, if a development to fourth order in eccentricities and inclinations is desired, the expansions O_4 and E_4 do not have to be obtained.

To carry out the actual development the eccentricities e and $\bar{e}$ and the inclinations i and $\bar{i}$ have been considered to be smaller than 1/10, which is in line with the physical facts about the moon's orbit and with the kind of orbits that are to be considered for m_1. Furthermore, the angle $\delta\tau$ is taken to be smaller than 1/10, and therefore (as will be seen later) it is reasonable to assume x_1 to be smaller than .01.

The general form of the disturbing function will be

$$m\bar{a}R = m\sum \mathrm{Coeff}(e, \gamma, x_1, \delta\tau, \bar{e}, \bar{\gamma}) \cdot \cos(j_1 l + j_2 \bar{l} + j_4 \varpi + j_5 \bar{\varpi} + j_7 h + j_8 \bar{h}). \tag{26}$$

The entire series is made up of cosines with arguments consisting of various combinations of the angular elements l, ϖ and h. The coefficients are mostly functions of the other elements, e, a through x_1 and i through γ, except that the angle $\delta\tau$ appears in some of the coefficients due to the expansion of Δ_0 in terms of $\delta\tau$. It is important to recognize two kinds of terms, those in which $j_1 + j_2 \neq 0$ and those in which $j_1 + j_2 = 0$. The former are called "short period" terms, the latter are "long period" terms. The long period terms are those which do not contain either l or $\bar{l}$ and those which have l and $\bar{l}$ only in the combination $(l - \bar{l})$; the short period terms are all others. The significance of and the reason for this distinction will become clear in the later treatments. The actual development has been carried out to obtain all short period terms of second order (18 different arguments are present) and all long period terms of second, third and fourth order (producing 14 different arguments). It must be noted that, because of the expansion of $\bar{a}/\Delta$ as $(\Delta^2/\bar{a}^2)^{-1/2}$ some of the numerical coefficients of the fourth order terms are so large that they are actually of third order. It may therefore be argued that the development has been obtained only to the third order. If this development may not lead to the ultimate accuracy which is desired, it is at least accurate enough to bring out the important features and difficulties of the problem; the results from this development may show how certain terms must be carried to higher order for greater accuracy.

b. *Direct Effect of the Sun.* In considering the sun's effect it will be assumed that the earth-moon barycenter describes a constant ellipse around the sun. It is thus reasonable to adopt the plane of that ellipse for the fundamental plane in the entire analysis. This is also the plane with respect to which the moon's coordinates are given in Brown's lunar tables. In the following the quantities relating to the sun's orbit will be indicated by a double bar.

The disturbing function due to the sun is given in the final bracket of equation (6). This may be written as

$$(m\bar{a}R)_{\text{sun}} = \frac{m_3}{m_0 + m_2} \left(\frac{\bar{a}}{\bar{\bar{a}}}\right)^3 \left[\left(\frac{a}{\bar{a}}\right)^2 \left(\frac{r}{a}\right)^2 \left(\frac{\bar{\bar{a}}}{\bar{\bar{r}}}\right)^3 \left(\frac{3}{2}\cos^2 S_{13} - \frac{1}{2}\right) \right.$$
$$+ \left(\frac{\bar{a}}{\bar{\bar{a}}}\right) \left(\frac{a}{\bar{a}}\right)^3 \left(\frac{r}{a}\right)^3 \left(\frac{\bar{\bar{a}}}{\bar{\bar{r}}}\right)^4 \left(\frac{5}{2}\cos^3 S_{13} - \frac{3}{2}\cos S_{13}\right) \tag{27}$$
$$+ \frac{1}{2} k_2 \left(\frac{\bar{a}}{\bar{\bar{a}}}\right) \left(\frac{a}{\bar{a}}\right)^2 \left(\frac{r}{a}\right)^2 \left(\frac{\bar{r}}{\bar{a}}\right) \left(\frac{\bar{\bar{a}}}{\bar{\bar{r}}}\right)^4 (3\cos S_{23}$$
$$\left. + 6\cos S_{13}\cos S_{12} - 15\cos^2 S_{13}\cos S_{23}) \right].$$

After $\cos S_{13}$ and $\cos S_{23}$ have been expressed in the true longitudes, nodal angles and inclinations of m_1, m_2 and m_3 (see [**13**], p. 34) the expansion of this disturbing function is obtained without great difficulties by referring to Cayley's tables for expressions as $(r/a)^p \cos nf$. The first and second terms of equation (27) do not contain the moon's coordinates and have therefore no long period terms with $(l - \bar{l})$ in the argument. The only long period terms are those in which neither l nor $\bar{l}$ appear. The third term of (27) does have the moon's coordinates but its coefficient is so small that even the first term of its expansion is of the fourth order, and this term is a short period term. The general form of the expansion of (27) is

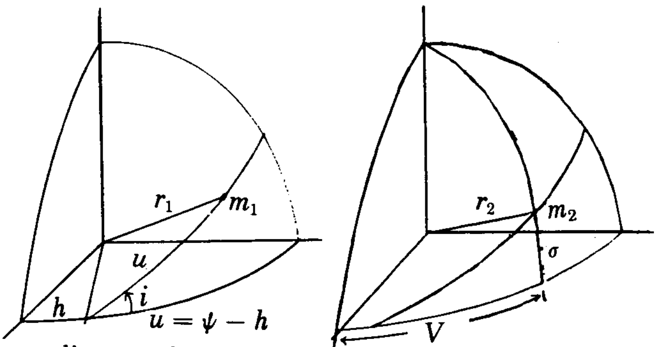
$$(m\bar{a}R)_{\text{sun}} = m_s \sum \mathrm{Coeff}(e, \gamma, x_1, \delta\tau, \bar{\bar{e}}, \bar{\bar{\gamma}}, \bar{\bar{a}}) \tag{28}$$
$$\times \cos(j_1 l + j_2 \bar{l} + j_3 \bar{\bar{l}} + j_4 \varpi + j_5 \bar{\varpi} + j_6 \bar{\bar{\varpi}} + j_7 h + j_8 \bar{h})$$

with

$$m_s = \frac{m_3}{m_0 + m_2} \left(\frac{\bar{a}}{\bar{\bar{a}}}\right)^3 \simeq .0052.$$

Terms of first order in eccentricities do now appear as long period as well as short period terms; there is even one short period term of zeroth order. If the development is carried out to second order in short periods and fourth order in long periods, there are 17 different short period arguments and 10 different long period arguments.

c. *Indirect Effect of the Sun.* In the development of the disturbing function for the main problem the moon's orbital elements were assumed to be constant. The indirect effect of the sun is taken care of by considering the elements of the moon's orbit as they are perturbed by the sun, that is by considering the actual motion of the moon. In the disturbing function due to the direct effect of the sun the moon's orbit is present with a very small coefficient so that there the elliptic values provide sufficient accuracy. Whereas here orbital elements are used to describe the motion of the moon, the best available information about the moon's actual motion uses coordinates. In E. R. Brown's *Theory of the motion of the moon* (Mem. Roy. Astronom. Soc. 57 (1906), 130-145), trigonometric sequences are given for $\sin(1/r_2)$ the sine of the parallax (i.e., the inverse of the earth-moon distance), v, the longitude in the fundamental plane and $s = \tan\sigma$, the tangent of the latitude (see Figure 2). The arguments contain the mean anomaly of the



coordinates of m_1 coordinates of m_2

FIGURE 2. Brown's Lunar Coordinates

moon, the longitudes of the moon's perigee and node and the mean longitude of the sun. These angles are to be taken as linear functions of the time, so that

$$
\begin{aligned}
&l = l_0 + \bar{n}t, && \bar{n} = 1,\\
&\varpi = \varpi_0 + n_\varpi t, && n_\varpi = 1/117.3159,\\
&\bar{h} = \bar{h}_0 + n_h t, && n_h = 1/246.5471,\\
&\bar{\bar{l}} = \bar{\bar{l}}_0 + \bar{\bar{n}}t, && \bar{\bar{n}} = 1/13.25575.
\end{aligned}
\tag{29}
$$

Brown's lunar coordinates consist of an elliptic part and a part due to the perturbations, as follows.

$$
\begin{gathered}
v = v_e + \delta v, \qquad \sigma = \sigma_e + \delta\sigma\\
\frac{1}{r_2} = \left(\frac{1}{r_2}\right)_e + \delta\left(\frac{1}{r_2}\right).
\end{gathered}
\tag{30}
$$

Brown's expressions are for v, $\tan\sigma$ and $\sin 1/r_2$, but because of the smallness of σ and $1/r_2$ the errors in putting $\delta\sigma = \delta s$ and $\delta(1/r_2) = \delta(\sin 1/r_2)$ will not show up in the development of the disturbing function. All the numerical coefficients of $\delta\sigma$ and $\delta(1/r_2)$ are of the second order or higher, and only one of the coefficients in δv is of first order, the others being of higher order. When the expressions (30) are substituted in the disturbing function of the main problem it, too, will consist of an elliptic part and a perturbed part. The expansion of the elliptic part is identical in form to the development which was obtained earlier, but for $\bar{l}$, $\bar{\varpi}$ and $\bar{h}$ the expressions (29) have to be used while the other elements, $\bar{e}$, $\bar{\gamma}$ and $\bar{a}$, are constant. The perturbed part of the disturbing function is best obtained by computing the additions to O_1, O_2, O_3 and O_4 (which are the first, second, third and fourth order parts of $\Delta^2/\bar{a}^2$) and to E_1, E_2 and E_3 (which are the various parts of $(\bar{a}/r_2)^3$) and substituting these in equation (25). The additions to $(\bar{a}/r_2)^3$ offer no difficulty, since only $\delta(1/r_2)$ is involved; the addition to E_1 is zero, because the largest term in $\delta(1/r_2)$ is of second order. The additions to $\Delta^2/\bar{a}^2$ are obtained from equation (4) as follows:

$$\begin{aligned}
\frac{\Delta^2}{\bar{a}^2} &= \left(\frac{\Delta^2}{\bar{a}^2}\right)_e + \delta\left(\frac{\Delta^2}{\bar{a}^2}\right) = \frac{r_1^2}{\bar{a}^2} + \frac{r_2^2}{\bar{a}^2} - \frac{2r_1r_2\cos S_{12}}{\bar{a}^2} \\
&= \left(\frac{\Delta^2}{\bar{a}^2}\right)_e + \left(\frac{r_{2e}}{\bar{a}}\right)^3 \left(\bar{a}\delta\left(\frac{1}{r_2}\right)\right) \left\{3\frac{r_{2e}}{\bar{a}}\delta\left(\frac{1}{r_2}\right) - 2\right\} \\
&\qquad - 2\frac{a}{\bar{a}}\frac{r_{2e}}{\bar{a}}\left(1 - \frac{r_{2e}}{\bar{a}}\bar{a}\delta\left(\frac{1}{r_2}\right)\right)\left\{\frac{r_1}{a}\delta\cos S_{12}\right. \\
&\qquad\qquad \left. - \frac{r_1}{a}\frac{r_{2e}}{\bar{a}}\bar{a}\delta\left(\frac{1}{r_2}\right)(\cos S_{12})_e\right\}
\end{aligned}$$

to high enough accuracy for the present purpose. The index e indicates the elliptic parts, the symbol δ indicates the perturbed parts. The addition to $\cos S_{12}$ is determined from the definition

$$\cos S_{12} = \mathbf{r}_1 \cdot \mathbf{r}_2/r_1r_2$$

in which now the angles i, ψ and h are used for $\mathbf{r}_1/r_1$ and the angles v and σ for $\mathbf{r}_2/r_2$, as follows:

$$\cos S_{12} = \begin{vmatrix} \cos h\cos u - \cos i\sin h\sin u \\ \sin h\cos u + \cos i\cos h\sin u \\ \sin i\sin u \end{vmatrix} \cdot \begin{vmatrix} \cos(\sigma_e + \delta\sigma)\cos(v_e + \delta v) \\ \cos(\sigma_e + \delta\sigma)\sin(v_e + \delta v) \\ \sin(\sigma_e + \delta\sigma) \end{vmatrix} .$$

After some algebra and the use of relations in spherical trigonometry to reintroduce the angles $\bar{\varpi}$, $\bar{h}$ and $\bar{i}$, the addition to $\cos S_{12}$ is found to be

$$\begin{aligned}
\delta \cos S_{12} = {} & \delta v \sin(\psi - \bar{\psi}) + \bar{\gamma}\delta\sigma \sin(\psi - 2\bar{\psi} + \bar{h}) \\
& - \bar{\gamma}\delta\sigma \sin(\psi - \bar{h}) + 2\gamma\delta\sigma \sin(\psi - h) \\
& + \delta v[-(\gamma^2 + \bar{\gamma}^2)\sin(\psi - \bar{\psi}) - \gamma^2 \sin(\psi + \bar{\psi} - 2h) \\
& + \bar{\gamma}^2 \sin(\psi + \bar{\psi} - 2\bar{h})] - \frac{1}{2}(\delta v)^2 \cos(\psi - \bar{\psi}).
\end{aligned} \tag{31}$$

The completion of the expansion presents no further difficulties. The development has been carried out to the third order for long period terms and to the second order for short period terms. In the short periods there are ten different arguments; the coefficients contain only one of the numerical coefficients in Brown's expressions for δv, besides the eccentricities (e and $\bar{e}$) and $\delta\tau$ (in two terms only). There are 15 different arguments in the long period terms; the coefficients contain two of the numerical coefficients from Brown's expression for δv and only one coefficient from the expression for $\delta(1/r_2)$. The angle $\delta\tau$ appears in six of the coefficients; the perturbation in latitude does not appear at all.

d. *General Appearance of the Development of the Disturbing Function*. The complete development of the disturbing function is the sum of the three developments which have just been outlined, those for the main problem, the direct effect of the sun and the indirect effect of the sun. When carried out to the second order (in eccentricities and inclinations and considering $\delta\tau$ of first order, x_1 of second order) there are 79 different arguments, including the argument zero. The arguments contain the eight quantities l, $\bar{l}$, $\bar{\bar{l}}$, ϖ, $\bar{\varpi}$, $\bar{\bar{\varpi}}$, h and $\bar{h}$ in various combinations; four of them, $\bar{l}$, $\bar{\bar{l}}$, $\bar{\varpi}$, and $\bar{h}$ are known linear functions of the time, their initial values being known for any initial time. The coefficients are functions of the variables e, i (through $\gamma = \sin \frac{1}{2} i$) and a (through $(a/\bar{a}) = (1 + x_1)^2$) and the corresponding numerically known quantities of the moon and sun; some of the coefficients also contain the angle $\delta\tau (= \tau - 60°)$. Some of the coefficients are simple functions, involving only a single variable; some are even purely numerical. A few of the coefficients, in particular that with zero argument (the "nonperiodic" term), are very complicated, but algebraic, expressions. For the further development it is important to recognize three kinds of terms. The classification of the terms depends on their arguments and is as follows.

Short period terms. Those in which l and $\bar{l}$ appear, but not in the combination $(l - \bar{l})$, so that $j_1 + j_2 \neq 0$.

Long period terms. Those in which l and $\bar{l}$ appear only in the combination $(l - \bar{l})$, so that $j_1 + j_2 = 0$, including the case where $j_1 = j_2 = 0$, but excluding the terms of the third class.

Libration terms. Those in which the argument is τ only, or multiples of τ; these include the term with argument zero (the "non-periodic" term).

Table I gives the number of terms in each classification, according to the lowest order of the separate terms in each coefficient; also indicated is the number of terms in which $\delta\tau$ does and does not appear in the coefficients.

TABLE I. Number of Terms in Each Classification

	ORDER				TOTAL
	0	1	2	3	
Short period					
$\delta\tau$ in coefficient		1	6		
No $\delta\tau$ in coefficient	1	3	31		42
Long period					
$\delta\tau$ in coefficient		0	7	7	
No $\delta\tau$ in coefficient		1	11	8	34
Libration					
$\delta\tau$ in coefficient			3		3
No $\delta\tau$ in coefficient					
					79

It must be noted that the nomenclature of "short" and "long" period is somewhat arbitrary. In some of the long period terms $j_1 = j_2 = 0$, $j_3 \neq 0$ and they have therefore a period which is only about 13.3 times longer than the short periods. Also, the libration term with argument has a period only 3 times longer than the short period, corresponding to the "long period" motion which was discussed in the preliminary work in rectangular coordinates.

VII. **Elimination of the short period terms.** The problem has been formulated in the canonical variables c_1, c_2, c_3, w_1, w_2 and w_3, with the Hamiltonian

$$H = \bar{n}\left(\frac{1}{2c_1^2} + m\bar{a}R\right),$$

where the disturbing function has the general form

$$m\bar{a}R = m\sum \mathrm{Coeff}(e, \gamma, \delta_T, x_1)\cos(j_1 l + j_2\bar{l} + j_3\bar{\bar{l}} + j_4\varpi + j_5\bar{\varpi} + j_6\bar{\bar{\varpi}} + j_7 h + j\bar{h}).$$

The relations between x, e, γ and c_1, c_2, c_3 and between l, ϖ, h and w_1, w_2, w_3 are according to equations (14) and (24).

The Hamiltonian contains the independent variable time explicitly because $\bar{l}$, $\bar{\varpi}$, $\bar{h}$ and $\bar{\bar{l}}$ are functions of the time by equations (29). This inconvenience is taken care of by introducing a fourth pair of variables

$$c_4 \text{ and } w_4 = \bar{n}t = \bar{l}.$$

The problem is then

$$\dot{c}_i = \frac{\partial F}{\partial w_i}, \quad \dot{w}_i = -\frac{\partial F}{\partial c_i}, \quad i = 1,2,3,4 \tag{32}$$
$$F = \bar{n}\left(\frac{1}{2c_1^2} - c_4 + m\bar{a}R\right)$$

in which now the Hamiltonian does not contain the independent variable explicitly any more. The general form of the argument is now, using the variables w_i,

$$j_1 w_1 + (j_2 + \bar{\nu}j_3 + \nu_\varpi j_5 + \nu_h j_8)w_4 + (j_4 - j_1)w_2 + j_7 w_3 + j_3\bar{\bar{l}}_0 + j_5\bar{\varpi}_0 + j_6\bar{\bar{\varpi}} + j_8\bar{h}_0, \tag{33}$$

where $\bar{\nu} = \bar{\bar{n}}/\bar{n}$, $\nu_\omega = n_\omega/\bar{n}$, $\nu_h = n_h/\bar{n}$.

The elimination of the short period terms will be accomplished by a canonical transformation. Let the generating function of this transformation be a power series in m,

$$S = S_0 + S_1 + S_2 + \cdots \tag{34}$$

the index indicating the power of m, and let S_0 be the generating function for the identity transformation

$$S_0 = c_1'w_1 + c_2'w_2 + c_3'w_3 + c_4'w_4,$$

where the new variables are indicated by primes. The relations between the old and the new variables are then

$$c_i = \frac{\partial S}{\partial w_i}, \quad w_i' = \frac{\partial S}{\partial c_i'}, \quad i = 1,2,3,4. \tag{35}$$

The new Hamiltonian may be written, also in the form of a power series in m, as

$$F^* = F_0^*(c_1', c_4') + F_1^*(c_i', w_i') + F_2^*(c_i', w_i') + \cdots.$$

Because the Hamiltonian does not contain the independent variable explicitly, the old and the new Hamiltonian are equal, so that

$$\begin{aligned} &F_0(c_1, c_4) + F_1(c_1, c_2, c_3, -, w_1, w_2, w_3, w_4) \\ &\qquad = F_0^*(c_1', c_4') + F_1^*(c_1', c_2', c_3', -, w_1', w_2', w_3', w_4') \end{aligned}$$

or, using equation (35),

$$\begin{aligned} &F_0\left(c_1' + \frac{\partial S_1}{\partial w_1} + \frac{\partial S_2}{\partial w_1}, \quad c_4' + \frac{\partial S_1}{\partial w_4} + \frac{\partial S_2}{\partial w_4}\right) \\ &\qquad + F_1\left(c_1' + \frac{\partial S_1}{\partial w_1}, \quad c_2' + \frac{\partial S_1}{\partial w_2}, \quad c_3' + \frac{\partial S_1}{\partial w_3}, -, w_1, w_2, w_3, w_4\right) \\ &\qquad = F_0^*(c_1', c_4') + F_1^*\left(c_1', c_2', c_3', \; w_1 + \frac{\partial S_1}{\partial c_1}, \; w_2 + \frac{\partial S_1}{\partial c_2}, \right. \\ &\qquad\qquad \left. w_3 + \frac{\partial S_1}{\partial c_3}, \; w_4 + \frac{\partial S_1}{\partial c_4}\right) + F_2^*(---). \end{aligned} \tag{36}$$

Expanding both sides of equation (36) in Taylor series and equating terms with equal powers of m produces the relations from which the generating function and the new Hamiltonian are to be determined. The zero order relation is

$$F_0^*(c_1', c_4') = F_0(c_1', c_4') = \frac{\bar{n}}{2c_1'^2} - \bar{n}c_4' \tag{37}$$

and gives immediately the zero order part of the new Hamiltonian. The first order relation is

$$\frac{\partial F_0}{\partial c_1'}\frac{\partial S_1}{\partial w_1}+\frac{\partial F_0}{\partial c_4'}\frac{\partial S_1}{\partial w_4}+F_1=F_1^*(c_1',c_2',c_3',-,w_1,w_2,w_3,w_4) \tag{38}$$

in which the notation $\partial F_0/\partial c_1'$ means $(\partial F_0/\partial c_1)_{c_1=c_1'}$. The second order relation is

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 F_0}{\partial c_1'^2}\left(\frac{\partial S_1}{\partial w_1}\right)^2+\left(\frac{\partial F_0}{\partial c_1'}\right)\frac{\partial S_2}{\partial w_1}+\frac{\partial F_0}{\partial c_4'}\frac{\partial S_2}{\partial w_4}&\\ +\frac{\partial F_1}{\partial c_1'}\frac{\partial S_1}{\partial w_1}+\frac{\partial F_1}{\partial c_2'}\frac{\partial S_1}{\partial w_2}+\frac{\partial F_1}{\partial c_3'}\frac{\partial S_1}{\partial w_3}&\\ =\frac{\partial F_1^*}{\partial w_1}\frac{\partial S_1}{\partial c_1'}+\frac{\partial F_1^*}{\partial w_2}\frac{\partial S_1}{\partial c_2'}+\frac{\partial F_1^*}{\partial w_3}\frac{\partial S_1}{\partial c_3'}+\frac{\partial F_1^*}{\partial w_4}\frac{\partial S_1}{\partial c_4'}+F_2^*.&\end{aligned} \tag{39}$$

If now F_1 is split in two parts

$$F_1=F_{1p}+F_{1s}, \tag{40}$$

equation (38) is satisfied by putting

$$F_1^*=F_{1s} \tag{41}$$

and

$$\frac{\partial F_0}{\partial c_1'}\frac{\partial S_1}{\partial w_1}+\frac{\partial F_0}{\partial c_4'}\frac{\partial S_1}{\partial w_4}=-F_{1p}. \tag{42}$$

Equation (41) defines the first order part of the Hamiltonian and equation (42) is a partial differential equation for the generating function S_1. Upon evaluation of $\partial F_0/\partial c_1'$ and $\partial F_0/\partial c_4'$ equation (42) becomes

$$n'\frac{\partial S_1}{\partial w_1}+\bar{n}\frac{\partial S_1}{\partial w_4}=F_{1p}, \tag{43}$$

where n' is symbolic for $\bar{n}/c_1'^3$.

In F_{1p} may be included all the short period terms, including those that have $\delta\tau$ in the coefficient. The new problem, with the Hamiltonian $F_1^*=F_{1s}$ has then only long period and libration terms; the short period terms have been "eliminated." The relation between the old and the new variables is completely specified by the determining function S_1 which is found by the integration of equation (43). In this integration no integration constants need be specified, since only the derivatives of S_1 are of interest.

With

$$\delta\tau = \tau - \tau_0 = w_1 - \overline{w}_1 - \tau_0 = w_1 - w_4 - \overline{w}_2 - \tau_0 \tag{44}$$

the general form of one term of F_{1p} is

$$\begin{aligned}\overline{n}(w_1 - w_4 - \overline{w}_2 - \tau_0)^i \cos\{ j_1 w_1 + (j_2 + \overline{\overline{\nu}} j_3 + \nu_\varpi j_5 + \nu_h j_8) w_4 \\ + (j_4 - j_1) w_2 + j_7 w_3 + j_3 \overline{\overline{l}}_0 + j_5 \varpi_0 + j_6 \overline{\overline{\varpi}} + j_8 \overline{h}_0 \},\end{aligned}$$

where i may have the values 0 or 1. For such a term the integral of equation (43) is for $i = 0$,

$$S_1 = \frac{\overline{n}}{n'} \frac{1}{p} \sin\{\quad\} \tag{45}$$

for $i = 1$,

$$S_1 = \frac{\overline{n}\delta\tau}{p} \sin\{\quad\} + \frac{\overline{n}(n' - \overline{n})}{p^2} \cos\{\quad\}, \tag{46}$$

where, of course, the arguments are unchanged and

$$p = n' j_1 + \overline{n} j_2 + \overline{\overline{n}} j_3 + n_\varpi j_5 + n_h j_8. \tag{47}$$

Although n' and $\overline{n}$ are nearly equal, the denominator p is never small, because only the short period terms have been included in F_{1p} so that $j_1 + j_2 \neq 0$.

The second order determining function S_2 is computed in similar manner by dividing equation (39) into two equations, one being a partial differential equation for S_2, the other determining the second order part, F_2^*, of the new Hamiltonian. The prodigious amount of labor to be expected by the appearance of equation (39) can be lessened considerably by the judicious choice of terms to be included. For instance, since this part of the work is concerned with contributions to the solution which have m^2 as a factor, it seems reasonable to include only zero and first order terms. The elimination of the short period terms is completed by (1) inverting equation (35) to express the old variables w_i in terms of the new w_i' and (2) expressing F_1^* and F_2^* as functions of the new variables w_i' (they were computed as functions of w_i). Both inversions can be performed by employing the Lagrange expansion theorem. In each case the first term of the new expression is obtained by simply switching the primes; it will probably be sufficient to compute only two terms.

It must still be noted that in the discussion of this work the canonical variables c_i, w_i are used, whereas the disturbing function has been expanded in terms of elliptic elements. In the actual execution of the work the transformation from elliptical elements to canonical variables does not need to be performed. In the first place, the arguments of the cosines in the expansion do not change and only the numerical coefficients of the w_i need to be considered for the computation of the coefficients in S_1 and S_2. Secondly, the variables x_1, e and γ present a far more efficient notation; it must only be remembered that they always stand for the following expressions in c_i:

$$x_1 = c_1 - 1 \left(= \sqrt{\frac{a}{\overline{a}}} - 1 \right), \quad e = \sqrt{\left(1 - \left(\frac{c_1 + c_2}{c_1} \right)^2 \right)},$$

(48)

$$\gamma = \sin \frac{1}{2} i = \sqrt{\left(- \frac{c_3}{2(c_1 + c_2)} \right)}.$$

Using (48) the partial derivatives to c_i can be computed in terms of partials to x, e and γ.

$$\frac{\partial}{\partial c_1} = \frac{\partial}{\partial x_1} - 3c_1^{-4} \frac{\partial}{\partial v} + \frac{1 - e^2 - \sqrt{(1 - e^2)}}{ec_1} \frac{\partial}{\partial e} - \frac{\gamma}{2c_1\sqrt{(1 - e^2)}} \frac{\partial}{\partial \gamma}$$

(49)
$$\frac{\partial}{\partial c_1} = - \frac{\sqrt{(1 - e^2)}}{ec_1} \frac{\partial}{\partial e} - \frac{\gamma}{2c_1\sqrt{(1 - e^2)}} \frac{\partial}{\partial \gamma}$$

$$\frac{\partial}{\partial c_3} = \frac{-1}{4\gamma c_1 \sqrt{(1 - e^2)}} \frac{\partial}{\partial \gamma}.$$

VIII. **Elimination of long period terms.** Because the new Hamiltonian

$$F^* = \frac{\overline{n}}{2c_1'^2} - \overline{n} c_4' + F_1^*(c_1', \cdots, w_1', \cdots)$$

contains w_1' and w_4' only in the combination $j_1(w_1 - w_4)$, a new variable is introduced

(50) $$y_1 = w_1' - (w_4' + \overline{\omega}) = l' - \overline{l} + \omega' - \overline{\omega}.$$

Let further

(51) $$y_2 = w_2', \quad y_3 = w_3', \quad y_4 = w_4'$$

and let x_1, x_2, x_3, x_4 be the variables conjugate to y_1, y_2, y_3, y_4. The transformation from c_i', w_i' to x_i, y_i is canonical if

$$c_1' = 1 + x_1, \quad c_2' = x_2, \quad c_3' = x_3, \quad c_4' = x_4 - (1 + x_1). \tag{52}$$

In writing $a/\overline{a} = (1 + x_1)^2$, the x_1 was of course introduced in anticipation of this last change of variables.

The problem is now with the canonical variables x_i, y_i, $i = 1, 2, 3, 4$ and the Hamiltonian

$$F^* = \frac{\overline{n}}{2(1 + x_1)^2} + \overline{n}(1 + x_1) - \overline{n}x_4 + F_1^* + F_2^*. \tag{53}$$

The general form of the arguments in F_1^* and F_2^* is

$$\begin{aligned}[j_1y_1 + (j_4 - j_1)\, y_2 + j_7y_3 + \{ j_3\overline{\overline{\nu}} + (j_1 + j_5)\, \nu_\varpi + j_8\nu_h \}\, y_4 \\ + j_3\overline{\overline{l}}_0 + (j_1 + j_5)\overline{\varpi}_0 + j_6\overline{\overline{\varpi}} + j_8\overline{h}_0].\end{aligned} \tag{54}$$

The method which was so successful in eliminating the short period terms will not quite take care of the long period terms. It is instructive to see why and how the method fails; this may help in formulating other approaches to deal with the long period terms and it will lead into the discussion of the equations of libration and semi-major axis.

Let the Hamiltonian F_1^* be divided in two parts F_{1s}^* and F_{1p}^*, in a way still to be determined. Let S_1^* be the generating function of a new canonical transformation from x_i, y_i to x_i', y_i'. Then, in precisely the same way as this was done in the previous section, a new Hamiltonian is determined by

$$F^{**} = F_0^* + F_{1s}^* \tag{55}$$

and the generating function S_1^* follows from the partial differential equation

$$\frac{\partial F_0^*}{\partial x_1'}\frac{\partial S_1^*}{\partial y_1} + \frac{\partial F_0^*}{\partial x_4'}\frac{\partial S_1^*}{\partial y_4} = F_{1p}^*. \tag{56}$$

From the general form of the arguments in F_{1p}^* (see (54)) it can be seen that the general form of terms in S_1^* is

$$S_1^* = \frac{\overline{n}\sin\{\quad\}}{(n'' - \overline{n})\, j_1 + \overline{\overline{n}}j_3 + n_\varpi(j_1 + j_5) + n_hj_8}, \tag{57}$$

where $n'' = (1 + x_1')^{-3}\overline{n}$.

The coefficient of j_1 is always small since the periods of m_1 and m_2 are quite close. The other terms in the denominator are also small, with $\bar{\bar{n}} = 1/13.4$, $n_{\varpi} = 1/117$ and $n_h = 1/248$. The smallness of the denominator may cancel two orders of magnitude; this is of course an indication of the difficulty caused by the long period terms of the disturbing function and it is the reason for carrying out the development of the long period terms to a higher order than the short period terms.

The canonical transformation under discussion here may still be successful for the terms in F_1^* in which $j_3 \neq 0$, because $\bar{\bar{n}}$ is not really very small. It is also conceivable that it may be used (probably at the expense of developing the disturbing function to higher order) for the terms in which $j_1 + j_5 \neq 0$ or $j_8 \neq 0$. This must still be investigated with great care; it is already clear that the second order part of the new Hamiltonian, F_2^{**}, must be computed and that the inversions required to express the old angular variables (as discussed before, relating to the short period terms) will cause additional difficulties with small denominators, especially in the case of y_1.

The terms in F_1^* for which this method is entirely powerless are those in which $j_3 = j_1 + j_5 = j_8 = 0$. Those are of course just the terms which were classified as "libration terms"; their treatment is the subject of the following section.

IX. **The equations of the libration.** In the previous section it was shown that the elimination of the long period terms by the same method as was used for the short period terms is difficult. It will not do to neglect this difficulty, which is quite basic to the whole problem, but in order to discuss the most fundamental aspects of the problem it will now be assumed that the long period terms have been eliminated by some canonical transformation. The problem which is then left consists of canonical equations in the variables x_i', y_i', $i = 1, 2, 3, 4$, with the Hamiltonian

$$F^{**} = \frac{\bar{n}}{2(1 + x_1')^2} + \bar{n}(1 + x_1') - \bar{n}x_4 \tag{58}$$
$$+ F_1^{**}(x_1', e'', \gamma'', y_1', -, -, -).$$

The first order part, F_1^{**}, consists of the terms in the disturbing function which previously have been identified as "libration terms," with double primes attached to their variables. As before, the variables e and γ, even though noncanonical, have been kept in favor of the canonical variables and according to previously introduced transformations

$$x_1' = \left(\frac{a}{\bar{a}}\right)^2 - 1$$

and

$$y_1' = w_1'' - (w_4'' + \bar{\varpi}).$$

It turns out that the new Hamiltonian, at least up to its first order part, is independent of y_2', y_3' and y_4'. Three pairs of the canonical equations can thus be integrated immediately, resulting in constant eccentricity and inclination and linear function for the apsidal and nodal angles, as follows.

$$x_2' = x_{2_0}', \quad y_2' = y_{2_0}' - \frac{\partial F^{**}}{\partial x_2'} t \tag{59}$$

$$x_3' = x_{3_0}', \quad y_3' = y_{3_0}' - \frac{\partial F^{**}}{\partial x_3'} t \tag{60}$$

and (trivially)

$$x_4' = x_{4_0}', \quad y_4' = -\frac{\partial F^{**}}{\partial x_4'} t = \bar{n} t = \bar{l}.$$

What is left is a one-dimensional problem in the variables x_1' (related to the semi-major axis) and y_1', the libration. With equations (14) and (52), considering the smallness of the eccentricity and the inclination and observing that these variables appear in F_1^{**} only squared, it follows that

$$e''^2 = 1 - \left(\frac{1 + x_1' + x_2'}{1 + x_1'}\right)^2 \cong -2 x_2' \left(1 - x_1' + \frac{1}{2} x_2'\right) \tag{61}$$

$$\gamma''^2 = \frac{-x_3'}{2(1 + x_1' + x_2')} \cong -\frac{1}{2} x_3' (1 - x_1' - x_2') \tag{62}$$

to sufficient accuracy.

F^{**} can thus be written easily as a function of x_1', x_2', x_3' and expanded in powers of y_1'. For convenience in notation the primes will now be dropped and in particular x_1' and y_1' will be written as x and y. The result is

$$F^{**} = \bar{n}\left[v_0 x + \frac{3}{2} m_1 x^2 - 2x^3 + U + xV\right], \tag{63}$$

where U and V are power series in y,

$$U = u_0 + u_1 y + u_2 y^2 + u_3 y^3 + \cdots$$
$$V = v_1 y + v_2 y^2 + v_3 y^3 + \cdots$$

the coefficients $u_0, u_1, \cdots, v_0, v_1, \cdots$ are functions of x_2 and x_3 only, and $m_1 = 1 - 3m + m_s$.

The equations for x and y are thus

$$\frac{dx}{\bar{n}dt} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} x \tag{64}$$

$$\frac{dy}{\bar{n}dt} = -(v_0 + 3m_1 x - 6x^2 + V). \tag{65}$$

If the first is substituted in the derivative of the second, there follows

$$\frac{d^2 y}{\bar{n}^2 dt^2} + (\ )\, y = (\ )\, xy + \text{terms with } x^2, y^2, y^3,$$

which is mainly a harmonic equation for y, as expected. The coupling with x, as presented by the first term of the right-hand side can be removed by a transformation of the time. Let the transformed time be

$$t_1 = \bar{n}t + \beta y \tag{66}$$

let also, for brevity, $(v_0 + 3m_1 x - 6x^2 + V) = f(t_1)$.

Equations (64) and (65) then become

$$\frac{dx}{dt_1} = \frac{\dfrac{\partial U}{\partial y} + \dfrac{\partial V}{\partial y}}{1 - \beta f}$$

$$\frac{dy}{dt_1} = \frac{-f}{1 - \beta f}.$$

Substitution of the first equation into the derivative of the second results in

$$\frac{d^2y}{dt_1^2} = -\frac{3}{(1-\beta f)^3}\left[(m_1 - 4x)\frac{\partial U}{\partial y} - \frac{1}{3}(v_0 + V + 6x^2)\frac{\partial V}{\partial y}\right] \tag{67}$$

the right-hand side of which can be worked out in terms of $u_0, u_1, u_2, \cdots, v_0, v_1, v_2, \cdots$. If now

$$\beta = \frac{4}{9m_1^2}$$

the variable x disappears from the coefficient of y. The libration equation becomes then

$$\frac{d^2y}{dt_1^2} + \frac{27}{4} mm_1 y = U_0 + U_1 y + U_2 y^2 + U_3 y^3 + \cdots. \tag{68}$$

The most important parts of the coefficients are

$$U_0 = \frac{3}{2}\sqrt{3}\, mm_s$$

$$U_1 = \frac{27}{4} m^2 + \frac{13}{4} mm_s$$

$$U_2 = \frac{81\sqrt{3}}{16} m - \frac{297\sqrt{3}}{16} m^2 + \frac{81\sqrt{3}}{16} mm_s$$

$$U_3 = -\frac{297}{32} m + \frac{213}{8} m^2 - \frac{297}{32} mm_s.$$

The appearance of the square of the small parameter is due to the time transformation which required the multiplication with the factor $(1 - \beta f)^{-3}$ (see equation (67)). The variable x appears in the coefficients U_i only with the square of the mass ratio as coefficient. The other variables of the problem are more prominent, but at this stage of the work they are all constants.

The solution of (68) begins with the first approximation, obtained by putting the right-hand side equal to zero,

$$y_0 = b\cos(\nu t_1 + \phi_0) \tag{69}$$

with $\nu^2 = (27/4)\, mm_1$. The integration constants are b and ϕ_0. For the second approximation the terms with coefficients U_0 and U_2 are introduced. The result is

$$y_1 = \frac{U_0}{\nu^2} + \frac{1}{2} U_2 \frac{b^2}{\nu^2} - \frac{1}{6}\frac{U_2}{\nu^2} b^2 \cos 2(\nu t_1 + \phi_0). \tag{70}$$

When this is substituted in the terms with coefficients U_1 and U_3, there will be a term with $\cos(\nu t_1 + \phi_0)$ in the right-hand side of the equation for the third approximation. This term is taken care of by introducing it in the left-hand side of the equation for the first approximation. The libration frequency is thus changed, and such a change must be made with every odd numbered approximation. The third approximation produces

$$\nu_1^2 = \nu^2 - \left[U_1 + 2\frac{U_0 U_2}{\nu^2} + \frac{5}{6}\frac{U_2^2}{\nu^2} b^2 + \frac{3}{4} U_3 b^2 \right] \tag{71}$$

and

$$y_2 = \frac{1}{6}\left[\frac{U_2^2}{\nu_1^2}\frac{b^3}{9\nu_1^2} + \frac{1}{4}\frac{U_3 b^3}{9\nu_1^2} \right] \cos 3(\nu_1 t_1 + \phi_0). \tag{72}$$

The magnitude of the coefficients $U_0, U_1, \cdots$ appears to be such that this process may be continued to obtain any desired degree of accuracy. The appearance of the libration amplitude b in the expression for ν_1 shows how the libration frequency depends on the amplitude. When a solution for y is obtained, x can be computed relatively easily; the differential equations for x and y are coupled, but because of the time transformation this coupling is at low enough order to expect that it introduces no great difficulties.

X. **Summary of major results.** The major results follow from the work on the libration equations which was outlined in the previous section. The variables x_2' and x_3', related to the eccentricity and inclination respectively, are constant. The actual eccentricity and inclination will show long period and short period variations. The short period variations have been found as a result of the elimination of the short period terms. The long period variations have not yet been determined. Short period and long period variations must also be added to linear functions y_2' and y_3' in order to find the actual motion of apse and node. It was found that

$$y_2' = y_{2_0}' - \frac{\partial F^{**}}{\partial x_2'} t$$

and

$$y_3' = y_{3_0}' - \frac{\partial F^{**}}{\partial x_3'} t,$$

where y'_{2_0} and y'_{3_0} are integration constants.

The rate of the apsidal advance is, in first instance,

$$\frac{\partial F^{**}}{\partial x'_2} = \bar{n}m \left[\left(-\frac{27}{8} - \frac{3}{4}\frac{m_s}{m} - \frac{1419}{64}\bar{e}^2 \right) + \left(\frac{10971}{64} - \frac{3}{4}\frac{m_s}{m} \right) x'_2 \right].$$

The effect of the sun, and the effect of the moon's orbital eccentricity are evident; they are of about the same magnitude and together decrease the time required for one revolution of the apse line from about 24 ($= 8/27m$ anomalistic months to about 23.5 months. Also, there is an important effect from the satellite's eccentricity; the coefficient of x'_2 ($= -(1/2)\bar{\bar{e}}^2$) is about 173, as compared to 1419/64, the coefficient of $\bar{e}^2$.

The rate of regression of the nodal line is

$$\frac{\partial F^{**}}{\partial x'_3} = \bar{n}m \left[\frac{3}{4}\frac{m_s}{m} - \frac{3}{4}\frac{m_s}{m} x'_2 \right]$$

and is thus, at least to this order of approximation, due to the sun's effect. The time required for one revolution is about 256 anomalistic months. The libration y'_1 and the variation of the semi-major axis x'_1 have been expressed as functions of a time-related variable t_1 which includes the libration itself as follows:

$$t_1 = t + \frac{4}{9m_1^2} y'_1, \quad m_1 = 1 - 3m + m_s.$$

The real time can be reintroduced later by inversion, using the Lagrange expansion theorem.

It was found that the libration is given by

$$y'_1 = b\cos(\nu_1 t_1 + \phi_0) + \frac{U_0}{\nu_1^2} + \frac{1}{2} U_2 \frac{b^2}{\nu_1^2} - \frac{1}{6}\frac{U_2}{\nu_1^2} b^2 \cos 2(\nu_1 t_1 + \phi_0)$$

$$+ \left[\frac{1}{6}\frac{U_2^2}{\nu_1^2}\frac{b^3}{9\nu_1^2} + \frac{1}{4}\frac{U_3 b^3}{9\nu_1^2} \right] \cos 3(\nu_1 t_1 + \phi_0),$$

where b and ϕ_0 are integration constants. The coefficients U depend especially on the mass ratios m and m_s (see the previous section) and contain also the other constants of the problem. The frequency ν_1 is given by

$$\nu_1^2 = \nu^2 - \left[U_1 + \frac{2U_0U_2}{\nu^2} + \frac{5}{6}\frac{U_2^2}{\nu^2} b^2 + \frac{3}{4} U_3 b^2 \right], \quad \nu^2 = \frac{27}{4} m_1 m$$

and decreases with increasing libration amplitude. The variation of the semi-major axis is, in first approximation,

$$x_1' = \frac{b\nu_1}{3m_1} \sin(\nu_1 t_1 + \phi_0).$$

The coefficient $\nu_1/3m_1$ is about $1/10$; it was thus indeed reasonable to assume that x_1 is of the order of .01 if the libration amplitude is of the order $1/10$.

References

1. J. L. Lagrange, *Oeuvres complètes,* Vol. VI, Gauthiers-Villars, Paris, 1867-1892, p. 277.

2. C. V. L. Charlier, *Meddelanden frän Lunds Astronomika Observatoricum,* No. 18.

3. H. C. Plummer, *On periodic orbits in the neighborhood of centers of libration,* Monthly Notices Roy. Astronom. Soc. **62** (1901), 6-17.

4. E. W. Brown, *On a new family of periodic orbits in the problem of three bodies,* Monthly Notices Roy. Astronom. Soc. **71** (1911), 438-454.

5. ———, *On the oscillating orbits about the triangular libration points in the problem of three bodies,* Monthly Notices Roy. Astronom. Soc. **71** (1911), 492-502.

6. H. R. Willard, *On a family of oscillating orbits of short periods,* Monthly Notices Roy. Astronom. Soc. **73** (1913), 471-484.

7. L. J. Linders, Ark. Mat. Astronom. Och. Fys. So. Vet. Ak. i Stockholm **4** (1908).

8. W. Smart, *Libration of the Trojan planets.* I, Mem. Roy. Astronom. Soc. **62** (1918), 79-112.

9. E. W. Brown, *The general orbits of the asteroids of the Trojan group,* Astronom. J. **35** (1923), 69-80; 124.

10. ———, *Theory of the Trojan group of asteroids,* Trans. Yale Observatory **3** (1926), 1-47; 87-133.

11. D. Brouwer, *Theory and tables of the motion of* (588) *Achilles,* Trans. Yale Observatory **6** (1933), Part VII, 177-188.

12. W. J. Eckert, *The general orbit of Hector,* Trans. Yale Observatory **6** (1933), Part VI, 161-172.

13. E. W. Brown and C. A. Shook, *Planetary theory,* Cambridge Univ. Press, Cambridge, England, 1933; Chapter IX.

14. H. G. Herz, *On the theory of Trojan asteroids,* Astronom. J. **50** (1943), 121-125.

15. A. Wilkins, *Methoden zur Ermittlung der Speziellen und der Absoluten Koordinatenstörungen der Planeten der Jupitergruppe durch Jupiter,* Astronom. Nachr. **205** (1917), 145-156.

16. ———, *Ueber die Integration der Differentialgliechungen der Koordinatenstörungen der Planeten der Jupitergruppe,* Astronom. Nachr. **206** (1918), 149-156.

17. ———, *Zur Theorie der Librationen im Sonnensystem,* Astronom. Nachr. **227** (1926), 337-348.

18. ———, *Untersuchungen zur Theorie der Jupitergruppe,* Astronom. Nachr. **245** (1932), 229-264.

19. B. Thuring, *Ueber die Planeten von Kommensurabilitats Typus,* 1:1, Astronom. Nachr. **238** (1930), 357-369.

20. ———, *Die Librationsperiode der Trojaner in ihrer Abhängigkeit von der Librationsamplitude*, Astronom. Nachr. **243** (1931), 181.

21. ———, *Numerische Untersuchungen zu dem Bewegungstheorie der Planeten der Jupitergruppen*, Astronom. Nachr. **279** (1950), 217-230.

22. ———, *Programmgesteuerte Berechnung von Librationsbahnen*, Astronom. Nachr. **285** (1959), 71-80.

23. E. Rabe, *Determination and survey of periodic Trojan orbits in the restricted problem of three bodies*, Astronom. J. **66** (1961), 500-513.

24. ———, *Additional periodic Trojan orbits and further studies of their stability features*, Astronom. J. **67** (1962), 382-390.

25. E. Rabe and A. Schanzle, *Periodic librations about the triangular solutions of the restricted earth-moon problem and their orbital stabilities*, Astronom. J. **67** (1962), 732-739.

26. K. Stumpff, *On Lagrange's theory of the three body problem*, NASA, TN D-1417, Jan. 1963.

27. W. B. Klemperer and E. T. Benedikt, *Selenoid satellites*, Astronaut. Acta. **4** (1958), 25-30; also, Proc. VIII International Astronaut. Congress, Springer, Berlin, 1958.

28. Anonymous, *More about the earth's cloud satellites*, Sky and Telescope (August 1961), 63.

29. E. T. Benedikt, *Exact determination of the lunar mass by means of selenoid satellites*, Nature **192** (1961), 442-443.

30. G. Colombo and E. T. Benedikt, *Letters to the editor*, Nature **193** (1962), 1063.

31. L. Sehnal, *The stability of the libration points L_4 and L_5 in the system earth-moon*, Bull. Astronom. Inst. Czechoslovakia (4) **11** (1960), 130-131.

32. G. Colombo, *The stabilization of an artificial satellite at the inferior conjunction point of the earth-moon system*, Smithsonian Institution, SAO Special Report, No. 80, Nov. 1, 1961.

33. G. Colombo, D. Lautman and C. Munford, *On the librations of a particle near the triangular point in the semi-restricted three body problem*, Smithsonian Institution, SAO Special Report, No. 108, Nov. 20, 1962.

34. G. A. Ellis, *Kenology*, Rome Air Development Command, Technical Memorandum RCLS-TM-60-2, Sept. 1960.

35. P. Pedersen, *On the periodic orbits in the neighborhood of the triangular equilibrium points in the restricted problem of three bodies*, Monthly Notices Roy. Astronom. Soc. **94** (1933), 167-185.

36. G. A. Ellis and A. C. Diana, *Motion of a satellite about the earth-moon sextile points*, Rome Air Development Command, Technical Memorandum RCLS-TM-60-1, Sept. 1960.

37. J. P. de Vries and W. M. Pauson, *A study of motion near libration points in the earth-moon system*, General Electric Company, TIS R 62SD5, Jan. 1962 (Class III).

38. J. P. de Vries, *The sun's perturbing effect on motion near a triangular Lagrange point*, Proc. of the XIII Internat. Astronautical Cong. (Varna, Bulgaria, Sept. 23-29, 1962) Springer, 1964, pp. 432-450.

39. W. H. Michael, Jr., *Considerations of the motion of a small body in the vicinity of the stable libration points of the earth-moon system*, NASA, TR R-160, 1963.

40. J. E. Littlewood, *On the equilateral configuration in the restricted problem of three bodies*, Proc. London Math. Soc. (9) **3** (1959), 343-372.

41. ———, *The Lagrange configuration in celestial mechanics*, Proc. London Math. Soc. (9) **3** (1959), 525-543.

42. A. M. Leontovic, *On the stability of the Lagrange periodic solutions of the reduced problem of the three bodies*, Dokl. Akad. Nauk SSSR **143** (1962), 525-528 = Soviet Math. Dokl. **3** (1962), 425-428.

43. E. Rabe, *Outline of a theory of nonperiodic motions in the neighborhood of the long-period librations about the equilateral points of the restricted problem of the three bodies*, Lectures in Applied Mathematics, Vol. 6, pp. 79-101, Amer. Math. Soc., Providence, R. I., 1966.

44. ———, *Elements of a theory of librational motions in the elliptical restricted problem*, Lectures in Applied Mathematics, Vol. 6, pp. 102-118, Amer. Math. Soc., Providence, R. I., 1966.

45. D. Brouwer and G. M. Clemence, *Methods of celestial mechanics*, Academic Press, New York, 1961.

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The Dominant Features of the Long-Period Librations of the Trojan Minor Planets

I. **The Trojan minor planets.** The problem of three bodies possesses a class of solutions in which the bodies move so that the triangle they define is always equilateral, as was shown by Lagrange. This type of solution found application in the study of the solar system with the discovery of minor planets moving so as to approximate such a configuration with the Sun and Jupiter. These planets are known as the "Trojan planets," and are names after heroes of the Trojan War. The present treatment seeks to present the long period features of motion in the vicinity of the equiangular triangle configurations, making use of the elements of an osculating orbit, and methods taken from the work on the motion of these planets by W. M. Smart (Mem. Roy. Astronom. Soc. **62** (1918); part 3), and H. G. Hertz (Astronom. J. **50** (1943), 121), taking into account only the gravitational attractions of the Sun and Jupiter, which of course dominate the motion.

II. **The equations of motion.** Consider the system comprising the bodies S and J, of masses m_S and m_J, and position vectors ρ_S and ρ_J in an inertial frame, and a third body P with position vector ρ_P, which has no attraction on the other two. The equations of motion are

$$\ddot{\rho}_S = Gm_J\frac{(\rho_J - \rho_S)}{(r')^3},$$

$$\text{(1)} \qquad \ddot{\rho}_J = Gm_S\frac{(\rho_S - \rho_J)}{(r')^3},$$

$$\ddot{\rho}_P = Gm_S\frac{(\rho_S - \rho_P)}{r^3} + Gm_J\frac{(\rho_J - \rho_P)}{\Delta^3},$$

where $r' = |\rho_J - \rho_S|, r = |\rho_S - \rho_P|, \Delta = |\rho_J - \rho_P|$. We use the relative position vectors

$$\text{(2)} \qquad \boldsymbol{r} = \rho_P - \rho_S, \quad \text{and} \quad \boldsymbol{r}' = \rho_J - \rho_S,$$

and the first two of (1) give

$$\text{(3)} \qquad \ddot{\boldsymbol{r}}' = -\frac{\mu \boldsymbol{r}'}{(r')^3},$$

where

$$\text{(4)} \qquad \mu = G(m_S + m_J).$$

This is the equation of the Keplerian two body problem, and we suppose that its solution is an ellipse of major semi-axis a', and eccentricity e', which is the orbit of J relative to S. The first and third equations give the equation for the relative motion of P and S as

$$\text{(5)} \qquad \ddot{\boldsymbol{r}} = -\frac{Gm_S\boldsymbol{r}}{r^3} + \frac{Gm_J(\boldsymbol{r}' - \boldsymbol{r})}{\Delta^3} - \frac{Gm_J\boldsymbol{r}'}{(r')^3}.$$

Now in the equiangular triangle configuration, the orbit of P relative to S is identical in size, shape and period to that of J relative to S, and therefore is a solution of the equation

$$\text{(6)} \qquad \ddot{\boldsymbol{r}} = -\frac{\mu \boldsymbol{r}}{r^3}.$$

So we rewrite (5) in the form

$$\text{(7)} \qquad \ddot{\boldsymbol{r}} = -\frac{\mu \boldsymbol{r}}{r^3} + \operatorname{grad} R,$$

where

$$\text{(8)} \qquad R = \mu m' \left\{ \frac{1}{\Delta} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{(r')^3} - \frac{1}{r} \right\},$$

with

$$(9) \qquad m' = \frac{m_J}{m_S + m_J}.$$

The solution of (6) will be regarded as the osculating orbit of P, and R is therefore the disturbing function for the action of J on P. Now if σ is the angle subtended by P and J at S, we have $\boldsymbol{r} \cdot \boldsymbol{r}' = rr' \cos\sigma$, and $\Delta^2 = r^2 + (r')^2 - 2rr' \cos\sigma$. From these we find that

$$\frac{\partial R}{\partial r} = \mu m' \left\{ -\frac{1}{\Delta^3}(r - r' \cos\sigma) + \frac{1}{r^2} - \frac{\cos\sigma}{(r')^2} \right\},$$

and

$$\frac{\partial R}{\partial(\cos\sigma)} = \mu m' \left\{ \frac{rr'}{\Delta^3} - \frac{r}{(r')^2} \right\}.$$

Both of these vanish if S, J and P form an equiangular triangle since then $r = r' = \Delta$, and $\sigma = \pi/3$. Therefore, since R only depends on the position of P through its dependence on r and $\cos\sigma$, grad R vanishes while such a configuration holds, and the motion of P is governed by equation (6). But one solution of this is the elliptical orbit identical with that of J, but oriented at $\pi/3$ to it in such a way that the equilateral configuration of SJP is always preserved, and this is therefore a solution of the original equations, confirming Lagrange's result for the case of the three body problem here considered.

We suppose the motion of P to take place entirely in the plane of the orbit of J, in which the true longitudes of P and J are ψ and ψ', respectively, and their mean longitudes are λ and λ', respectively. If the elements of the osculating orbit of P are a, e, ϖ, ϵ, then $\lambda = nt + \epsilon$, where $\mu = n^{1/2}a^{-3/2}$, and we use variables

$$\delta a = a - a',$$

$$(10) \qquad \phi = \lambda - \lambda',$$

$$k = e \cos\varpi$$

and

$$h = e \sin\varpi,$$

which satisfy the equations, derived easily from the Lagrange equations for the elements,

$$\frac{d}{dt}(\delta a) = \frac{2}{na}\frac{\partial R}{\partial \phi},$$

(11) $$\frac{d\phi}{dt} = n - n' - \frac{2}{na}\frac{\partial R}{\partial(\delta a)} + \frac{B}{2na^2}\left(k\frac{\partial R}{\partial k} + h\frac{\partial R}{\partial h}\right),$$

$$\frac{dk}{dt} = -\frac{A}{na^2}\frac{\partial R}{\partial h} - \frac{B}{2na^2}k\frac{\partial R}{\partial \phi},$$

and

$$\frac{dh}{dt} = \frac{A}{na^2}\frac{\partial R}{\partial k} - \frac{B}{2na^2}h\frac{\partial R}{\partial \phi},$$

where

(12) $$A = \sqrt{(1-e^2)} = 1 - \frac{1}{2}(k^2 + h^2) + O(k^4, h^4, k^2h^2),$$

and

(13) $$B = \frac{2}{e^2}\{\sqrt{(1-e^2)} - 1 + e^2\} = 1 - \frac{1}{4}(k^2 + h^2) + O(k^4, h^4, k^2h^2).$$

III. **The disturbing function.** The disturbing function takes the form

(14) $$R = \mu m'\left\{\frac{1}{\Delta} - \frac{r}{(r')^2}\cos(\psi - \psi') - \frac{1}{r}\right\}.$$

We expand it, making use of the following expressions, in which $M = \lambda - \varpi$ is the mean anomaly,

$$r = a\left\{1 + \frac{1}{2}e^2 - e\cos M - \frac{1}{2}e^2\cos 2M + O(e^3)\right\},$$

(15) $$r\cos(\psi - \varpi) = a\left\{-\frac{3}{2}e + \left(1 - \frac{3}{8}e^2\right)\cos M + \frac{1}{2}e\cos 2M + \frac{3}{8}e^2\cos 3M + O(e^3)\right\},$$

$$r\sin(\psi - \varpi) = a\left\{\left(1 - \frac{5}{8}e^2\right)\sin M + \frac{1}{2}e\sin 2M + \frac{3}{8}e^2\sin 3M + O(e^3)\right\};$$

and their counterparts for J. Making use of these expansions, we find for the secular and long-period part of R, that is, the part which does not involve λ or λ', and the part which involves them only in the slowly varying combination $\phi = \lambda - \lambda'$, respectively,

$$\bar{R} = \frac{\mu m'}{a'}\left\{\frac{1}{\sqrt{(2(1-\cos\phi))}} - \cos\phi + X\right\}, \tag{16}$$

where

$$\begin{aligned} X = {} & \left(\frac{\delta a}{a'}\right)\left\{1 - \cos\phi - \frac{1}{2\sqrt{(2(1-\cos\phi))}}\right\} \\ & + \left(\frac{\delta a}{a'}\right)^2\left\{-1 + \frac{3}{8\sqrt{(2(1-\cos\phi))}} - \frac{1}{4\sqrt{2}(1-\cos\phi)^{3/2}}\right\} \\ & + g_1(\phi)(k^2+h^2) + g_2(\phi)(kk'+hh') + g_3(\phi)(hk'-kh'), \end{aligned} \tag{17}$$

where

$$\begin{aligned} g_1(\phi) &= \frac{7}{8\sqrt{2}(1-\cos\phi)^{3/2}} - \frac{5}{16\sqrt{2}\sqrt{(1-\cos\phi)}} + \frac{1}{2}\cos\phi, \\ g_2(\phi) &= -\frac{7}{4\sqrt{2}(1-\cos\phi)^{3/2}} + \frac{11}{8\sqrt{2}\sqrt{(1-\cos\phi)}} \\ &\qquad - \frac{\sqrt{(1-\cos\phi)}}{8\sqrt{2}} - \cos 2\phi, \\ g_3(\phi) &= \left\{-\frac{5}{4\sqrt{2}(1-\cos\phi)^{3/2}} + \frac{1}{8\sqrt{2}(1-\cos\phi)^{1/2}} \right. \\ &\qquad \left. - 2\cos\phi\right\}\sin\phi. \end{aligned} \tag{18}$$

Terms of the third and higher degrees in $\delta a/a'$, k, h, k' and h' have been neglected.

IV. **The relative equilibrium solutions.** We suppose that the long period part of the problem has been separated from the short period part by Von Zeipel's transformation or an equivalent procedure, and proceed to solve the equations for the mean and long period parts of the elements. The transformation will add to the disturbing function terms proportional to $(m')^2$ and higher powers of m', but we will work now only to the first order in m'. The equations then take the form

$$
\begin{aligned}
\frac{d}{dt}(\delta a) &= \frac{m'na^2}{a'}\left[\sin\phi\left\{2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right\}+2\frac{\partial X}{\partial\phi}\right]\\
&= m'na'\sin\phi\left[2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right.\\
&\quad\left.+3\left(\frac{\delta a}{a'}\right)\left\{2-\frac{1}{2\sqrt{2}(1-\cos\phi)^{3/2}}\right\}\right]+O((m')^2);
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\frac{d\phi}{dt} &= n-n'-\frac{2m'na^2}{a'}\frac{\partial X}{\partial\delta a}+\frac{m'na}{2a'}\left(k\frac{\partial X}{\partial k}+h\frac{\partial X}{\partial h}\right)\\
&= n-n'+m'n\left[\frac{1}{\sqrt{(2(1-\cos\phi))}}-2+2\cos\phi\right.\\
&\quad+\left(\frac{\delta a}{a'}\right)\left\{-1+\frac{1}{2\sqrt{(2(1-\cos\phi))}}+\frac{1}{\{2(1-\cos\phi)\}^{3/2}}\right.\\
&\quad\left.\left.+2\cos\phi\right\}\right]+O((m')^2);
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\frac{dk}{dt} &= -\frac{m'na}{a'}\frac{\partial X}{\partial h}-\frac{m'na}{2a'}k\left[\sin\phi\left\{2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right\}+\frac{\partial X}{\partial\phi}\right]\\
&= -m'n\{2hg_1(\phi)+h'g_2(\phi)+k'g_3(\phi)\}\\
&\quad-\frac{1}{2}m'nk\sin\phi\left\{2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right\}+O((m')^2);
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
\frac{dh}{dt} &= \frac{m'na}{a'}\frac{\partial X}{\partial k}-\frac{m'na}{2a'}h\left[\sin\phi\left\{2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right\}+\frac{\partial X}{\partial\phi}\right]\\
&= m'n\{2kg_1(\phi)+k'g_2(\phi)-h'g_3(\phi)\}\\
&\quad-\frac{1}{2}m'nh\sin\phi\left\{2-\frac{1}{\sqrt{2}(1-\cos\phi)^{3/2}}\right\}+O((m')^2).
\end{aligned}
\tag{22}
$$

The equation (19) shows that δa is constant only if $\phi=\pi$, (which is the collinear relative equilibrium configuration with P and J on the opposite side of S), or if $\delta a=0$ and $2-1/\sqrt{2}(1-\cos\phi)^{3/2}=0$. The latter requires $\cos\phi=1/2$, that is, $\phi=\pm\pi/3$. This is the equiangular triangle configuration. Substituting in (20) shows that ϕ is constant, since $n=n'$, and, putting $\phi=\pm\pi/3$ in (18), (21) and (22) give

$$
\frac{dk}{dt} = -m'n\left\{\frac{27}{8}h-\frac{27}{16}h'\mp\frac{27}{16}\sqrt{3}k'\right\}, \tag{23}
$$

and

$$\frac{dh}{dt} = m'n\left\{\frac{27}{8}k - \frac{27}{16}k' \pm \frac{27}{16}\sqrt{3}h'\right\}. \tag{24}$$

We can have k and h constant provided

$$h = \frac{1}{2}h' \pm \frac{\sqrt{3}}{2}k' = e'\sin\left(\varpi' \pm \frac{\pi}{3}\right), \tag{25}$$

and

$$k = \frac{1}{2}k' \mp \frac{\sqrt{3}}{2}h' = e'\cos\left(\varpi' \pm \frac{\pi}{3}\right). \tag{26}$$

Thus $e = e'$, and $\varpi = \varpi' \pm \pi/3$, confirming that the orbit of P is congruent to that of J, but inclined to it at an angle $\pi/3$.

V. **Librations about the relative equilibrium positions.** Put $\phi = \pi/3 + \delta\phi$. Then, to first order in m', $\delta\phi$, and δa, equations (19) and (20) lead to

$$\frac{d}{dt}(\delta a) = m'na'\left\{\frac{9}{2}\delta\phi \pm 3\frac{\sqrt{3}}{2}\left(\frac{\delta a}{a'}\right)\right\} + O\{(m')^2\} \tag{27}$$

and

$$\frac{d}{dt}(\delta\phi) = -\frac{3n}{2a'}\delta a + O(m'n),$$

from which

$$\frac{d^2}{dt^2}(\delta\phi) \mp 3\frac{\sqrt{3}}{2}m'n\frac{d}{dt}(\delta\phi) + \frac{27}{4}m'n^2\delta\phi = O(m'n^2). \tag{28}$$

A trial solution with $\delta\phi$ proportional to $\exp(\alpha t)$ leads to

$$\alpha^2 \mp 3\frac{\sqrt{3}}{2}m'n\,\alpha + \frac{27}{4}m'n^2 = 0,$$

so that

$$\alpha = \pm\frac{3\sqrt{(3m')}}{2}ni + O(m'n). \tag{29}$$

Thus the second term in (28) is of order $(m')^{3/2}n^2$, and so is of an order to which this equation has not been completely derived. Thus the expression (29) cannot be extended to higher powers in m' without computing some of the neglected powers of m' in (27), which would require knowledge of terms of order $(m')^2$ in $\overline{R}$.

To our accuracy, then, the solution for a and ϕ is

$$(30)\qquad \begin{aligned} \delta\phi &= A\sin(\nu t+\beta), \\ \delta a &= -\sqrt{(3m')}\,Aa'\cos(\nu t+\beta), \end{aligned}$$

where A and β are disposable constants, and $\nu = (3\sqrt{(3m')}/2n$. Now for Jupiter, $m' = 1/1047$, and hence $\nu = 0.08028n$. The orbital period of Jupiter is 11.862 years, and so the period of the libration in a and ϕ is $11.862/0.08028 = 147.8$ years. The amplitudes of the oscillations in $a'\delta\phi$ and δa are in the ratio $1:\sqrt{(3m')}$, that is $18.6:1$, and these correspond approximately to oscillations in the transverse and radial directions, so that this libration, when its amplitude A is small, is approximately an ellipse, with its centre at the equiangular triangle point, whose axes are in the ratio of $18.6:1$, the minor axis being in the direction towards S.

For the eccentricity and apse, put

$$(31)\qquad \begin{aligned} k &= e'\cos\left(\varpi' \pm \frac{\pi}{3}\right) + \delta k, \\ h &= e'\sin\left(\varpi' \pm \frac{\pi}{3}\right) + \delta h. \end{aligned}$$

The equations (23) and (24) now give

$$(32)\qquad \begin{aligned} \frac{d}{dt}(\delta k) &= -\frac{27}{8}\,m'n\,\delta h, \\ \frac{d}{dt}(\delta h) &= \frac{27}{8}\,m'n\,\delta k. \end{aligned}$$

The solution of these is

$$(33)\qquad \begin{aligned} \delta k &= C\cos(\gamma t+\delta), \\ \delta h &= C\sin(\gamma t+\delta), \end{aligned}$$

where C and δ are disposable constants, and

$$(34)\qquad \gamma = \frac{27}{8}\,m'n = 0.003222\,n,$$

substituting the value for Jupiter. The period of this motion is

$2\pi/\gamma = 3682$ years. Thus the eccentricity and apse longitude are given by

$$
\begin{aligned}
e\cos\varpi &= e'\cos\left(\varpi' \pm \frac{\pi}{3}\right) + C\cos(\gamma t + \delta) \\
e\sin\varpi &= e'\sin\left(\varpi \pm \frac{\pi}{3}\right) + C\sin(\gamma t + \delta).
\end{aligned}
\tag{35}
$$

If $C < e'$, ϖ librates about $\varpi' \pm \pi/3$, if $C > e'$, ϖ increases monotonically through all values.

The treatment of these librations in rotating rectangular coordinates in the restricted problem does not exhibit this very long period oscillation directly, but shows a short period oscillation corresponding to a small eccentricity, but with period differing from that of Jupiter by an amount corresponding to the motion of the apse given by (35) when $e' = 0$.

The relative equilibrium positions may be considered as a special case of periodic solutions associated with a commensurability of period, but differ from other such cases in that there are here two independent free librations about the solution, in place of only one, as in the other cases, and also in that the mean orbital period in librating solutions in the present case is always exactly equal to that of Jupiter, while the librating and periodic solutions associated with other commensurabilities in general have periods not exactly commensurable with that of Jupiter, since the exact linear relation that exists involves the apse motion as well as the mean motions in longitude.

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Outline of a Theory of Nonperiodic Motions in the Neighborhood of the Long-Period Librations about the Equilateral Points of the Restricted Problem of Three Bodies

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I. **Summary.** In the rotating coordinate frame of the restricted problem of three bodies, all motions which are nonperiodic, but of a librational nature relative to one of the equilateral points, are treated as oscillations about given periodic solutions of long period.* On the basis of the Fourier series representation of the periodic reference orbit, the displacements from this intermediate orbit take the form of infinite series of periodic terms, with coefficients proceeding essentially in powers of those of the principal terms of short period established in the first, linear approximation. It appears that the stability of such nonperiodic librations will be endangered only when the predominant oscillations are so large as to prevent the convergence of the series proceeding in powers of their amplitudes. The results also prove the "higher order stability" of the periodic orbits themselves, beyond the first-order stability previously proved on the basis of Hill's (linearized) equation. Author

II. **Introduction.** It has been known for a long time, from the integrals of the linearized, approximating differential equations,

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that for a sufficiently small ratio μ of the two finite masses of the plane restricted three body problem the librational motion of any particle in the immediate vicinity of either one of the two equilateral Lagrangian points is characterized by the superposition of two independent periodic oscillations. The smaller one of the two periods is of the order of the period of the relative orbital motion of the two finite masses, while the larger one amounts to about 150 years in the case of the typical sun-Jupiter system, and goes to infinity with $\mu \to 0$. Either one of the two oscillations can be reduced to zero by an appropriate choice of the starting conditions or constants of integration, so that the remaining motion is periodic in the rotating frame of reference. For particles permitted to depart to noninfinitesimal distances from the equilateral center of libration, the two superposed elliptic solutions of the simplified equations cannot be expected to represent their more complicated motions. Even the existence of rigorously valid periodic solutions of large amplitude and long period has been doubted (see [**5**] and [**6**]), but with the aid of electronic computers such rather asymmetrical periodic librations have recently been established for the astronomically interesting sun-Jupiter case (see [**1**] and [**2**]) and earth-moon case (see [**3**]) of the restricted problem.

Subsequent to the numerical determination in [**2**] of a whole series of conveniently selected long-periodic libration orbits of hypothetical "Trojan"-planets in the restricted sun-Jupiter problem, additional numerical work has been devoted to the study of motions deviating from a given periodic Trojan orbit by specified initial displacements or velocity differences. The resulting nonperiodic trajectories have the general appearance of a series of short-period fluctuations superposed on a predominant libration of long period, but the Jacobi or energy constant C of the nonperiodic orbit differs from that of the most "similar" periodic reference orbit, and increasingly so with an increasing amplitude of the principal short-period oscillations. Various such trajectories have been computed over one or several librations on the SIEMENS 2002 electronic computer of the Astronomisches Rechen-Institut at Heidelberg, Germany, in cooperation with J. Schubart, during the summer of 1962. From the results then obtained, but especially from those of a more systematic and extensive survey undertaken by A. Schanzle on the IBM 1620 computer of the University of Cincinnati (see [4]), the following principal findings emerged rather clearly.

As long as a nonperiodic trajectory has a very small initial deviation from the periodic orbit (of long period and small, moderate or large amplitude) with an identical value of the Jacobi constant C, the nonperiodic Trojan continues to oscillate in a vine-like fashion about this reference orbit, with principal fluctuation periods of the general order of Jupiter's orbital period. With starting conditions producing more substantial fluctuations, however, the displacements towards the outside of the periodic orbit increase more rapidly (with increasing initial deviations) than those towards the inside, and already for rather moderate initial displacements the "vine" detaches itself from the periodic orbit. This detachment from the periodic solution with the same value of C occurs first in the vicinity of the "turning points" of these rather elongated libration orbits. For even larger initial displacements from the isoenergetic periodic libration, the nonperiodic orbits exhibit libration amplitudes which may be several times larger than those of the reference orbit. Obviously, then, any theory of such fluctuations about the periodic librations should not be based on the assumption that the Jacobi constant of the nonperiodic trajectory is identical with that of the reference orbit. This assumption is sometimes made when the ordinary stability of a periodic orbit is studied by means of Hill's equation, but no harm will normally be done then because only infinitesimally small displacements are envisioned in such first-order proofs of orbital stability. Any theory considering more substantial deviations and a more rigorous proof of stability, however, has to discard this restrictive assumption with regard to the values of C.

All the nonperiodic orbits, computed over extended periods of time, suggest that "librational stability" exists for relatively large superposed short-period fluctuations. Furthermore it became evident that the geometrical picture of vine-like fluctuations about some *suitable* periodic reference orbit can always be restored by association of the nonperiodic Trojan under consideration with such a periodic Trojan orbit which, in the nonrotating heliocentric coordinate system, varies its osculating semimajor axis a, as a long-period function of time, in close synchronization with that of the nonperiodic planet. The fluctuations of the nonperiodic Trojan about such a periodic orbit, in the rotating frame of reference, are then closely related to, and in their total amplitude roughly proportional to, the nearly constant eccentricity e of the osculating heliocentric orbit of the nonperiodic planet. Since the heliocentric eccentricity of any periodic Trojan is rather close to zero at all times (see [1]), the close association of the nonperiodic Trojan's principal short-periodic fluctuations with its eccentricity e is not surprising. However, the very pronounced noninterference of the more or less constant total fluctuation amplitude and eccentricity with the libration of long period, even in the cases of very large libration and fluctuation amplitudes, is a phenomenon which could hardly be anticipated with certainty. This common feature of all the computed trajectories indicates that, in spite of the more complicated shape of the periodic librations of large amplitude, and in spite of the varying orientation of the superposed "epicycles" as they move around on the reference orbit, the combined nonperiodic motion can still be conveniently described in terms of two such basic periods. All these orbital characteristics, as revealed or confirmed by the numerical survey, can be utilized now in devising the most convenient and suitable analytical approach, in order to deepen our insight into the nature and stability of such nonperiodic librations and to develop a theory which may eventually be extended to deal with the still more involved motions of the actual Trojan planets.

III. **The periodic reference orbits.** The theory to be presented is applicable not only to the sun-Jupiter or Trojan case, but to nonperiodic librations for all mass ratios permitting stable periodic orbits about the equilateral points. For the Trojan case, however, the application is greatly facilitated by the availability of a sufficiently dense net of periodic solutions, given in [2]. The Fourier expansions representing these periodic librations converge very satisfactorily for those amplitudes which are of main interest for the real Trojan planets. Any desired periodic orbit can readily be obtained by interpolation between the tabulated data. With the immediate application to the Trojan case in mind, as well as for the sake of a simplified terminology, all the subsequent considerations and derivations will be expressed in terms of a theory of nonperiodic Trojans, even though the analysis will be valid, except for the particular numerical data, also for other mass ratios $\mu < 0.04$. The convergence of the series expansions involved must be expected to deteriorate, however, for μ-values approaching the critical one of 0.0401.

The numerical results described in §II suggest the representation of any nonperiodic librational motion by a series of Fourier terms of short periods, superposed on that periodic solution which is most closely approximated with regard to amplitude and period of the librational behavior. Therefore, if

$$(1)\qquad \begin{aligned} x &= x_{c,0} + \sum_{j=1}^{\infty} x_{c,j}\cos(j\sigma) + \sum_{j=1}^{\infty} x_{s,j}\sin(j\sigma), \\ y &= y_{c,0} + \sum_{j=1}^{\infty} y_{c,j}\cos(j\sigma) + \sum_{j=1}^{\infty} y_{s,j}\sin(j\sigma), \end{aligned}$$

with

$$(2)\qquad \sigma = \frac{2\pi}{T}(t - t_0) = n(t - t_0),$$

represents a given periodic libration of period T, all the nonperiodic trajectories with this librational component or basis should be representable in the form

$$(3)\qquad x^* = x + u, \qquad y^* = y + v,$$

where u, v consists of periodic and constant terms only. If solutions u, v of this nature can be found to satisfy rigorously the complete differential equations of motion, then such results would indeed constitute the desired theory of all those motions which have a stable, permanent association with one or both equilateral libration centers. In this theory, the role played by the periodic reference orbit will be similar to that of Hill's variation orbit in the lunar theory.

To facilitate a later extension of the theory to the case of an elliptic orbit of Jupiter, the origin of the rotating x, y coordinate system will be identified with the center of mass of sun and Jupiter. The orientation of the axes and the choice of the units of mass, distance and time, however, will be the same as in the earlier investigations dealing with the periodic orbits, specifically in [1] and

[2]. Consequently, the x, y coordinates used here differ from the earlier p, q only by the constant $x - p = -1/(1+\mu)$. As an illustration of the convergence of the series representing the periodic librations according to Equations (1), Table I lists the Fourier coefficients of the particular orbit which intersects the straight line connecting the sun with the libration center L_5 at a solar distance $d_0 = 1.02$. This periodic solution has a total amplitude of about 43° in longitude, larger than those of almost all the real Trojan planets, so that it may serve also to test the convergence and usefulness of the subsequent analysis of nonperiodic motions for a rather extreme case. The coefficients given in Table I are

TABLE I. Fourier Coefficients of Selected Periodic Orbit

j	$x_{c,j}$	$x_{s,j}$	$y_{c,j}$	$y_{s,j}$
0	− .403 1971		+ .877 1222	
1	− 114 1352	+ .308 9355	− 34 3026	+ .155 1070
2	+ 8 2615	+ 9 5859	+ 36 3535	+ 25 0714
3	− 2049	+ 8844	+ 4 7200	− 3 7777
4	+ 2146	+ 1308	− 3633	− 8710
5	+ 282	− 466	− 1869	+ 191
6	− 119	− 72	− 88	+ 434
7	− 24	+ 33	+ 102	+ 56
8	+ 10	+ 9	+ 24	− 23
9	+ 4	− 3	− 4	− 9
10	− 1	− 2	− 4	0
11	− 1	0	0	+ 1
12	0	0	0	0
13	− 2	0	0	+ 1

based on an epoch t_0 coinciding with the periodic Trojan's intersection of the sun, L_5 line at the solar distance $d_0 = 1.02$, and on $\mu = 1/1047.355$ for the mass of Jupiter in terms of the solar unit mass. The period of this selected libration is $T = 80.26303$, as compared to Jupiter's orbital period $P = 2\pi/(1+\mu)^{1/2} = 6.28019$.

The related frequency or mean motion n of σ, as defined in Equation (2), amounts to

$$n = 0.0782\ 8243. \tag{4}$$

Any expansions proceeding in powers of n should benefit, as far as convergence is concerned, from the first-order smallness of this quantity as illustrated by (4).

The periodic solution (1) satisfies the differential equations

$$\begin{aligned} \ddot{x} - 2N\dot{y} &= \Omega_x, \\ \ddot{y} + 2N\dot{x} &= \Omega_y, \end{aligned} \tag{5}$$

where

$$N = (1+\mu)^{1/2} = 1.0004\ 7728 \tag{6}$$

denotes the angular rate of Jupiter's circular motion around the sun, and Ω_x and Ω_y are the partial derivatives with respect to x and y of the function

$$\Omega = \frac{1}{\Delta_1} + \frac{\mu}{\Delta_2} + \frac{1}{2}(\Delta_1^2 + \mu\Delta_2^2). \tag{7}$$

Ω is a function of the Trojan's distances Δ_1 and Δ_2 from the sun and Jupiter, respectively, which depend on x and y through the relations

$$\begin{aligned} \Delta_1^2 &= (x - m\mu)^2 + y^2, \\ \Delta_2^2 &= (x + m)^2 + y^2. \end{aligned} \tag{8}$$

The auxiliary quantity

$$m = 1/(1+\mu) \tag{9}$$

simplifies the Equations (8) and some subsequent relations.

IV. **The differential equations for the variations** u, v. The differential equations (5) have to be satisfied not only by any periodic solution x, y as given by Equations (1), but also by the nonperiodic solutions x^*, y^* according to Equations (3). If the latter expressions for x^*, y^* are substituted into Equations (5), the partials $\Omega_x(x^*, y^*)$, $\Omega_y(x^*, y^*)$ may be expanded as Taylor series in u, v, provided that these variations or displacements from the periodic reference orbit will remain small enough to permit convergence. After $\Omega_x(x^*, y^*)$

and $\Omega_y(x^*, y^*)$ have been expanded about Ω_x and Ω_y, the differential equations satisfied by x, y may be subtracted from the complete equations satisfied by x^*, y^*, leading to the following equations for u and v:

$$
\begin{aligned}
\ddot{u} - 2N\dot{v} &= \Omega_{xx} u + \Omega_{xy} v \\
&\qquad + \frac{1}{2}\,\Omega_{xxx} u^2 + \frac{1}{2}\,\Omega_{xyy} v^2 + \Omega_{xxy} uv + \cdots, \\
\ddot{v} + 2N\dot{u} &= \Omega_{xy} u + \Omega_{yy} v \\
&\qquad + \frac{1}{2}\,\Omega_{xxy} u^2 + \frac{1}{2}\,\Omega_{yyy} v^2 + \Omega_{xyy} uv + \cdots.
\end{aligned} \tag{10}
$$

In these differential equations, all the higher-order partials Ω_{xx}, Ω_{xy}, etc., are functions of the periodic Trojan's coordinates x, y, and thus periodic functions of σ and of the time t. Assuming therefore the second-order derivatives of Ω to be given by

$$
\begin{aligned}
\Omega_{xx} &= A_{c,0} + 2\sum_{r=1}^{\infty} [A_{c,r}\cos(r\sigma) + A_{s,r}\sin(r\sigma)], \\
\Omega_{yy} &= B_{c,0} + 2\sum_{r=1}^{\infty} [B_{c,r}\cos(r\sigma) + B_{s,r}\sin(r\sigma)], \\
\Omega_{xy} &= C_{c,0} + 2\sum_{r=1}^{\infty} [C_{c,r}\cos(r\sigma) + C_{s,r}\sin(r\sigma)],
\end{aligned} \tag{11}
$$

and the third-order partials by similar Fourier series, the coefficients of these expansions have to be determined on the basis of the particular periodic solution (1). Two entirely different procedures are available to find the numerical values of the coefficients $A_{c,r}$, $A_{s,r}$, etc.

First, the equations resulting from the repeated differentiation of Equations (5) with respect to the time t constitute a number of relations connecting the Ω_{xx}, Ω_{xy}, etc., with the $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, etc., where the latter group of functions is easily obtained from the corresponding differentiations of the solution (1) with respect to time. It will be found, however, that the number of equations established in this manner is not yet sufficient to permit their solution for Ω_{xx}, Ω_{xy}, etc. The additional relations required are those represented by the equation

$$y\,[\Omega_{xx}+\Omega_{yy}]=3N^2y-2N\dot{x}-\ddot{y}, \tag{12}$$

and its first and higher derivatives with respect to t and y. Equation (12) follows from the differential equations valid for three coordinates x, y, z when the specification $z\equiv 0$, as adopted in this study, is introduced. The three second-order partials of Ω can be obtained now, in principle, from Equation (12) and the two equations based on Equations (5):

$$\begin{aligned}\Omega_{xx}\dot{x}+\Omega_{xy}\dot{y}&=\ddot{x}-2N\dot{y},\\ \Omega_{xy}\dot{x}+\Omega_{yy}\dot{y}&=\ddot{y}+2N\ddot{x}.\end{aligned} \tag{13}$$

The four third-order partials Ω_{xxx}, Ω_{xxy}, Ω_{xyy}, Ω_{yyy} are determinable from the three equations obtained by differentiation of Equations (12) and (13) with respect to t, and from the differentiation of (12) with respect to y. The procedure for finding yet higher derivatives of Ω is clear from this. The main disadvantage of the method thus outlined probably lies in the many multiplications and divisions by Fourier series containing cosine- and sine-terms, so that an alternative method, disregarding the availability of the Fourier expansions for x and y, may actually be preferable.

This second method takes advantage of the availability of the special values of x and y, for a set of equidistant values of σ, the harmonic analysis of which had produced the Fourier coefficients of x and y in the first place. All the partial derivatives of Ω are of course expressible as explicit functions of x and y, obtainable by the necessary differentiations of Equation (7) with respect to x and y. In this fashion one finds

$$\begin{aligned}\Omega_{xx}&=(1-\Delta_1^{-3})+3\Delta_1^{-5}(x-m\mu)^2+\mu\,[(1-\Delta_2^{-3})+3\Delta_2^{-5}(x+m)^2],\\ \Omega_{yy}&=(1-\Delta_1^{-3})+3\Delta_1^{-5}y^2+\mu\,[(1-\Delta_2^{-3})+3\Delta_2^{-5}y^2],\\ \Omega_{xy}&=3\Delta_1^{-5}(x-m\mu)\,y+\mu 3\Delta_2^{-5}(x+m)\,y,\end{aligned} \tag{14}$$

$$\begin{aligned}\Omega_{xxx}&=9\Delta_1^{-5}(x-m\mu)-15\Delta_1^{-7}(x-m\mu)^3\\ &\quad+\mu\,[9\Delta_2^{-5}(x+m)-15\Delta_2^{-7}(x+m)^3],\\ \Omega_{yyy}&=9\Delta_1^{-5}y-15\Delta_1^{-7}y^3+\mu\,[9\Delta_2^{-5}y-15\Delta_2^{-7}y^3],\\ \Omega_{xxy}&=3\Delta_1^{-5}y-15\Delta_1^{-7}(x-m\mu)^2y+\mu\,[3\Delta_2^{-5}y-15\Delta_2^{-7}(x+m)^2y],\\ \Omega_{xyy}&=3\Delta_1^{-5}(x-m\mu)-15\Delta_1^{-7}(x-m\mu)\,y^2\\ &\quad+\mu\,[3\Delta_2^{-5}(x+m)-15\Delta_2^{-7}(x+m)\,y^2],\ \text{etc.}\end{aligned} \tag{15}$$

From the harmonic analysis of the special values of the three second-order partials of Ω, as computed with the corresponding values of x and y by means of Equations (14), the coefficients $A_{c,r}$, $A_{s,r}$, etc., of Equations (11) can be determined on an electronic computer with a routine program. Similarly the Fourier coefficients of Ω_{xxx} etc., will be obtained from their special values computed according to Equations (15), and those of any higher-order partials of Ω on the basis of additional equations resulting from the continued differentiation of Equations (15).

Rather well known are those constant values of the lower-order partials of Ω which are valid for a particle resting in one of the triangular points, say L_5. For this case, with $x \equiv -m + 0.5$, $y \equiv 3^{1/2}/2$, the Equations (5), (14), and (15) yield the results

$$\Omega_x = 0, \qquad \Omega_y = 0, \tag{16}$$

$$\Omega_{xx} = +\frac{3}{4}(1+\mu), \quad \Omega_{yy} = +\frac{9}{4}(1+\mu), \quad \Omega_{xy} = -\frac{3}{4}3^{1/2}(1-\mu), \tag{17}$$

$$\begin{aligned} \Omega_{xxx} &= -\frac{21}{8}(1-\mu), & \Omega_{yyy} &= -\frac{9}{8}3^{1/2}(1+\mu), \\ \Omega_{xxy} &= -\frac{3}{8}3^{1/2}(1+\mu), & \Omega_{xyy} &= +\frac{33}{8}(1-\mu). \end{aligned} \tag{18}$$

For periodic solutions with small libration amplitudes, the constant terms $A_{c,0}$, $B_{c,0}$, $C_{c,0}$ of Equations (11) should approximate the values listed on the right sides in (17). For large amplitudes, however, as in the case of the periodic orbit represented in Table I, the constant terms may differ substantially from the values at L_5.

It will be convenient to assume the solution of the differential equations (10) in the form of an exponential series. To this end, let the various partials of Ω be expressed in the same form,

$$\Omega_{xx} = \sum_{-\infty}^{\infty} \alpha_r \exp(ir\sigma), \quad \Omega_{yy} = \sum_{-\infty}^{\infty} \beta_r \exp(ir\sigma), \quad \Omega_{xy} = \sum_{-\infty}^{\infty} \gamma_r \exp(ir\sigma), \tag{19}$$

etc., where $i = (-1)^{1/2}$, by putting for $r < 0$:

$$\alpha_r = A_{c,r} + iA_{s,r}, \quad \beta_r = B_{c,r} + iB_{s,r}, \quad \gamma_r = C_{c,r} + iC_{s,r}, \tag{20}$$

and for $r > 0$:

$$\alpha_r = A_{c,r} - iA_{s,r}, \quad \beta_r = B_{c,r} - iB_{s,r}, \quad \gamma_r = C_{c,r} - iC_{s,r},$$

etc. As far as the constant terms of the Equations (19) are concerned, one has $\alpha_0 = A_{c,0}$, $\beta_0 = B_{c,0}$, $\gamma_0 = C_{c,0}$.

As long as approaches to the sun and close approaches to Jupiter are excluded, as in all the periodic reference orbits considered here, all the partials of Ω involved in the right-hand sides of Equations (10) are periodic functions of the general order of unity. Consequently, the expansions in powers of u and v may be expected to converge as long as $|u| < 1$, $|v| < 1$. For $|u| \ll 1$, $|v| \ll 1$, the convergence should be rapid, and a good approximation to the complete solution should be obtainable from the consideration of only the linear terms in u and v. In this case, the problem is still distinctly different from that of the infinitesimally small oscillations about L_5, in so far as the coefficients Ω_{xx} etc., are not constants, but periodic functions of perhaps considerable amplitudes. Thus even the reduced differential equations

$$
\begin{aligned}
\ddot{u} - 2N\dot{v} &= \Omega_{xx} u + \Omega_{xy} v, \\
\ddot{v} + 2N\dot{u} &= \Omega_{xy} u + \Omega_{yy} v,
\end{aligned}
\tag{21}
$$

encompass, at least for very small variations u, v, all the essential aspects of the dynamical problem at hand.

For the integration of the reduced Equations (21), the convergence of the series for Ω_{xx}, Ω_{yy}, and Ω_{xy} will be of primary significance. To investigate this convergence, the Fourier coefficients of the relevant Equations (11) have been determined for the selected large reference orbit of §III. For this purpose, the harmonic analysis has been based on only 12 special values of σ, and the results, rounded to five decimals, are listed in Table II. Since all the coefficients are multiplied by the small displacements u, v on the right-hand sides of Equations (21), a considerably reduced numerical accuracy is justified anyway, compared to the seven-decimal precision for the reference orbit in Table I.

The corresponding expansion coefficients of Ω_{xxx}, Ω_{yyy}, Ω_{xxy}, Ω_{xyy}, which may be denoted by $a_{c,r}$, $a_{s,r}$; $b_{c,r}$, $b_{s,r}$; $c_{c,r}$, $c_{s,r}$; $d_{c,r}$, $d_{s,r}$, have also been computed with those of the second-order partials of Ω. They are given in Table III, rounded to three decimals because of their multiplication with u^2, v^2, or uv in the rigorous Equations (10). The coefficients of the sine-terms involving 6σ remained undetermined in Tables II and III because only twelve points were used in the harmonic analysis.

TABLE II. Fourier Coefficients of Ω_{xx}, Ω_{yy}, Ω_{xy} for the Reference Orbit of Table I

r	$A_{c,r}$	$A_{s,r}$	$B_{c,r}$	$B_{s,r}$	$C_{c,r}$	$C_{s,r}$
0	+.65263		+2.35084		−.98331	
1	+ 14004	−.37980	− 16742	+.37015	− 7556	+.31672
2	− 8376	− 4827	+ 8114	+ 5229	− 3083	− 4483
3	− 74	+ 759	+ 139	− 683	− 1505	+ 923
4	+ 96	− 99	− 73	+ 87	+ 123	+ 255
5	− 22	− 21	+ 19	+ 15	+ 40	− 13
6	− 4	..	+ 3	..	0	..

TABLE III. Fourier Coefficients of Ω_{xxx}, Ω_{yyy}, Ω_{xxy}, Ω_{xyy} for the Reference Orbit of Table I

r	$a_{c,r}$	$a_{s,r}$	$b_{c,r}$	$b_{s,r}$	$c_{c,r}$	$c_{s,r}$	$d_{c,r}$	$d_{s,r}$
0	−1.645		−2.750		+.112		+2.857	
1	+ 62	+.056	+ 713	−1.771	− 540	+1.581	+ 51	−.538
2	− 385	− 315	− 292	− 134	+ 242	+ 88	+ 353	+ 329
3	− 74	+ 77	+ 60	− 8	− 67	+ 15	+ 79	− 73
4	+ 10	+ 3	− 4	− 21	+ 5	+ 22	− 9	− 4
5	− 1	− 2	− 3	+ 1	+ 4	− 1	+ 1	+ 2
6	0	.	0	.	0	.	0	.

It can be seen that the general convergence of the Fourier series for these second- and third-order derivatives of Ω closely parallels that of the series representing the periodic reference orbit itself, as exhibited in Table I. This might have been anticipated on the basis of the equations obtainable from the differentiation of Equations (5), as relations connecting the Ω_{xx} etc., with the various time derivatives of x and y. In view of the quite rapid convergence of the Ω_{xx}, Ω_{yy}, and Ω_{xy} even for a reference orbit of such rather large dimensions, a good convergence may be expected also for the solutions of the reduced differential equations (21).

V. **The principal characteristics of the solution.** A particular solution of the reduced Equations (21) may be assumed to have the form

$$u = \sum_{-\infty}^{\infty} u_r \exp[i(r+c)\sigma], \qquad v = \sum_{-\infty}^{\infty} v_r \exp[i(r+c)\sigma], \tag{22}$$

where the unknown coefficients u_r, v_r and the characteristic exponent or stability constant c have to be determined from the identities resulting from the substitution of these u, v into the two Equations (21). This leads to the following pairs of equations, for $r = 0, \pm 1, \pm 2, \cdots$:

$$\begin{aligned} &[\alpha_0 + n^2(r+c)^2]u_r + [\gamma_0 + 2iNn(r+c)]v_r \\ &\qquad = -\sum_{s=1}^{\infty}(\alpha_s u_{r-s} + \alpha_{-s}u_{r+s}) - \sum_{s=1}^{\infty}(\gamma_s v_{r-s} + \gamma_{-s}v_{r+s}), \\ &[\gamma_0 - 2iNn(r+c)]u_r + [\beta_0 + n^2(r+c)^2]v_r \\ &\qquad = -\sum_{s=1}^{\infty}(\gamma_s u_{r-s} + \gamma_{-s}u_{r+s}) - \sum_{s=1}^{\infty}(\beta_s v_{r-s} + \beta_{-s}v_{r+s}). \end{aligned} \tag{23}$$

If Equations (23) are compared with those arising from Hill's equation for the determination of c (see [1]), then it is seen that twice as many conditions have to be satisfied here in consequence of the simultaneous involvement of both coordinates, u and v. On the other hand, the coefficients α_r, β_r, γ_r appearing in Equations (23) converge rapidly with increasing absolute values of r, in contrast to the very poor convergence of the coefficients θ_r previously used in Hill's equation for the transversal displacement η. This more satisfactory feature of the present approach is closely related to the fact, as revealed by the many nonperiodic trajectories obtained by numerical integration, that the sharp curvature of the periodic orbits in the region of the two turning points does *not* affect the shape and orientation of the superposed fluctuations to any comparable extent. Since the geometrical parameters of the principal "loops" of short period fail to synchronize themselves with the sudden changes in the direction of the normal to the periodic orbit, it is clear now that the sharp and deep dips of the

$f(u)$ function at the turning points noted in [1] are due entirely to the unsuitability of Hill's equation for such strongly curved orbits, but not to any intrinsic stability anomalies.

If the periodic reference orbit is one of small amplitude, the first approximation to the solution of Equations (21) may be found by neglecting in the system (23) of linear equations all α_r, β_r, γ_r except α_0, β_0, γ_0, and all u_r, v_r except u_0, v_0. This reduces the Equations (23) to

$$(24) \qquad \begin{aligned} (\alpha_0 + n^2c^2)\,u_0 + (\gamma_0 + 2iNnc)\,v_0 &= 0, \\ (\gamma_0 - 2iNnc)\,u_0 + (\beta_0 + n^2c^2)\,v_0 &= 0. \end{aligned}$$

Either one of these two equations determines the ratio u_0/v_0, provided the determinant of the coefficients vanishes, or

$$(25) \qquad (n^2c^2)^2 + (\alpha_0 + \beta_0 - 4N^2)(n^2c^2) + (\alpha_0\beta_0 - \gamma_0^2) = 0.$$

In the case of an infinitesimally small periodic reference orbit, the α_0, β_0, γ_0 reduce to the values of Ω_{xx}, Ω_{yy}, Ω_{xy} valid at L_5, as listed on the right sides in (17), and Equation (25) becomes identical with the well-known equation for the two frequencies of such very small oscillations about L_5. For noninfinitesimal amplitudes, however, Equation (25) determines the first approximation to c as a function of α_0, β_0, γ_0, and of the libration period $T = 2\pi/n$.

For α_0, β_0, γ_0 not too different from the values listed on the right sides in (17), i.e., for small libration amplitudes, Equation (25) will be satisfied by two real roots (nc) in the vicinity of ± 1, or by c-values of the order of ± 12. Such values of c are indeed representative of a short-period fluctuation, with a period of the order of Jupiter's orbital period of revolution. If the solution of Equation (25) is attempted for the α_0, β_0, γ_0 listed (as $A_{c,0}$, $B_{c,0}$, $C_{c,0}$) in Table II, however, no real root (nc) will be obtained. This finding does not indicate instability, but simply reveals the inadequacy of the approximating Equations (24) and (25) in the case of such a large libration. Table II shows that the neglected Fourier coefficients $2A_{c,1}, \cdots, 2C_{s,1}$ are indeed of the order of ± 1, so that the α_1, β_1, γ_1 should be considered, together with the α_0, β_0, γ_0, already in the first approximation to the solution of Equations (21). This more reasonable approach requires the consideration of u_0, v_0 in the *six* Equations (23) for r-values of $-1, 0, +1$, and consequently the simultaneous determination of the coefficients u_{-1}, v_{-1}, u_1, v_1, in addition to u_0, v_0. For this selected reference orbit of Tables I—III, the c-value causing the determinant of these six equations to vanish is found to be

$$c = \pm 12.056.$$

With either one of the two roots for c, the six equations under consideration can now be solved for five of the u_r, v_r involved in terms of the sixth one, say v_0. A second approximation may be based next on the ten equations (23) for the r-values -2, -1, 0, $+1$, $+2$, leading to an improved value of c and to a new solution for the u_r, v_r, including now those with the subscripts -2, $+2$. The convergence of these successive approximations, involving the addition of four equations and four unknowns in each subsequent step, should be rather good, thanks to that of the α_r, β_r, γ_r.

If v_0 serves as the arbitrary constant of integration, two particular solutions with different and independent constants v_0 will satisfy the Equations (21) in consequence of the existence of two real roots c, as illustrated by the approximation (26) for the selected reference orbit. The two v_0 may be chosen conjugately complex, so that the sum of the two particular solutions represents real displacements u, v, depending on two real constants of integration. While two integration constants have thus been identified, the general solution of the Equations (21) should involve four such constants. This raises the question of the existence of additional particular solutions.

It is obvious, now, that

$$u = k\dot{x}, \quad v = k\dot{y} \qquad (k = \text{const.}) \tag{27}$$

represents another particular solution of the Equations (21), because the substitution of these expressions results in two equations which are identical with those obtained by differentiation with respect to time and subsequent multiplication by k of the Equations (5) valid for the periodic orbit. Since $\dot{x}$ and $\dot{y}$, as the derivatives with respect to t of the Equations (1), are of the order of n, the solution (27) may be interpreted as a small displacement *in* the periodic reference orbit. Actually, the displacement represented by Equations (27) lies in the direction of the tangent to the periodic orbit, but this degree of approximation is consistent with the linearized nature of the differential equations (21) themselves. Evidently any periodic Trojan moving in the same given reference orbit, even at a more substantial distance from the adopted "reference Trojan," describes an orbit relative to the reference body which must be representable as a specific *periodic* solution of the differential equations for u and v. To find the rigorous solution valid for such Trojans moving in, in the reference orbit, one has to go back to the complete Equations (10). It is easily verified that they will be satisfied by

$$u = k\dot{x} + \frac{1}{2}k^2\ddot{x} + \frac{1}{6}k^3\dddot{x} + \cdots,$$

(28)

$$v = k\dot{y} + \frac{1}{2}k^2\ddot{y} + \frac{1}{6}k^3\dddot{y} + \cdots,$$

as expressions representing that arc in the periodic orbit through which the reference Trojan moves in the time $\Delta t = k$. If the solutions (27) and (28) are compared with the assumed form (22) of the particular solutions of Equations (21), it is seen that any periodic solution of period T must be associated with an integral value of c. In the case of the selected reference orbit, where the approximation $c = \pm\ 12.056$ was found for the nonperiodic fluctuations, the six equations involved at this approximation can indeed be found to be satisfied also when evaluated with the principal coefficients of solution (27), and with $c = 0$. These periodic solutions *in* the reference orbit, with $u_0 = v_0 = 0$, are of no interest, however, for the present study of the nonperiodic fluctuations *about* x, y.

Once the determination of c and of the necessary number of coefficients u_r, v_r has been achieved to the required degree of numerical accuracy, the solution representing the nonperiodic fluctuations has been completed as far as the reduced differential equations (21) are concerned. This solution can be applied to any initial displacements $(u)_0$, $(v)_0$ at the zero-epoch t_0, by means of the relevant relations with the original two constants of integration. The same value of c is valid in combination with all possible amplitudes or starting values $(u)_0$, $(v)_0$, as indicated already by the fact that the solution of the linear equations (23) will not be affected by the application of any common factor f to all the u_r, v_r

involved. It will be seen that this noninterference between c and the integration constants ceases to exist when the linearized Equations (21) are replaced by the complete Equations (10) for the motion of nonperiodic Trojans.

As long as the second- and higher-order terms of the rigorous Equations (10) are small compared to the linear ones considered in the reduced Equations (21), the solution (22) of the latter system can serve as the first approximation to the solution of the complete equations. The second approximation has to consider the presence of the terms involving u^2, v^2, and uv on the right-hand sides of Equations (10). If these previously neglected terms are simply evaluated with the u, v represented by the first-order solution (22), exponential terms with exponents of the forms

$$E_s = i(j+2c)\sigma, \quad E_l = ij\sigma \qquad (j = \text{integer}), \tag{29}$$

but with known coefficients, appear. The terms with exponents E_s have short periods, except possibly for those where $(j+2c)$ is a small quantity, while the periods associated with the exponents E_l are long, at least for small integers j. The occurrence of $j = 0$ is not excluded, and will give rise to small constant terms in the E_l category. Since all the exponents are different from those appearing in the first approximation as represented by Equations (22), the second approximation to the solution of Equations (10) can be achieved by simply adding to the first solution the necessary second-order increments u_{II}, v_{II}, to be determined from the equations

$$\begin{aligned} \ddot{u}_{\text{II}} - 2N\dot{v}_{\text{II}} - \Omega_{xx}u_{\text{II}} - \Omega_{xy}v_{\text{II}} &= P, \\ \ddot{v}_{\text{II}} + 2N\dot{u}_{\text{II}} - \Omega_{xy}u_{\text{II}} - \Omega_{yy}v_{\text{II}} &= Q, \end{aligned} \tag{30}$$

where P and Q represent the sums of all the exponential terms created by the substitution of the first approximation for u, v into the second-order terms of Equations (10). The Equations (30) can be solved by substituting an assumed solution, in the form of series of exponential terms involving all the exponents listed in (29), but with unknown coefficients, and by determining these coefficients from the resulting system of identities.

The sums of terms with exponents of the type E_l can be considered, of course, as periodic functions of period T. The corresponding parts of the solution u_{II}, v_{II} may be added, therefore, to the basic reference orbit, so that all the remaining fluctuations are again of short periods, but are referred to a modified reference orbit which is *not* a periodic solution of the differential equations. Furthermore, since the first-order part of the solution is proportional to the arbitrary v_0 and the conjugate $\overline{v_0}$ of Equations (22), all these second-order increments are consequently proportional to $v_0\overline{v_0}$ so that such a modified reference orbit depends on the constants of integration.

The emergence of certain constant parts, from the E_l-terms with $j = 0$, is not endangering the stability of the solution. They will be absorbed by correspondingly small constant terms u_{00}, v_{00} in the increments u_{II}, v_{II} as obtained from Equations (30). If the constant terms in P, Q are denoted by κ, λ, respectively, the first approximation to the constant members of u and v will be obtained from the relevant parts of Equations (30):

$$\begin{aligned} \alpha_0 u_{00} + \gamma_0 v_{00} &= -\kappa, \\ \gamma_0 u_{00} + \beta_0 v_{00} &= -\lambda. \end{aligned} \tag{31}$$

If terms of the order of the cubes of the principal terms in the original solution (22) have to be considered, the necessary third-order additions u_{III}, v_{III} to the present solution $u + u_{\text{II}}$, $v + v_{\text{II}}$ have to be determined on the basis of the third-order terms created on the right-hand sides of the Equations (10) by the substitution of $u + u_{\text{II}}$ and $v + v_{\text{II}}$ into the terms of second and third order. The third-order terms created by this substitution will have exponents of the forms $i(j + 3c)\sigma$ and $i(j + c)\sigma$. The latter type of exponents is identical with those occurring in the first-order solution (22), so that it becomes necessary to readjust the earlier determination of c and of the coefficients u_r, v_r for the effect of the corresponding third-order increments to the Equations (23). This can be done differentially, considering only sensitive terms and neglecting the higher-order effects of any resulting coefficient-corrections Δu_r, Δv_r on the third-order terms in Equations (10) which are the basis of this readjustment. It is clear that the changes produced in c and in the u_r, v_r associated with the exponents $i(r + c)\sigma$ will depend on the value of the integration constant v_0, which as the only one of the u_r, v_r should be considered as fixed in the adjustment procedures. Since the modifications are functions of v_0, the interdependence of c and v_0 has been established. The c-value originally established on the basis of the Equations (23) is rigorously valid only for infinitesimally small displacements from the reference orbit.

From the preceding considerations it is evident that no principal obstacle stands in the way of an extension of the solution of the complete Equations (10) to the incorporation of any desired powers of u and v. A considerable variety of smaller and smaller periodic and constant terms enters the results for u and v as the refinement of the solution progresses by means of the procedures just described in connection with the second- and third-order parts of Equations (10), but convergence should be expected as long as the first-order solution (22) consists of terms whose amplitudes add up to amounts well below the order of unity. If the integration constant v_0 is assumed to be so large, however, that the total displacement may from time to time approach this order of unity, then the convergence of the expansions on the right-hand sides of Equations (10) may obviously be endangered. It is intended to establish the actual limits of stability from a third-order determination of c.

Except for the possibility of instability caused by excessively large initial displacements or velocity deviations, the general form of the solution is such that it actually provides proof of the non-existence of secular terms of any order, and of orbital stability beyond the so-called first-order stability ordinarily established on the basis of Hill's equation. While the actual proof of convergence of the solution will be more numerical in nature than analytical in any given case, such proof would not appear to be less satisfactory than the somewhat similar one of the existence of the periodic reference orbits in the first place.

VI. **The relationship between C and v_0.** For nonvanishing integration constants v_0, or nonvanishing solutions u, v, the Jacobi constant C associated with the solution is always smaller than that of the reference orbit, and increasingly so with increasing values of v_0. This statement is true at least for the first-order solution (22), and can be proved as follows.

In this section, let u, v represent the solution in terms of real quantities, as obtained by an appropriate choice of the two complex constants of integration. Neglecting all but the principal terms with $r = 0$ in the original Equations (22) for the first-order solution, the resulting oscillation takes the form

$$u = 2u_c \cos(c\sigma) + 2u_s \sin(c\sigma),$$
$$v = 2v_c \cos(c\sigma) + 2v_s \sin(c\sigma), \tag{32}$$

where now the real coefficients v_c, v_s may be considered as constants of integration. The other two coefficients, u_c, u_s, depend on v_c, v_s through relations which are the equivalent of the earlier Equations (24) and (25) for the complex u_0, v_0.

Since the Jacobi integral

$$V^2 = 2\Omega - C, \tag{33}$$

with

$$V^2 = \dot{x}^2 + \dot{y}^2, \tag{34}$$

is valid for the nonperiodic Trojan as well as for the periodic reference planet, the difference of the two integrals may be written in the form

$$C - C_0 = 2(\Omega - \Omega_0) - (V^2 - V_0^2), \tag{35}$$

where the subscript zero indicates the quantities representing the periodic Trojan. When the right-hand side of Equation (35) is expressed in terms of the relevant trigonometric series, all periodic terms can be disregarded, because $C - C_0$ must be equal to the sum of all constant terms involved. As far as the last term of Equation (35) is concerned, one has

$$V^2 - V_0^2 = 2(\dot{x}\dot{u} + \dot{y}\dot{v}) + \dot{u}^2 + \dot{v}^2. \tag{36}$$

With the $\dot{x}$, $\dot{y}$ and $\dot{u}$, $\dot{v}$ resulting from the differentiation of Equations (1) and (32), respectively, the part $2(\dot{x}\dot{u} + \dot{y}\dot{v})$ of Equation (36) is found to consist of periodic terms only, while the remaining terms produce a constant contribution

$$\overline{V^2 - V_0^2} = 2(nc)^2(u_c^2 + u_s^2 + v_c^2 + v_s^2). \tag{37}$$

It remains to find the constant part of $\Omega - \Omega_0$, which function can be expanded in the form

$$\Omega - \Omega_0 = \Omega_x u + \Omega_y v + \frac{1}{2}\Omega_{xx}u^2 + \frac{1}{2}\Omega_{yy}v^2 + \Omega_{xy}uv + \cdots, \tag{38}$$

where the second-order partials of Ω are those previously used in the expansions of Ω_x and Ω_y. Again, all these partials of Ω have to be evaluated as periodic functions of the coordinates x, y of the reference-Trojan. Since the Ω_x etc., in turn have the form of Fourier series involving only integral multiples of σ, including zero, the first two terms of Equation (38) produce no constant parts. This is true even when the constant terms u_{00}, v_{00} of the second-order parts of the solution u, v are considered, because the coefficients Ω_x and Ω_y have no constant terms, according to the Equations (5) for these derivatives of Ω. The second-order terms of Equation (38), however, contribute constant terms. When the constant terms of the second-order partials of Ω are approximated by their values at L_5, as listed in (17), and terms of the order of μ are neglected, the constant part of $\Omega - \Omega_0$ takes the form

$$\overline{\Omega - \Omega_0} = \frac{3}{4}(u_c^2 + u_s^2) + \frac{9}{4}(v_c^2 + v_s^2) - \frac{3}{2}\, 3^{1/2}(u_c v_c + u_s v_s). \tag{39}$$

Since the approximation (32) represents an elliptic fluctuation, u_c and u_s may be expressed in terms of v_c and v_s, taking advantage of the relevant relations based on Equations (24) and (25). If this is done in Equations (37) and (39), approximating again α_0, β_0, γ_0 by their values valid at L_5 and omitting terms of the order of μ, substitution into Equation (35) leads to

$$\begin{aligned}\left(n^2c^2 + \frac{3}{4}\right)(C - C_0) &\\ = -\Bigg[\left(2n^2c^2 - \frac{3}{2}\right)\left(n^2c^2 + \frac{9}{4}\right) &\\ + \left(2n^2c^2 - \frac{9}{2}\right)\left(n^2c^2 + \frac{3}{4}\right) + \frac{27}{4}\Bigg]&(v_c^2 + v_s^2).\end{aligned} \tag{40}$$

In this expression, $(nc)^2$ differs from 1 by a quantity of the order of μ, which may be neglected in line with the previous approximations. Then the difference between the two Jacobi constants is reduced to

$$C - C_0 = -\frac{16}{7}(v_c^2 + v_s^2), \tag{41}$$

so that indeed C is always smaller than C_0 by an amount of the order of the square of the amplitude of the principal short-periodic oscillation.

Equation (41) represents the first approximation to the rigorous expression for the Jacobi constant C of a nonperiodic orbit as a function of the constants of integration. The result can be refined, on the basis of the appropriate higher-order terms in u and v. According to the numerical evidence as discussed in §II, and in agreement with the theory outlined in this investigation, the nonperiodic Trojan with a Jacobi constant C exhibits a librational behavior equal to that of a periodic Trojan with the larger Jacobi constant $C_0(C, v_c, v_s)$, as approximated by Equation (41). Since two periodic orbits with only slightly different Jacobi constants may have substantially different libration amplitudes, the result (41) explains the sometimes very large displacements of a given nonperiodic trajectory from the periodic orbit with the *same* Jacobi constant C.

According to the Tisserand criterion,

$$C = (1 + \mu) \left[\frac{1}{a} + 2a^{1/2}(1 - e^2)^{1/2} \right], \tag{42}$$

as an approximate equivalent of the Jacobi integral, any bounded periodic fluctuation of the osculating semi-major axis a about Jupiter's $a = 1$ has a very slight effect on the value of the right-hand side of this equation. This explains the fact that indeed the periodic Trojans are able to reconcile their substantial periodic variations of a with the condition (42), even in connection with an eccentricity e which remains close to zero at all times. On the other hand, for any nonperiodic Trojan synchronizing its behavior of a with that of a certain periodic orbit (as suggested by all the numerical integrations of such trajectories), a certain only slightly variable eccentricity e will be required (again in complete agreement with the numerical evidence), in order to satisfy Equation (42). Since the eccentricities of the periodic Trojans of the restricted problem differ from zero only by amounts of the order of μ, comparison of Equations (41) and (42) finally confirms the empirical finding that the oscillation amplitude is roughly proportional to the mean value of the nearly constant eccentricity e.

References

1. E. Rabe, *Determination and survey of periodic Trojan orbits in the restricted problem of three bodies,* Astronom. J. **66** (1961), 500-513.

2. ———, Astronom. J. **67** (1962), 382-390.

3. E. Rabe and A. Schanzle, Astronom. J. **67** (1962), 732-739.

4. A. Schanzle, Dissertation, 1964.

5. B. Thüring, *Die Librationsbahnen der Trojaner als nicht-geschlossene Bahnkurven,* Astronom. Nachr. **280** (1952), 226-232.

6. ———, Astronom. Nachr. **285** (1959), 71-80.

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Elements of a Theory of Librational Motions in the Elliptical Restricted Problem

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I. **Summary.** The author's earlier treatment of the nonperiodic librational motions in the restricted problem of three bodies given in [**3**], as oscillations about a given periodic reference orbit, is extended here to the case where the relative motion of the two finite masses is elliptic.* The new reference orbit combines a periodic solution of the restricted problem with a periodic scale factor as determined by the periodically changing linear dimensions of the equilateral triangles. These "pulsating" reference or intermediate orbits are nonperiodic, except for the special cases of commensurability between the two periods involved, but they are suitable because of their representation of the two most predominant features of such librational motions in the elliptical problem. The equations of motion are referred to a nonuniformly rotating coordinate system, the x-axis of which coincides permanently with the straight line connecting the two finite masses, but the integrations are facilitated by expansions in powers of the eccentricity e of the fundamental elliptic orbit. While the chosen reference orbit as such is not a particular solution of the differential equations, the remaining superposed oscillations are found to consist of a forced and a free

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part, where only the latter one can be reduced to zero by an appropriate choice of the constants of integration. A07

II. **Introduction.** A theory of all the nonperiodic librational motions about the equilateral points of the restricted problem of three bodies has been outlined in [3]. The periodic solutions, as previously established by numerical methods for the Sun-Jupiter or "Trojan" case in [1] and [2] as well as for the earth-moon case in [4], are the basis of this theory and thus are the equivalent of Hill's so-called variation orbit in the lunar theory.

Since a particle located in one of the two equilateral points can remain there forever even in the case of elliptic relative motion of the two finite masses, it seems logical to suspect that any periodic libration of the restricted problem, transposed into the elliptic system simply by multiplication with the proper variable scale factor, would produce a reference orbit not too different from an actual (particular) solution of the differential equations. The present study is concerned with the derivation of the relevant equations of motion, referred to the nonuniformly rotating coordinate system associated with the elliptic motion of the two finite masses about their center of mass, and with an outline of the theory resulting from their integration.

III. **The nonperiodic reference orbits.** As in the preceding study of the nonperiodic trajectories in the restricted problem given in [3], only motions in the plane determined by the motion of the two finite masses will be considered. For the sake of a simplified terminology, these two nonzero masses will be identified again with the sun and with Jupiter, and the vanishingly small mass will be referred to as a "Trojan" planet, even though the theory will be applicable to a wide range of values for the ratio μ between the smaller and the larger one of the finite masses.

Let x, y denote the rectangular coordinates, in the uniformly rotating frame, of a Trojan moving in one of the periodic libration orbits which exist in the restricted problem of three bodies. The relevant periodic series

$$
\begin{aligned}
x &= x_{c,0} + \sum_{j=1}^{\infty} x_{c,j} \cos(j\sigma) + \sum_{j=1}^{\infty} x_{s,j} \sin(j\sigma), \\
y &= y_{c,0} + \sum_{j=1}^{\infty} y_{c,j} \cos(j\sigma) + \sum_{j=1}^{\infty} y_{s,j} \sin(j\sigma),
\end{aligned} \tag{1}
$$

with

$$\sigma = \frac{2\pi}{T}(t - t_0) = n(t - t_0), \tag{2}$$

represent a particular solution of the differential equations

$$\begin{aligned} \ddot{x} - 2N\dot{y} &= \Omega_x, \\ \ddot{y} + 2N\dot{x} &= \Omega_y. \end{aligned} \tag{3}$$

In these equations, T denotes the period and n the related frequency of the solution, t_0 the arbitrarily chosen epoch at which this periodic Trojan intersects the straight line connecting the sun and the equilateral point L_5 on the outside of Jupiter's heliocentric orbit, and N the constant angular motion of Jupiter as given by

$$N = (1 + \mu)^{1/2}. \tag{4}$$

Ω_x and Ω_y are the partials with respect to x and y of the function

$$\Omega = \frac{1}{\Delta_1} + \frac{\mu}{\Delta_2} + \frac{1}{2}(\Delta_1^2 + \mu\Delta_2^2). \tag{5}$$

Ω depends on x and y through the Trojan's distances Δ_1 and Δ_2 from sun and Jupiter, respectively, as determined by

$$\begin{aligned} \Delta_1^2 &= (x - m\mu)^2 + y^2, \\ \Delta_2^2 &= (x + m)^2 + y^2, \end{aligned} \tag{6}$$

where m is the auxiliary quantity

$$m = 1/(1 + \mu). \tag{7}$$

All units, definitions etc. are identical with those of [**3**].

The coordinates

$$x^* = x + u, \qquad y^* = y + v \tag{8}$$

of any nonperiodic Trojan, with a motion representable in the form of such oscillations u, v about the periodic reference orbit given by Equations (1), have to satisfy the differential equations (3), too. In this case, the $\Omega_x(x^*, y^*)$, $\Omega_y(x^*, y^*)$ may be expanded in the form

$$\begin{aligned} \Omega_x(x^*, y^*) &= \Omega_x + \Omega_{xx}u + \Omega_{xy}v + \frac{1}{2}\Omega_{xxx}u^2 + \frac{1}{2}\Omega_{xyy}v^2 + \Omega_{xxy}uv + \cdots, \\ \Omega_y(x^*, y^*) &= \Omega_y + \Omega_{xy}u + \Omega_{yy}v + \frac{1}{2}\Omega_{xxy}u^2 + \frac{1}{2}\Omega_{yyy}v^2 + \Omega_{xyy}uv + \cdots, \end{aligned} \tag{9}$$

where the Ω_x, Ω_{xx}, Ω_{xxx}, etc., are the corresponding first, second, and higher order partials of Ω with respect to the x and y of the periodic orbit, and thus are periodic functions of σ and of the time t. The differential equations for the oscillations u, v of such nonperiodic restricted problem Trojans take the form

$$\begin{aligned} \ddot{u} - 2N\dot{v} &= \Omega_x(x^*, y^*) - \Omega_x, \\ \ddot{v} + 2N\dot{u} &= \Omega_y(x^*, y^*) - \Omega_y, \end{aligned} \tag{10}$$

and their integration has been treated in [**3**].

To proceed to the more general case of the elliptical restricted problem, let e denote the eccentricity of the orbit of Jupiter around the sun. If f denotes the true anomaly of Jupiter in this orbit,

$$r = \frac{1 - e^2}{1 + e\cos f} \tag{11}$$

represents the radius vector r as a function of f and thus of the time t. Jupiter's semi-major axis a is equal to unity. The well-known elliptic relations connecting f with the mean anomaly

$$M = M_0 + N(t - t_0) \tag{12}$$

involve the mean angular motion N as given by Equation (4), while the true angular rate of motion $\dot{f}$, as the nonuniform rate of rotation of the ξ, η coordinate frame to be used in the elliptical problem, is determined by

$$r^2\dot{f} = N(1 - e^2)^{1/2}. \tag{13}$$

The new rectangular system ξ, η may be chosen so that its origin coincides with that of the x, y-system, while the ξ-axis coincides with the straight line connecting sun and Jupiter. Consequently, the coordinates ξ_1, η_1 and ξ_2, η_2 of the sun and of Jupiter, respectively, shall be given by

$$\begin{aligned} \xi_1 &= \frac{\mu}{1 + \mu} r, & \eta_1 &\equiv 0, \\ \xi_2 &= -\frac{1}{1 + \mu} r, & \eta_2 &\equiv 0. \end{aligned} \tag{14}$$

For $e = 0$, of course, the Equations (11), (13), and (14) reduce to $r = 1$, $\dot{f} = N$, $\xi_1 = \mu/(1 + \mu)$, $\xi_2 = -1/(1 + \mu)$, or to the appropriate *constant* values of the restricted problem.

If the periodic solution (1) of the restricted Trojan problem is transposed into the ξ, η-reference frame of the elliptical problem as proposed in §II, the resulting intermediate orbit is represented by the expressions

$$(\xi) = rx, \qquad (\eta) = ry. \tag{15}$$

In general, any point in the x, y-frame will be transformed into its "image" in the ξ, η-system by the multiplication of both coordinates with r, so that any nonperiodic solution x^*, y^* of the restricted problem, as represented by Equations (8), will be transposed into a certain curve in the ξ, η plane, given by

$$\xi = rx^* = r(x+u), \qquad \eta = ry^* = r(y+v). \tag{16}$$

If the u, v appearing in Equations (8) and (16) are solutions of the differential equations (10) of the restricted problem, they cannot be expected to satisfy the differential equations of the elliptical problem, as considered in the following section. On the other hand, the as yet unknown solution of the elliptical problem may be *assumed* in the form of the Equations (16), with unknowns u, v instead of ξ, η, and the differential equations for ξ and η may then be transformed into equations for the determination of u and v. The u, v resulting from the integration of these new equations can be substituted into Equations (16), so that now indeed the true motion in the ξ, η-plane will be obtained in the form

$$\xi = (\xi) + ru, \qquad \eta = (\eta) + rv, \tag{17}$$

where ru and rv represent the components of the total displacement of the actual Trojan planet from the position of the fictitious or reference Trojan of Equations (15).

IV. **The differential equations for u, v.** Since the ξ, η-system is rotating about its origin at the center of mass with the variable angular velocity $\dot{f}$, the differential equations of motion for the Trojan of negligible mass take the form

$$\begin{aligned} \ddot{\xi} - 2\dot{f}\dot{\eta} - \dot{f}^2\xi - \ddot{f}\eta &= R_\xi, \\ \ddot{\eta} + 2\dot{f}\dot{\xi} - \dot{f}^2\eta + \ddot{f}\xi &= R_\eta, \end{aligned} \tag{18}$$

where R_ξ and R_η are the partials with respect to ξ and η of

$$R = \frac{1}{\Delta_1} + \frac{\mu}{\Delta_2}, \tag{19}$$

with Δ_1 and Δ_2 now determined by

$$\Delta_1^2 = (\xi - \xi_1)^2 + \eta^2, \qquad \Delta_2^2 = (\xi - \xi_2)^2 + \eta^2. \tag{20}$$

The ξ_1, ξ_2 involved in Equations (20) are functions of r and thus of f and of the time through the Equations (14).

On the basis of Jupiter's elliptic motion, one has

$$\dot{r} = Ne(1 - e^2)^{-1/2} \sin f, \tag{21}$$

while $\dot{f}$ is obtained from Equation (13). The second derivatives with respect to time, as needed in Equations (18), are easily obtained:

$$\ddot{r} = N^2 e r^{-2} \cos f, \tag{22}$$

$$\ddot{f} = -2N^2 e r^{-3} \sin f. \tag{23}$$

For the first and second derivatives of ξ and η, the following expressions follow immediately from the Equations (16):

$$\begin{aligned} \dot{\xi} &= \dot{r}x^* + r\dot{x}^*, \\ \dot{\eta} &= \dot{r}y^* + r\dot{y}^*, \end{aligned} \tag{24}$$

$$\begin{aligned} \ddot{\xi} &= \ddot{r}x^* + 2\dot{r}\dot{x}^* + r\ddot{x}^*, \\ \ddot{\eta} &= \ddot{r}y^* + 2\dot{r}\dot{y}^* + r\ddot{y}^*. \end{aligned} \tag{25}$$

Let R^* denote the function to which R is reduced when the variable r appearing in the expressions for ξ, η, ξ_1, ξ_2, and thus in Δ_1 and Δ_2, is replaced by the constant value 1. It is easily verified that

$$R_\xi = r^{-2} R_x^*(x^*, y^*), \qquad R_\eta = r^{-2} R_y^*(x^*, y^*), \tag{26}$$

where the partials R_x^* and R_y^* are identical with the corresponding partials with respect to x^* and y^*, respectively, in consequence of the definitions (8) of x^* and y^*.

If all the relevant substitutions are made, the differential equations (18) will be transformed into the following two equations for x^* and y^*:

$$\begin{aligned} r^3\ddot{x}^* - 2N(1 - e^2)^{1/2} r\dot{y}^* + 2N(1 - e^2)^{-1/2} r^2 e \sin f \dot{x}^* &= \Omega_x^*, \\ r^3\ddot{y}^* + 2N(1 - e^2)^{1/2} r\dot{x}^* + 2N(1 - e^2)^{-1/2} r^2 e \sin f \dot{y}^* &= \Omega_y^*. \end{aligned} \tag{27}$$

Here the function $\Omega^*(x^*,y^*)$ is determined by the relation

$$(28)\quad 2\Omega^*(x^*,y^*) = 2R^*(x^*,y^*) + N^2(x^{*2}+y^{*2}) + \mu/(1+\mu),$$

and this $\Omega^*(x^*,y^*)$ is identical with the $\Omega(x^*,y^*)$ resulting from the earlier restricted-problem Equation (5) for Ω when x is simply replaced by x^*, and y by y^*.

To develop the left-hand sides of the Equations (27) into Fourier series proceeding in powers of e, with arguments involving multiples of the mean anomaly M of Jupiter's orbit, the elliptic expansions

$$(29)\quad \begin{aligned} r &= 1 + \frac{1}{2}e^2 - \left(e - \frac{3}{8}e^3\right)\cos M - \frac{1}{2}e^2\cos 2M - \frac{3}{8}e^3\cos 3M \cdots, \\ r^2 &= 1 + \frac{3}{2}e^2 - 2\left(e - \frac{1}{8}e^3\right)\cos M - \frac{1}{2}e^2\cos 2M - \frac{1}{4}e^3\cos 3M \cdots, \\ r^3 &= 1 + 3e^2 - \left(3e + \frac{9}{8}e^3\right)\cos M + \frac{1}{8}e^3\cos 3M \cdots, \\ &(1-e^2)^{-1/2}e\sin f \\ &= \left(e - \frac{3}{8}e^3\right)\sin M + e^2\sin 2M + \frac{9}{8}e^3\sin 3M \cdots, \end{aligned}$$

together with the corresponding expansions of $(1-e^2)^{1/2}$, may be substituted. For the right-hand sides of Equations (27), since

$$(30)\qquad \Omega_x^* = \Omega_x(x^*,y^*), \qquad \Omega_y^* = \Omega_y(x^*,y^*),$$

the restricted-problem expansions (9) in powers of u and v are immediately applicable.

After both sides of the Equations (27) have been expanded as indicated, the principal terms of the two equations are identical with those constituting the differential equations (3), as satisfied by the periodic reference Trojan. Therefore, substraction of these earlier equations finally results in the two equations which have to be satisfied by the unknowns u, v of the assumed solution (16) of the elliptical problem. With the aid of convenient auxiliary quantities or definitions, these differential equations take the form

$$(31)\qquad \begin{aligned} \ddot{u} - 2N\dot{v} &= R_1 + e(E_1 + F_1), \\ \ddot{v} + 2N\dot{u} &= R_2 + e(E_2 + F_2), \end{aligned}$$

where

$$(32)\quad \begin{aligned} R_1 &= \Omega_{xx} u + \Omega_{xy} v + \frac{1}{2}\,\Omega_{xxx} u^2 + \frac{1}{2}\,\Omega_{xyy} v^2 + \Omega_{xxy} uv + \cdots, \\ R_2 &= \Omega_{xy} u + \Omega_{yy} v + \frac{1}{2}\,\Omega_{xxy} u^2 + \frac{1}{2}\,\Omega_{yyy} v^2 + \Omega_{xyy} uv + \cdots, \end{aligned}$$

$$(33)\quad \begin{aligned} E_1 &= A\dot{u} + B\dot{v} + C\ddot{u}, \\ E_2 &= -\,B\dot{u} + A\dot{v} + C\ddot{v}, \end{aligned}$$

$$(34)\quad \begin{aligned} F_1 &= A\dot{x} + B\dot{y} + C\ddot{x}, \\ F_2 &= -\,B\dot{x} + A\dot{y} + C\ddot{y}, \end{aligned}$$

and

$$(35)\quad \begin{aligned} A &= -\,2N\left[\left(1 + \frac{3}{8}\,e^2\right)\,\sin M - \frac{1}{8}\,e^2\sin 3M + \cdots\right], \\ B &= -\,2N\left[\left(1 - \frac{7}{8}\,e^2\right)\,\cos M + \frac{1}{2}\,e\cos 2M \right. \\ &\qquad\qquad \left. + \frac{3}{8}\,e^2\cos 3M + \cdots\right], \\ C &= -\,3e + \left(3 + \frac{9}{8}\,e^2\right)\,\cos M - \frac{1}{8}\,e^2\cos 3M + \cdots. \end{aligned}$$

Except for the additional terms $e(E_1 + F_1)$ and $e(E_2 + F_2)$, the differential equations (31) are identical with the equivalent Equations (10) of the restricted problem, and for $e = 0$ the Equations (31) reduce to that earlier system. For $e \neq 0$, of course, the u, v appearing in Equations (31) are different by definition from those in Equations (10) and only for $e = 0$ do the two definitions become identical. It is evident that, for $e \neq 0$, the Equations (31) will *not* be satisfied by $u \equiv 0$, $v \equiv 0$, because of the nonvanishing functions F_1, F_2 which are independent of u, v and their derivatives. Therefore, the reference orbit (ξ), (η) of Equations (15) is not a solution of the elliptical problem. Apparently, then, any solution u, v satisfying the equations (31) must involve fluctuations of certain minimum amplitudes, depending on the given value of e. It will be seen that two of the four constants of integration of the general solution of the Equations (31) are associated with this "forced" part of the solution, while the remaining two constants are related to the "free" part, or to the arbitrary initial deviations from the trajectory represented by the forced solution. In this respect, the forced solution emerges as the equivalent of the periodic reference orbit in the restricted problem.

V. **The principal features of the solution.** Since the terms eF_1 and eF_2 in Equations (31) are independent of u and v, they have to be considered only in the determination of the forced part, say u_0, v_0, of the complete solution

$$u = u_0 + u_f, \qquad v = v_0 + v_f, \tag{36}$$

which includes the free oscillation u_f, v_f. According to Equations (34) and (35), the basic expansions for the periodic solution x, y, as represented by Equations (1), as well as the elliptic expansions for A, B, C, will affect the functions F_1 and F_2, and thus the forced solution u_0, v_0. Consequently, arguments of the form $jM + k\sigma$ will be characteristic for the resulting expansions of u_0, v_0.

In the differential equations, all terms involving u, v and their derivatives will affect the forced solution u_0, v_0, aside from their role in the free solution u_f, v_f. Therefore, the complete Equations (31) have to be considered in the successive approximations for u_0, v_0. However, since the partials Ω_{xx} etc., in the linear terms of R_1 and R_2 are of zero order, while in eE_1 and eE_2 the coefficients of $\dot{u}$ etc., are at least of the order of e, the first approximation for u_0, v_0 may be obtained from the reduced equations:

$$\begin{aligned} \ddot{u} - 2N\dot{v} &= R_1 + eF_1, \\ \ddot{v} + 2N\dot{u} &= R_2 + eF_2. \end{aligned} \tag{37}$$

If complex variables u_0, v_0 are introduced, as was done in [**3**], in the integration procedures for the restricted problem Trojans, the solution of Equations (37), as well as of the complete Equations (31), may be assumed in the form:

$$\begin{aligned} u_0 &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} u_{j,k} \exp[i(jM + k\sigma)], \\ v_0 &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} v_{j,k} \exp[i(jM + k\sigma)]. \end{aligned} \tag{38}$$

The unknown coefficients $u_{j,k}$, $v_{j,k}$ will be obtained from the identities resulting from the substitution of the assumed u_0, v_0, and of their first and second derivatives, into the Equations (37). For the first approximation, only those terms involving u and v linearly should be considered in R_1 and R_2, and only the first power of e in eF_1 and eF_2. As to the coefficients Ω_{xx}, Ω_{yy}, and Ω_{xy} of u and v in R_1 and R_2, their expansions may be taken from [**3**]. If the basic periodic libration x, y is of small amplitude, the use of the constant terms of Ω_{xx} etc., may be sufficient for the integration of Equations (37). If the amplitude of the periodic orbit is large, however, the principal periodic terms of Ω_{xx} and of the other two partials of Ω will have to be considered, too, in the first approximation from Equations (37).

For the second approximation, terms of the order e^2 will have to be considered on the right-hand sides of the differential equations. Depending on the size of the amplitudes of the principal u_0, v_0-terms obtained from the first approximation, the previously neglected parts eE_1 and eE_2 of the complete Equations (31) may have to be included in this second approximation, and even the terms involving u^2, v^2, uv, through R_1 and R_2, may become significant now or later on. No principal difficulty stands in the way, however, of the required successive determinations and refinements of the coefficients $u_{j,k}$, $v_{j,k}$ of the solution (38), until the solution satisfies the rigorous Equations (31) to the desired degree of numerical precision, considering such powers of e, n, u, v, and such coefficients in the expansions for x, y, Ω_{xx} etc., as may be required to achieve this accuracy. It is evident, from the form of the Equations (31) to (35), that all exponents or arguments created in these successive approximations will be of the same general form $jM + k\sigma$. The integers j and k may be positive or negative, and terms where j or k or both are zero will be encountered in the higher approximations.

To illustrate this method of solution, and to indicate the nature of the principal terms of the forced oscillations u_0, v_0, the reduced Equations (37) will be integrated, considering only the first power of e and assuming a small libration amplitude. The latter assumption will justify the consideration of only the principal terms of $\dot{x}$ and $\dot{y}$, and only the constant terms α_0, β_0, γ_0 of Ω_{xx}, Ω_{yy}, Ω_{xy}, respectively. Since $\ddot{x}$ and $\ddot{y}$ contain n^2 as a factor, compared to the corresponding factor n in $\dot{x}$ and $\dot{y}$, $\ddot{x}$ and $\ddot{y}$ may also be omitted in Equations (34) for F_1 and F_2. Consequently, the Equations (37) are further reduced to

$$\begin{aligned} \ddot{u} - 2N\dot{v} &= \alpha_0 u + \gamma_0 v - 2Ne(\dot{x}\sin M + \dot{y}\cos M), \\ \ddot{v} + 2N\dot{u} &= \gamma_0 u + \beta_0 v + 2Ne(\dot{x}\cos M - \dot{y}\sin M), \end{aligned} \tag{39}$$

with

$$\begin{aligned} \dot{x} &= -\,n x_{c,1}\sin\sigma + n x_{s,1}\cos\sigma, \\ \dot{y} &= -\,n y_{c,1}\sin\sigma + n y_{s,1}\cos\sigma. \end{aligned} \tag{40}$$

The differential equations (39), as well as the original system (37), have been given without the subscript zero for the variables u, v. It is understood, however, as implied by the inclusion of the eF_1 and eF_2-terms, that these are the equations for the determination of the forced solution u_0, v_0. Instead of the exponential solution (38), the trigonometric form

$$\begin{aligned} u_0 &= a_1\cos(M+\sigma) + b_1\sin(M+\sigma) \\ &\qquad + a_2\cos(M-\sigma) + b_2\sin(M-\sigma), \\ v_0 &= c_1\cos(M+\sigma) + d_1\sin(M+\sigma) \\ &\qquad + c_2\cos(M-\sigma) + d_2\sin(M-\sigma) \end{aligned} \tag{41}$$

may be assumed at once in this simple case. It is easily seen that indeed all terms involving the first power of e should be associated with the two arguments considered in Equations (41), because $2M$ for instance enters eF_1 and eF_2 only in association with coefficients involving e^2. Similarly, 2σ can enter only in connection with the supposedly second-order coefficients $x_{c,2}$, $x_{s,2}$, $y_{c,2}$, $y_{s,2}$ of the periodic libration x, y.

Substitution of the assumed solution (41) into the two Equations (39) produces eight identities, namely two for each cosine- and sine-function of the two arguments involved. The following four identities must be satisfied by the coefficients a_1, b_1, c_1, d_1:

$$\begin{array}{lllll} [\alpha_0 + (N+n)^2]a_1 & & +\;\gamma_0 c_1 + & 2N(N+n)d_1 = & enNP_1, \\ & [\alpha_0 + (N+n)^2]b_1 - & 2N(N+n)c_1 + & \gamma_0 d_1 = & enNQ_1, \\ \gamma_0 a_1 - & 2N(N+n)b_1 + & [\beta_0 + (N+n)^2]c_1 & = & -enNQ_1, \\ 2N(N+n)a_1 + & \gamma_0 b_1 & & +\,[\beta_0 + (N+n)^2]d_1 = & enNP_1, \end{array} \tag{42}$$

with the P_1 and Q_1 given below in Equations (44). The a_2, b_2, c_2, d_2 have to be determined from the very similar equations

$$
(43)\quad
\begin{aligned}
&[\alpha_0+(N-n)^2]a_2 + \gamma_0 c_2 + 2N(N-n)d_2 = enNP_2,\\
&[\alpha_0+(N-n)^2]b_2 - 2N(N-n)c_2 + \gamma_0 d_2 = enNQ_2,\\
&\gamma_0 a_2 - 2N(N-n)b_2 + [\beta_0+(N-n)^2]c_2 = -enNQ_2,\\
&2N(N-n)a_2 + \gamma_0 b_2 + [\beta_0+(N-n)^2]d_2 = enNP_2.
\end{aligned}
$$

In these two systems of linear equations, the P_1, Q_1, P_2, Q_2 involved in the absolute terms are given by

$$
(44)\quad
\begin{aligned}
P_1 &= x_{c,1} + y_{s,1}, & P_2 &= -x_{c,1} + y_{s,1},\\
Q_1 &= x_{s,1} - y_{c,1}, & Q_2 &= x_{s,1} + y_{c,1}.
\end{aligned}
$$

The coefficients of the unknowns of Equations (43) differ from the corresponding coefficients in Equations (42) only by the appearance of $-n$ instead of $+n$, wherever n is involved.

To permit unique solutions, the determinants D_1 and D_2 of the systems (42) and (43), or

$$
(45)\quad D_{1,2} = \begin{vmatrix}
\alpha_0+(N\pm n)^2 & 0 & \gamma_0 & 2N(N\pm n)\\
0 & \alpha_0+(N\pm n)^2 & -2N(N\pm n) & \gamma_0\\
\gamma_0 & -2N(N\pm n) & \beta_0+(N\pm n)^2 & 0\\
2N(N\pm n) & \gamma_0 & 0 & \beta_0+(N\pm n)^2
\end{vmatrix},
$$

should not vanish. The symmetrical determinant (45) reduces to the expression

$$
(46)\quad D_{1,2} = \{[\alpha_0 + (N\pm n)^2][\beta_0 + (N\pm n)^2] - \gamma_0^2 - 4N^2(N\pm n)^2\}^2,
$$

and each nonvanishing subdeterminant of order 3 is found to contain the factor $(D_{1,2})^{1/2}$. For small periodic librations x, y, the constants α_0, β_0, γ_0 may be approximated by the values of Ω_{xx}, Ω_{yy}, Ω_{xy} at L_5, as listed in Equations (17) of [**3**]. In this case, $D_{1,2}$ will be found to be approximated by $4n^2$, and all the nonvanishing minors of $D_{1,2}$ have values of the order of $2n$. Consequently, when the linear Equations (42) and (43) are solved for the coefficients $a_1, \cdots, d_2$ of the solution (41), the absolute terms, $enNP_1$ etc., will be divided by quantities of the order $2n$. Therefore, the resulting amplitudes $a_1, \cdots, d_2$ of the forced oscillation (41) will be of the general order of eL, where L simply represents the amplitude (expressed in units of the mean distance sun-Jupiter) of the basic periodic libration.

The preceding considerations and results show that the rigorous Equations (31) will indeed be satisfied by such a specific solution u_0, v_0, produced by the eccentricity of Jupiter's orbit. The principal terms of this impressed solution are those of the first approximation (41), with coefficients of the order of eL, but this result is based on the assumption of a small libration amplitude L. For L-values exceeding substantially Jupiter's orbital eccentricity e, the principal *periodic* terms of Ω_{xx} etc., namely the terms depending on $\sin\sigma$ and $\cos\sigma$, will also have to be considered as factors of u and v in Equations (39), in addition to the constant parts α_0 etc., of these partials of Ω. This in turn will necessitate the consideration of the additional arguments $M-2\sigma$, M, $M+2\sigma$, and of the resulting additional identities for the determination of an increased number of unknowns. There is no real difficulty, however, which would prevent the gradual refinement of the initial solution u_0, v_0 until it satisfies the complete Equations (31) to the desired degree of perfection. Constant terms as well as terms of long period will appear in the higher approximations, because j and k may be zero simultaneously or separately in the general form (38) of the solution. No secular terms, however, enter the picture. Small divisors are possible and should be given special attention in any detailed application of this theory, but normally these may be expected to occur in connection with higher powers of e, and with higher-order coefficients $x_{c,j}$ etc.

The solution (38) will become periodic in the case of a commensurability between N and n, or between the two fundamental periods. In the sun-Jupiter case, the only $j/1$ type commensurability within the period range of the actual Trojan planets is represented by the ratio

$$T/P = N/n = 13/1. \tag{47}$$

Additional "simple" solutions, of the type $j/1$, exist for the similar commensurabilities

$$T/P = 14/1,\ 15/1,\ 16/1, \cdots, \tag{48}$$

but the corresponding periods T are longer than those of all the known real Trojan planets. Since the sequence (48) of commensurability ratios represents an infinite number of orbits still inside of the so-called limiting orbit with $T=\infty$, the "density" of these periodic solutions apparently increases with $T \to \infty$. In the restricted problem, periodic solutions exist for all T-values exceeding the minimum value associated with the librations of infinitesimally small dimensions. In the elliptic problem, simple periodic solutions exist only for the T/P ratios $13/1, 14/1, \cdots$, and solid coverage of the x, y plane is approached only as these orbits converge towards the limiting orbit with $T = \infty$.

If the commensurability is of the more general type

$$T/P = j/k, \qquad k \neq 1, \qquad j > k, \tag{49}$$

periodic solutions of the elliptic problem exist for periods $T^* = 2T$, $3T, 4T, \cdots$, but again only for specific values of T^* or T. Such periodic orbits are different from the simple librations of period T, in so far as such Trojans would return to their starting position and velocity components only after $2, 3, \cdots$ revolutions about the equilateral point.

According to Equations (16), the Trojan's coordinates ξ, η are functions not only of u, v, but also of x, y, and r. Only integral multiples of M and of σ are involved in all these quantities, however, and therefore all arguments appearing in the resulting expansions of ξ and η will again be of the form $jM + k\sigma$. Consequently, if the forced solution u_0, v_0 is periodic because of a commensurability between N and n, the related coordinates ξ_0, η_0 are periodic, too.

The forced solution u_0, v_0 of the Equations (31) is unique in so far as, for given values of the constants e and M_0 associated with Jupiter's orbit, one and only one such solution (38) exists. This solution is an intrinsic part of the real equivalent of the related periodic orbit in the restricted problem, because the "elliptic equations" (18) can not be satisfied without it, and it may now serve as a reference solution for the superposed free oscillation u_f, v_f.

It is evident that the separation of u, v into the forced part u_0, v_0 and the free part u_f, v_f leads immediately to a corresponding separation of the left-hand sides of the differential equations (31), as a reference solution for the superposed free oscillation u_f, v_f. v and their derivatives are linearly involved. The only complication comes from those parts of R_1 and R_2 which involve u^2, v^2, uv, u^3, etc. All these higher-order powers and products can be expanded, however, into polynomials proceeding in powers of u_f and v_f, with coefficients depending on u_0 and v_0. The simple example

$$u^2 = u_0^2 + 2u_0 u_f + u_f^2 \tag{50}$$

is representative of the essential features of all these polynomials. One term is independent of u_f and v_f, while the others involve their various powers (and products of such powers, in the more general case), with coefficients which are known functions of the forced oscillation u_0, v_0.

If the forced oscillation u_0, v_0 is considered as a given reference solution, then the differential equations (31) as satisfied by u_0, v_0 may be subtracted from the same equations for $u = u_0 + u_f$, $v = v_0 + v_f$. The resulting differential equations for u_f, v_f are of the same form as the original Equations (31) for u, v, except for the additional terms originating from the second and higher order involvement of u and v through the parts R_1 and R_2, and except for those terms canceled out because of their independence of u_f and v_f. It is easily verified that consequently u_f and v_f must satisfy the equations

$$\begin{aligned} \ddot{u}_f - 2N\dot{v}_f &= S_1 + eE_1, \\ \ddot{v}_f + 2N\dot{u}_f &= S_2 + eE_2, \end{aligned} \tag{51}$$

where E_1 and E_2 are still given by the earlier Equations (33), but for $u \equiv u_f$, $v \equiv v_f$, while S_1 and S_2 stand for

$$\begin{aligned} S_1 &= (\Omega_{xx} + \Omega_{xxx}u_0 + \Omega_{xxy}v_0)\, u_f + (\Omega_{xy} + \Omega_{xyy}v_0 + \Omega_{xxy}u_0)\, v_f \\ &\quad + \frac{1}{2}\,\Omega_{xxx}u_f^2 + \frac{1}{2}\,\Omega_{xyy}v_f^2 + \Omega_{xxy}u_f v_f + \cdots, \\ S_2 &= (\Omega_{xy} + \Omega_{xxy}u_0 + \Omega_{xyy}v_0)\, u_f + (\Omega_{yy} + \Omega_{yyy}v_0 + \Omega_{xyy}u_0)\, v_f \\ &\quad + \frac{1}{2}\,\Omega_{xxy}u_f^2 + \frac{1}{2}\,\Omega_{yyy}v_f^2 + \Omega_{xyy}u_f v_f + \cdots. \end{aligned} \tag{52}$$

Any third and higher order terms of Equations (52) are easily established, too, if required, on the basis of the corresponding Taylor expansions (32) of R_1 and R_2.

Since no terms independent of u_f and v_f are involved in Equations (51), the particular solution $u_f \equiv 0$, $v_f \equiv 0$ exists, as it should. For u_f, v_f of any amplitude, the terms eE_1, eE_2 will be smaller by one order of e than the linear parts of S_1 and S_2, so that the first approximation to the solution can be obtained from the equations

$$\begin{aligned} \ddot{u}_f - 2N\dot{v}_f &= \Omega_{xx}u_f + \Omega_{xy}v_f, \\ \ddot{v}_f + eN\dot{u}_f &= \Omega_{xy}u_f + \Omega_{yy}v_f, \end{aligned} \tag{53}$$

with appropriately reduced expansions for Ω_{xx} etc. These Equations (53) are identical with the corresponding approximation in the restricted problem, so that the first order result for the free oscillations in the elliptical problem is also expressible in the same exponential form,

$$u_f = \sum_{-\infty}^{\infty} u_k \exp[i(k+c)\sigma], \qquad v_f = \sum_{-\infty}^{\infty} v_k \exp[i(k+c)\sigma]. \tag{54}$$

Here c is the characteristic exponent, to be determined from the condition that the determinant of the system of identities resulting from the substitution into Equations (53) should vanish. In terms of real variables, two of the coefficients of the solution, or the amplitude and initial phase of the most significant term, may be considered as constants of integration. With e and M_0 as the constants determining the elliptic motion of Jupiter in relation to the periodic reference orbit x, y of Equations (1), four arbitrary constants are thus involved in the complete solution u, v. The forced part u_0, v_0 depends on e and M_0 alone, but beginning with the second approximation, u_f, v_f involve all four constants, because of the appearance of u_0 and v_0 in the second and higher order terms of S_1 and S_2 in Equations (51).

The second and higher approximations for u_f, v_f will have only certain terms in common with the corresponding solution of the restricted problem, but many additional terms arise in consequence of the eccentricity of Jupiter's orbit, through the eE_1 and eE_2 terms of Equations (51) as well as through the appearance of u_0 and v_0 in the higher order parts of S_1 and S_2. For $e = 0$, of course, the u_f, v_f solution reduces to the comparable solution of the restricted problem.

When the first-order solution for u_f, v_f, with arguments of the form $(k + c)\sigma$, is substituted into the parts eE_1 and eE_2 of Equations (51), new terms with arguments of the type $jM \pm (k + c)\sigma$ are created. Since the combination $j = 1$, $k = 0$ is admissible, the second approximation to u_f, v_f will contain certain terms, factored by e

and by the relevant coefficient from the first-order solution, with arguments S of the form

$$S = M_0 + (N - cn)(t - t_0). \tag{55}$$

For small periodic librations, cn may be approximated by the corresponding result for the short-period librations of infinitesimal amplitude, namely

$$cn \approx 1 - \frac{23}{4}\mu. \tag{56}$$

Since $N \approx 1 + (1/2)\mu$, Equation (55) takes the approximate form

$$S \approx M_0 + \frac{25}{4}\mu(t - t_0), \tag{57}$$

indicating that the period of such terms will be very long, of the order of about 170 revolutions of Jupiter, or about 13 complete libration periods. The amplitude of such terms may exceed that of the causative principal term of the first approximation by a factor of the order of 10, because a closer analysis discloses the involvement of a small divisor of the order of 7μ, together with a multiplicator of the order of e. Nevertheless, all such terms are part of the *free* solution, which may be reduced to zero by an appropriate choice of the constants of integration. If the starting conditions are such that the deviation from the forced solution is not relatively small, the Trojan planet will have to compensate for this departure with substantial terms of the long period associated with the argument S of Equation (55).

References

1. Eugene Rabe, *Determination and survey of periodic Trojan orbits in the restricted problem of three bodies*, Astronom. J. **66** (1961), 500-513.

2. ———, Astronom. J. **67** (1962), 382-390.

3. ———, *Outline of a theory of nonperiodic motions in the neighborhood of the long-period librations about the equilateral points of the restricted problem of three bodies*, Lectures in Applied Mathematics, Vol. 6, Amer. Math. Soc., Providence, R. I., 1966; pp. 79-101.

4. Eugene Rabe and A. Schanzle, Astronom. J. **67** (1962), 732-739.

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The Equilibrium Shape of the Earth in the Light of Recent Discoveries in Space Science

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I. **The physical setting of geodesy.** The study of the figure of the earth has its historical roots in studies made by geodesists. These studies came from two sources: One was the detached scientific desire to know more about the figure of the earth which moved Eratosthenes and Snell; the other was the practical urge to produce adequate maps which moved the Cassinis (see [1]) and Digges. The scientific motivation for the study of the earth is relatively easy to understand, but I should like to point out some of the practical reasons which have powerfully reinforced scientific motivations.

The practical surveyor is attempting to construct a map which will serve the ordinary purposes of daily life. For some of them, such as hiking or automobile travel, an accuracy of 1 percent is more than sufficient. For others, including the problem of artillery firing, the laying out of pipe lines, the emplacement of micro-wave antennae and the putting in of telephone lines an accuracy of a tenth of 1 percent would be desirable so far as the paper stability permits it. These accuracies would not by themselves justify the precision which is lavished on first order triangulation. It might appear possible to make relatively crude surveys and patch them together. In practice, however, it is found that this policy is extremely expensive and that it is far more satisfactory to have an underpinning of precise survey. What happens when you have a set of inaccurate maps is that in the compilation room the conflicts between the maps appear. For example, suppose that the maps are in error by 1 percent; then along the junction between two individual sheets you may have an error of a few tenths of an inch, which might be tolerable; but when you have joined together 20 or 30 such maps to form a loop or an area, then you find that there are discrepancies of many times this amount where the loops close. Since the mapping of even so small an area as France involves several hundred map sheets, this procedure is evidently very unsatisfactory.

Theoretically one could go into the compilation room and say to the other compilers that they should distort their sheets in such a way as to produce a unified whole and that you don't care how they do it. If this is resorted to, then enormous waste and delays will ensue. The compilers will want to work on the area a little at a time. Left to themselves they will crowd all the errors into one area where they become intolerable, or they will start in two different areas and when these two areas join an intolerable discrepancy will be found. In the meantime endless discussions will rage among the compilers as to how this problem is to be met. Since the compilers are very numerous compared to the first order triangulators, the net loss is very large indeed.

Just prior to the German invasion of France in 1940 there was a conference among the allies about the problem of the adjustment of the Dutch, Belgian and German map and survey data to agreement with the French. The plan called for the recalculation of the Belgian and Dutch triangulation starting from French triangles. German triangulation was adjusted by applying blanket corrections to the latitudes and longitudes. Since these corrections left a discrepancy on the order of 11 meters between certain points of Holland and Germany, a graph was prepared. This graph was intended to adjust not the map data but only the lists of surveyed points which were supplied to the artillery for their purposes.

Using everything except the graph, the U. S. Army Map Service prepared a series of maps of Holland. The maps were compared with the coordinate lists. Since the Dutch maps were on the stereographic projection, there was felt to be some uncertainty about putting them into the framework of the Lambert projection used by the French. These worries became acute when it was discovered that the original Dutch stereographic coordinates could not be converted to satisfactory agreement with the British coordinate lists by the aid of information available to the U. S. In the meantime the invasion of France by the Germans resulted in the loss of important portions of the records. For several months, efforts continued at the Army Map Service and in the Corps of Engineers to discover some mathematical discrepancy which would explain the difference between the American coordinates lists and the British lists. During this time the printing of the maps was delayed. The discrepancy was finally explained when the British produced the graph, but the dislocation of the map production program had serious effects on the later conduct of the war. Had there been an orderly and well understood program, this delay would not have occurred.

It turns out that the only way of adjusting a whole series of maps to agreement with one another is to provide a precise framework for the area as a whole and to pin each map to that framework. Of course the framework itself must suffer arbitrary adjustments which are disguised as least-squares solutions, but the magnitude of the discrepancies which are tolerated here can be kept below the level which is detected by the compilers. As a result the inevitable squabbling about how those discrepancies are to be adjusted can be confined to a relatively small number of people. Here the sternest practicality indicates the need for triangulation data.

When it is a matter of adjusting the triangulation between several countries, it is an enormous advantage if there exists a framework so precise that each of the several countries involved will accept it as superior to its own. The reason is that when a staff conference is held, each of the military officers in the conference is representing a group of civilian employees whom he cannot easily consult. A few of them may be sitting back of the conference table at his elbow, but the great majority are necessarily left at home. He cannot easily make concessions. The question of national pride is deeply involved. To adopt the proposal of another country when it is obviously unscientifically constructed and to distort one's native maps and surveys to fit it is felt as humiliating and is resisted. If, on the other hand, the proposal for survey unification is scientifically drawn and will represent an overall improvement in the survey situation even in the separate countries, then acceptance is much more readily secured.

Thus we see that *precision in survey is a tool of the high command.*

In securing survey precision one obstacle is more serious than any other and sets a limit to the precision that is reached. This obstacle is the crookedness of the path of light through the atmosphere. Let us remember that at the moment when we see the sun's lower limb touch the horizon, the whole of the sun is below the horizon, and would so appear if there were no atmosphere. That is to say the refraction amounts to one-half a degree on long rays through the atmosphere. If we compare the curvature of some 1800 seconds of arc with the desired angular precision, which is less than a single second of arc, we see the magnitude of the problem which the geodesists must face.

It is characteristic of geodesy that the method by which this problem is attacked is the use of the gravitational field of the earth. In the determination of height above sea level, in the determination of position on sea level, and in the exploration of the sea level surface itself, the geodesist takes advantage of the gravitational field of the earth to correct the errors arising from atmospheric refraction.

The first example is the measurement of height. When it is impossible to avoid it, vertical angles are sometimes measured between points whose relative elevation is to be found. The inevitable effects of the curvature of the ray are minimized so far as possible by measuring reciprocally over the line; that is, measuring the angular elevation of B as seen from A and the elevation of A as seen from B simultaneously. It turns out that this procedure eliminates the effect of the mean curvature over the line. It does not, however, eliminate higher order difficulties, and the angular accuracy which is attainable is on the order of one ten-thousandth or one twenty-thousandth of the distance. Here it will be noted that by referring the angles to the zenith at both ends of the line, some use was made of the earth's gravitational field.

A far more effective use arises when the line is cut up into a large number of small pieces and the relative elevations are determined section-by-section. The best instrument for this purpose is the spirit level. In practice, the surveyor puts the spirit level at the center of the small section which he is measuring; he sets the optical axis level and points first at the rod ahead and then at the rod in back or vice versa. Through his telescope he can read the height of the mark on the rod to which his telescope is pointing. The difference of the two rod readings is a very good approximation to the difference between the heights of the feet of the rods. The curvature of the ray is much less troublesome on a short section since its effects increase with the square of the distance. Thus a section one kilometer long cut into 100 meter bits will have only one-tenth the total amount of curvature that the whole kilometer piece would have had. Moreover, by measuring both forward and backward from the middle of the line, the surveyor is able to make the effects of curvature cancel on each separate line. The ray curves downward from the instrument toward the mark by the same amount in both cases. By this method of spirit leveling it is possible, for example, to determine the heights of points in the center of the United States with an accuracy of a few tenths of a meter referred to tide gauges on the coast. At a distance of a few thousand kilometers these tenths of a meter subtend angles of only a small fraction of a second of arc. We see that the curvature of the ray has in a certain sense been straightened out by continual reference to the direction of the vertical.

In measurements of horizontal position, again we find that the properties of the gravitational field are used. It turns out that the ray of light is curved in a direction perpendicular to the stratification of the atmosphere. This stratification is in nearly horizontal layers. If, therefore, the geodesist measures angles in the horizontal plane his angles will be nearly free of the effects of refraction. It turns out that on a day when vertical angles are distorted by many minutes of arc, the horizontal angles as measured will be accurate within a fraction of a second of arc.

Since the days of Pierre Bouguer, in the middle of the 18th century, it has been customary to represent the results of such angle measurements as these by supposing them to have been measured on an imaginary prolongation of the sea level surface under the land (see [**6**]). This prolongation is called the geoid. In order to bring the measured lengths into the same intellectual framework, it has been customary since the time of Bouguer to reduce the lengths to the values which they would have had if measured at sea level between the points vertically below the actual ends of the measured pieces. Thus the net result of an extensive triangulation measurement is the fixing of angles and lengths as if they had been measured on the geoid. They are accompanied at the same time by spirit leveling measurements which give heights above the geoid.

In all of the above the question of the exact form of the geoid is systematically ignored. For local surveys it is possible to get by with the assumption that the earth is flat. No significant distortions of horizontal angles will appear unless the triangle approaches an area of 100 square kilometers. For more extensive surveys, up to the size of a state of the U.S., it is often possible to get by with the assumption that the earth is a sphere. Even in national surveys it is possible to make a precise computation assuming that the earth is an ellipsoid of revolution, but not troubling to get the exact parameters of the ellipsoid. These methods are perfectly adequate as long as the measurements are only those of horizontal angles or lengths along the surface, and as long as the results which are desired from the measurements are of the same kind. In particular, the heights which are wanted for the construction of dams or the laying of pipes or other hydraulic problems are of just this kind. The notion of the true form of the geoid is merely parasitic in most ordinary engineering applications of geodesy.

The mathematicians have been confronted with a situation which they thoroughly enjoy. The problem is to devise coordinate systems and methods of thought in which it will be possible to move about over the surface of the earth in the spirit of a two dimensional being who does not know that there is such a thing as up and down. The problem is one of great mathematical interest. Some of the most beautiful of the papers of Gauss concerned themselves with this problem, and the modern theory of relativity inherits its point of view and many of its mathematical techniques from Gauss, his pupil, Riemann, and his successors, the founders of tensor analysis.

The geophysicists never really liked this situation and were constantly endeavoring to find out something about the form of the geoid. They got very little support from the practical people until the modern age of the intercontinental ballistic missiles, the earth satellite and the space probe. For each of these, what is needed is the true x, y, z coordinate of the tracking station referred to the center of the earth. To convert the measurements made on the geoid to measurements referred to the center requires a knowledge of the shape of the geoid, and it is with this we will concern ourselves next.

The first approximation to the form of the geoid which is in practical use today is the assumption that it is an ellipsoid of revolution with a semi-major axis a, and a semi-minor axis b. Instead of giving b, it is more customary to give the quantity $(a - b)/a$ which is called ϵ the flattening. The measurement of these two quantities was originally made by determining the radius of curvature at various latitudes. The first determination was made in the 18th century by the expeditions of the French academy to Peru and Lapland. The method has remained in vogue with improvements right up to the work of Chovitz and Fischer on the Hough spheroid in 1956. In recent times, however, there has been a tendency to rely on measurements of gravity for the determination of the flattening. There has also been a tendency to obtain the flattening from the relationship between the constant of precession and the hydrostatic theory. It turns out, in fact, that measurements of the radius of curvature do not give particularly reliable measures of both quantities a and ϵ. Instead, they give a relation between the two.

Once an ellipsoid has been assumed, the geodesists concern themselves with the deviations between the actual shape of the geoid and that of the assumed ellipsoid. Several methods of measuring these undulations of the geoid are in use.

In the first place, it is possible to make astronomic measurements of latitude and longitude along a triangulated arc. Each measurement of latitude and longitude amounts to a determination of the direction of the vertical at that point. When this is compared with the calculated direction of the vertical, the so-called geodetic latitude and longitude, the differences which appear are called the deflection of the vertical or perhaps the deflection of the plumb, depending on whether we think of ourselves as looking upward or downward along the vertical. Each deflection of the vertical can be thought of as giving the slope of the geoid with respect to the ellipsoid at a particular point. If we combine these deflections, we can build up a picture of the height of the geoid above the ellipsoid in much the same way as a picture is built up of the form of the topography by clinometric measurements, i.e., measurements of the slope. The process is called astronomical leveling, and it is found that with a reasonable distribution of the astronomical stations, a precision of the order of a few meters can be reached. The weakness of this method lies in the fact that only relative heights are determined. An initial height above the ellipsoid must be quite arbitrarily assumed. Hayford arbitrarily assumed a height of + 10 meters at Calais, Maine. It was also necessary to make a more or less arbitrary assumption about the place at which the slope of the geoid matches that of the ellipsoid. For the United States, the *average* slope of the geoid matches that of the ellipsoid very closely; for France the two are made equal for five astronomic stations near Paris; for England they are equated at the old Greenwich Observatory; for Spain at the observatory in Madrid, and so on.

Another method, having a different set of troubles, relies upon gravity. If gravity data were available for the whole earth then it would be possible, according to a theorem worked out by G. G. Stokes, to determine the gravitational potential at every point. The underlying idea can perhaps be put in the following way. The intensity of gravity as it is measured at any point depends essentially on the integrated mass in a unit column under the station. In its effect on the gravity meter, a layer which is at a depth of several kilometers has no less effect than one which is only a few meters down. The reason is that while a single gram would be much more effective when nearby than when far away, yet in terms of its contribution to the vertical component of gravity it is only the chunks which are within a reasonable angle from the vertical that matter. The amount of any layer which is within a cone of, say, 45° from the vertical will be proportional to the square of the distance from the station, and this increase in the amount of material balances the decrease in the effectiveness per gram, so that in a horizontally stratified earth the intensity of gravity is a fair measure of the column integral of the mass. As a consequence, it is possible in many cases to formulate the application of Stokes' principle by imagining the earth to consist of a shell with a surface distribution of matter which is proportional to the intensity of gravity at the point. The elaborate integrals which appear in Stokes' equation are, in fact, not much more than the expression of this idea.

It will be seen at once that the effectiveness of Stokes' theorem depends on a reasonably complete knowledge of the intensity of gravity over the earth. Any gaps in our knowledge will inevitably falsify the potential, not only as far as the absolute value of the slope is concerned, but even the shape of the geoid. On the whole, the dimensions of the geoid from gravity are usually found to be more accurate in local details but less accurate in overall shape than the dimensions found by astronomical leveling.

The end result, therefore, of the geodetic surveys of the earth is a set of x, y, z coordinates in which we have superposed the measured heights and measured horizontal coordinates on a geoid whose general shape was found by the methods of astronomy and gravity (for sample heights see Figures 3 and 4). It is a long detour to get a simple result, and many modern geodesists have suggested that this detour is not really necessary. In particular, Martin Hotine has suggested that surveyors should regard their measured angles in the same way that a photogrammetrist regards the angles which he can obtain from a single photograph. Hotine suggests that triangulation nets should be built up by the step-wise accumulation of sets of angles, a procedure which may be called *three-dimensional geodesy*. The comparison is very instructive but, in fact, it is found that when Hotine's procedure is carried out, the results are inferior to those produced by ordinary techniques of calculation.

The reasons for the failure of three-dimensional geodesy are two-fold. First, in an ordinary photogrammetric survey most of the angles are nearly vertical, which means that the refraction of light along the lines is relatively small. In the second place, the requirements for precision in photogrammetric surveys are much less than the requirements in geodetic surveys. As a consequence of these two facts, the photogrammetrist is justified in considering that any direction which he measures is in error by a small solid angle whose trace on the sphere is nearly circular. The geodesist, on the other hand, considers that his angles are likely to have errors in the vertical direction which are orders of magnitude larger than those in the horizontal direction. It is for this reason that the techniques of geodesy are so entirely alien to those of photogrammetry.

On the other hand, it is a consequence of this thought that when we observe targets which are very high above the earth, such as satellites, instead of the conventional geodetic targets, which are lights around the horizon, then the mathematical situation in geodesy becomes very much like that in photogrammetry. Since the future is likely to bring us more high targets to observe on, and since the mathematics required to deal with these problems is much simpler than that required in the usual geodetic methods, it is likely that this whole fragile web of thought which I have been describing for you is one whose practical significance will become less every year.

It is still, however, the best way to obtain precise positions. Finally, its historic importance as the parent of differential geometry and so of the theory of relativity will give it a place in the hearts of mathematicians for years to come.

II. **The physical significance of the flattening of the earth.** It was Newton who first pointed out that, as a consequence of the rotation of the earth, it was necessary to conclude that the earth is flattened. He showed that, if the earth were not flattened, then the seas in the equatorial regions would be more than six miles deep, and the land would protrude in a corresponding way in polar regions. Newton calculated, on the basis of the assumption of a homogeneous earth, that the flattening ϵ should be about 1/230. A few years later, Domenique Cassini announced that the remeasurement of the meridian of France from Dunkirk south toward the Pyrenees indicated that the length of a degree of latitude tended to increase as one went southward. If the earth were really flattened, then the length of a degree of latitude should have decreased going southward, as may be seen from Figure 1. (It is to be remembered that geodetic latitudes and longitudes represent angles between the local vertical and the reference planes respectively of the equator and the meridian of Greenwich. If, on the other hand, they were geocentric angles, then the length of a degree of latitude would be greatest at the equator and least at the poles.) The discrepancy between Newton's prediction and Cassini's observations led to a bitter quarrel between the French and the English mathematicians. The quarrel has been caricatured by Swift in *Gulliver's Travels*. In the end, the measurements carried out by Maupertuis in Lapland (1736) and by Bouguer and de la Condamine (1735) in Peru showed that, in fact, Newton was right, and the earth was flattened rather than football-shaped.

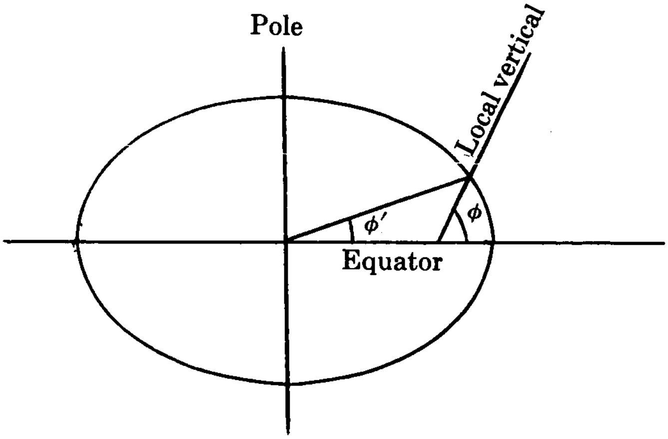

FIGURE 1. Relation of Geocentric Latitude (ϕ') to Geodetic Latitude (ϕ)

From the latter part of the 18th century on, it became clear that the measured value of the flattening of the earth was inconsistent with the idea that the earth is homogeneous. The measured values were much nearer to 1/300 than to the value of 1/230 which would have been required if the earth had been homogeneous.

In the early stages of the measurements, it was enough to measure the flattening without specific reference to the surface that was involved; later on, after the introduction of the idea of the geoid, it became clear that the best surface to discuss was the sea level surface of the earth. Once the idea had been introduced, it was possible to give a precise meaning to the idea of the flattening of the earth, and to calculate the expected value on various assumptions about the interior.

A number of particular hypotheses were discussed: the possibility that the earth was homogeneous, the possibility that it consisted of a nucleus which contained nearly all of the mass plus a sort of atmosphere, and the possibility of various smooth distributions of density which would interpolate between these. A very important result was shown by Radau about 1880, namely, that the predicted value of the flattening of the earth depended on its moment of inertia around the polar axis, and that all distributions of density having the same moment of inertia would have almost the same flattening. The error of this assumption is in the fourth significant figure, provided that the density always decreases outward. Thus the kernel of the problem of predicting the flattening of the earth is the problem of the calculation of the flattening of a body whose polar moment of inertia C is given.

The theory of this calculation will be given below. For the moment it is important to view this problem as it was seen up to 1958. During that time, the problem of determining the earth's flattening was thought to be best treated by thinking of three unknowns. These were the polar moment of inertia C, the difference between C and the axial moment of inertia A, i.e., the quantity $C - A$, and the hydrostatic value of the flattening ϵ. From hydrostatic theory, as mentioned, it was possible to find an equation between C and ϵ. From the theory of the luni-solar perturbations, it was possible to determine the quantity $H = (C - A)/C$, which is called the dynamical flattening. In addition, it was known that the quantity $(C - A)/Ma^2 = J_2$ was equal to $2/3(\epsilon - \frac{1}{2}m)$, where m is the ratio of centrifugal force at the equator to gravity at the equator and M is the mass of the earth. This relation is somewhat approximate, since there are small higher-order terms of the order of a fraction of a percent, but it is also purely mathematical, and depends in no way on assumptions about hydrostatic equilibrium. This equation related $C - A$ to ϵ, but it should be noted that the ϵ here is the real flattening of the earth and not necessarily the one predicted by hydrostatic theory. Before 1958, it was customary to make the assumption that the real ϵ equaled the hydrostatic ϵ. One then had three relations among the three unknowns, and the solution was possible. In recent years, the determination of J_2 directly from satellite orbits has furnished a new relation in this problem. At the same time, the recognition that the hydrostatic flattening is not necessarily equal to the actual flattening means that we have a new unknown. However, with one more relation and one more unknown, the solution is still possible. The point which is not clear from the older discussions is that the hydrostatic flattening of the earth depends *only* on the assumed value of the polar moment of inertia. This is directly determinable now, since we can measure $(C - A)/Ma^2$ and also $(C - A)/C$; the quotient of these is evidently C/Ma^2. From this, the hydrostatic flattening is directly determinable. I repeat, formerly it was impossible to obtain C/Ma^2 with adequate accuracy unless one made the auxiliary assumption that the hydrostatic and the actual flattening were equal. Thus it is the older situation which is complicated and the newer one which is simple.

III. **The hydrostatic flattening.** I shall now give the theory of the relation between C/Ma^2 and ϵ, the flattening, as it would be in a plastic or liquid body. I shall follow Jeffreys' theory as stated in [2]. My excuse for giving a long commentary on section 4.03 of his book, which covers only 8 pages, is that I have found these pages very difficult. Since there are 2 errors on these pages which appear in the 1952 edition and were reprinted in the 1958 edition, it is just possible that I am not the only person who has had trouble reading these pages. (Since 1959, both errors have been spotted by others beside myself.)

My equations will be numbered in accordance with his; those with letters following are interpolated.

The theory of the interior of the earth starts from the assumption that the earth is in hydrostatic equilibrium—that is to say, that it is in equilibrium under the action of forces which cause no motion and which produce pressures acting equally in all directions, as in a fluid. Under these circumstances, we will expect that the density will be stratified in layers such that the surfaces of constant density will also be surfaces of constant potential. The result is intuitively obvious; it means only that a fluid seeks its level. If there were a place where the density above an equipotential surface exceeded the density below it, then the heavier fluid above would tend to displace the lighter fluid below the surface. The point can be proved analytically, but it is one which is too simple physically to be worth such a discussion. The fact that an analytic proof can be given reinforces our confidence that the mathematical model is a good description of the physical situation.

It is important to remember that the potential which is involved here is not the true gravitational potential of the body, but rather the geopotential. The difference is the centrifugal force which arises from the rotation of the body. This force is included in the geopotential, on exactly the same footing as the true gravitational force. Once again, this is a matter of ordinary experience; the force which we call gravity in daily life is 99 percent the real gravitational attraction of the earth, but the remaining fraction is the centrifugal force of the earth's rotation. The difference is quite perceptible in ordinary life. The flow of the Mississippi requires a drop of about one foot per mile, which is less than one minute of arc. The maximum inclination between surfaces of true gravitational potential and geopotential is of the order of 5 or 10 minutes of arc, so that without centrifugal force the flow of the Mississippi would be reversed.

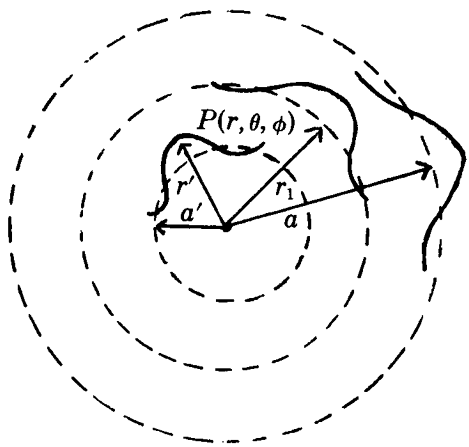

FIGURE 2. Surfaces of Constant Density and Constant Geopotential

We shall follow Jeffreys in this derivation and designate the density by the symbol ρ, and the geopotential by the symbol Ψ. The surfaces of constant Ψ will be surfaces of constant ρ. We consider a homogeneous, nearly spherical body whose surface is given by the equation

$$r = a\left(1 + \sum_{n=1}^{\infty} \epsilon_n S_n\right), \tag{2}$$

according to Jeffreys, where S_n is a surface harmonic, a is the earth's mean radius, and ϵ_n is a small numerical coefficient (Figure 2). Notice that Jeffreys has written this equation as a single summation over n; this is merely a convenience to avoid the ugliness of a double summation. In fact, the S_n's must be considered as functions not only of the degree n of the harmonic but also of its order m. Since we shall get rid of all these harmonics except S_2 at an early stage in the game, it is not important to distinguish between tesseral and zonal harmonics, and hence m may be omitted.

We now consider the gravitational potential due to this body. In calculating the potential, Jeffreys makes the assumption that all of the ϵ's are so small that we can neglect second order terms. Under these circumstances, we can represent the attraction of the body as that of a sphere combined with the attraction of an infinitely thin surface distribution of matter painted on the outside of the sphere. What is neglected here is the fact that a real 3-dimensional bulge would attract, not toward a point right on the sphere, but toward a point half way up through the bulge. The neglect of second order terms is fully justified for all harmonics except the second. In the case of the second harmonic, quadratic terms have been calculated by Darwin. They represent an enormous increase in the difficulty of the computation without any real increase in the accuracy with which the computation represents physical reality. The effects of lack of fluidity in the earth are large enough so that the use of second order terms is not justified even for the second harmonic.

For the potential outside the body, Jeffreys gives

$$U_0 = \frac{4}{3}\pi f \rho a^3 \left(\frac{1}{r} + \sum_{n=1}^{\infty} \frac{3}{2n+1} \frac{a^n}{r^{n+1}} \epsilon_n S_n\right), \tag{3}$$

where f is the absolute constant of gravitation. This equation may be derived from Equation 1 on p. 395 of [**3**], namely:

$$V = \sum_{m=0}^{\infty} S_m(\phi_0, \theta_0) \left(\frac{a}{r}\right)^{m+1}.$$

Here V is the potential; m is Jeffreys' n; ϕ_0, θ_0 are the coordinates of the point at which the potential is being evaluated; and $S_m(\phi_0, \theta_0)$ is a surface harmonic, multiplied by its coefficient, defined by the following equation for the surface density σ:

$$\sigma = \frac{1}{4\pi a} \sum_{m=0}^{\infty} (2m + 1) S_m(\phi, \theta),$$

where ϕ, θ are the coordinates of any point. In this case, the mass distribution corresponding to the mth harmonic will be

$$\sigma_m = \frac{2m + 1}{4\pi a} S_m(\phi, \theta).$$

For Jeffreys, this surface distribution of mass is produced by additional thickness of the homogeneous body. It is thus

$$\sigma_n = \rho a \epsilon_n S_n.$$

Equating σ_n to σ_m,

$$\rho a \epsilon_n S_n \cdot \frac{4\pi a}{2m + 1} = S_m(\phi, \theta).$$

Substituting in the equation above for V, and multiplying by f (which was taken equal to unity in the equation for V) we obtain, for the nth term

$$\frac{4}{3} \pi f \rho a^3 \cdot \frac{3}{2n + 1} \frac{a^n}{r^{n+1}} \epsilon_n S_n,$$

as for Jeffreys. In (3), the first term is nothing but the Newtonian attraction of a sphere.

For the interior attraction, Jeffreys gives the following equation:

$$(4) \qquad U_1 = \frac{4}{3} \pi f \rho a^3 \left(\frac{3a^2 - r^2}{2a^3} + \sum_{n=1}^{\infty} \frac{3}{2n + 1} \frac{r^n}{a^{n+1}} \epsilon_n S_n \right).$$

This equation is obtainable from the equation,

$$V = \sum_{m=0}^{\infty} S_m(\phi_0, \theta_0) \left(\frac{r}{a}\right)^m \qquad \text{if } r < a$$

(which is given in [**3**]) with the same substitutions for σ except for the first term inside the parentheses. The first term represents the potential at a point in the interior of a sphere. It consists of two contributions. The first is that due to the portion of the sphere interior to the point in question, which is clearly

$$\frac{4}{3}\,\pi f \rho r^3,$$

where r is the radius from the center of the sphere to the point in question. The potential due to the portion of the sphere outside the point in question is given by [3, p. 38]:

$$2\pi\rho f\,(a^2 - r^2),$$

and the combined effect is

$$\frac{4}{3}\,\pi\rho f a^3\,\frac{(3a^2 - r^2)}{2a^3},$$

which is the first term inside the parentheses of Jeffreys' Equation (4). We now consider a heterogeneous body. The density is constant and equal to ρ' over a surface given by Jeffreys' Equation (5):

$$r' = a'(1 + \sum \epsilon_n S_n), \tag{5}$$

where ρ' and ϵ_n are functions of a'. In order to keep straight the varying meanings and kinds of radii which are involved in this situation, let us look at Figure 2. First we have a, which is the mean radius of the outer surface of the body. It is thus approximately the semi-major axis of the earth. Next we have a', which is the mean radius of any interior surface. We can describe a point of the equal density surface by giving r and S_n, since S_n will contain the angular variables. The mean radius of that surface which passes through the interior point $P(r, \theta, \phi)$, where the potential is to be found, is defined by Jeffreys as r_1.

To calculate the potential, Jeffreys proceeds to take the difference between two homogeneous bodies, one having the outer surface corresponding to the density ρ, and the other having a surface corresponding to

$$\rho' + \Delta\rho'.$$

The external potential is therefore clearly given by Equation (6):

$$U_0 = \frac{4}{3}\,\pi f \int_0^a \rho'\,\frac{\partial}{\partial a'}\left(\frac{a'^3}{r} + \sum \frac{3}{2n+1}\,\frac{a'^{n+3}}{r^{n+1}}\,\epsilon_n S_n\right) da'. \tag{6}$$

The quantity ρ' is not differentiated because while the gravitational attraction of the thin spherical shell is proportional to the difference in radius da' between its two sides, it is proportional to ρ' itself and not to $d\rho'$. The integration is extended over a' up to a rather than to ∞, clearly because beyond a there is no density.

For an internal point, we calculate the potential U_1 in two parts. The first term is due to the matter which is interior to the point under consideration. For this, an explanation exactly like Equation (6) applies, except that the integral extends only up to the mean radius r_1 through the point in question. For matter external to the point, we differentiate and integrate Equation (4) in an entirely similar way:

$$U_1 = \frac{4}{3}\pi f \int_0^{r_1} \rho' \frac{\partial}{\partial a'}\left(\frac{a'^3}{r} + \sum \frac{3}{2n+1}\frac{a'^{n+3}}{r^{n+1}} \epsilon_n S_n\right) da' \tag{7}$$
$$+ \frac{4}{3}\pi f \int_{r_1}^{a} \rho' \frac{\partial}{\partial a'}\left(\frac{3}{2}a'^2 + \sum \frac{3}{2n+1}\frac{r^n}{a'^{n-2}} \epsilon_n S_n\right) da'.$$

Note that in these differentiations and integrations, the only variable is a'; r is the radius to the point P at which the potential is being evaluated; r_1 is the mean value of r on the equipotential through P, i.e.,

$$r = r_1(1 + \sum \epsilon_n S_n).$$

To obtain Ψ, the geopotential, we must add the contribution from the centrifugal force. Thus

$$\Psi = U + \frac{1}{2}\omega^2 r^2 \cos^2\phi' = U + \frac{1}{3}\omega^2 r^2 + \frac{1}{2}\omega^2 r^2 \left(\frac{1}{3} - \sin^2\phi'\right). \tag{8}$$

Let us note that, after Equation (8), Jeffreys mentions that he can ignore the difference between ϕ and ϕ'. The next sentence, which discusses the behavior of ρ and Ψ over the equipotential surfaces, contains the word "then", which appears to refer back to the remark about ϕ and ϕ'. I have been unable to make sense out of this relation, and I believe that the sentence about ϕ and ϕ' is simply misplaced. In fact, Jeffreys continues to use ϕ' until after his Equation (12). The justification for ignoring the difference is the fact that trigonometric functions of ϕ' occur only with the small coefficient ω^2 or one of the epsilons.

Jeffreys proceeds to point out that, in his Equations (7) and (8), Ψ can be a function only of r_1. This is because the value of r_1 is constant over an equipotential surface. In particular, Ψ cannot be a function of the S_n's, which are functions of the coordinates ϕ,

λ. Jeffreys next defines $\bar{\rho}$, the mean density in the body, by means of his Equation (9), namely

$$M = 4\pi \int_0^a \rho' a'^2 da' = \frac{4}{3}\pi a^3 \bar{\rho}. \tag{9}$$

He defines the mean density ρ_0 within a surface whose mean radius is r_1 by Equation (10), namely

$$\rho_0 = \frac{3}{r_1^3} \int_0^{r_1} \rho' a'^2 \, da'. \tag{10}$$

Jeffreys then proceeds to substitute for $1/r$ in his Equations (7) and (8). It is important to notice that r has small coefficients except in the first term. In this term, therefore, we must retain first order of small quantities. Elsewhere we can replace r by r_1. We notice also that r can be taken out from under the integral sign and from the differentiation, since both of these refer to the running variable a' rather than to the point at which the potential is being evaluated.

The quantities ϵ_n and ρ' are to be regarded as functions of a'. In obtaining Equation (11), namely

$$\begin{aligned} &\frac{4}{3}\pi f \Bigg[\frac{1 - \sum \epsilon_n S_n}{r_1} \int_0^{r_1} 3\rho' a'^2 da' \\ &+ \sum \frac{3}{2n+1} S_n \left\{ \frac{1}{r_1^{n+1}} \int_0^{r_1} \rho' d(a'^{n+3}\epsilon_n) + r_1^n \int_{r_1}^a \rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right) \right\} \\ &+ \frac{1}{3}\omega^2 r_1^2 + \frac{1}{2}\omega^2 r_1^2 \left(\frac{1}{3} - \sin^2\phi'\right) \Bigg] \\ &= \text{function of } r_1 \text{ only,} \end{aligned} \tag{11}$$

Jeffreys has twice preferred to replace expressions of the form $(\partial f/\partial a')\, da'$ by df. The function of r_1 to be used on the right of (11) is

$$\Psi - \frac{4}{3}\pi f \int_{r_1}^a \rho' d\left(\frac{3}{2}a'^2\right).$$

Since the left-hand side of Equation (11) must be constant for a given r_1, the coefficients of all of the S_n's, where n is greater than or equal to 1, must vanish because the S_n's contain the angle variables. If their coefficients did not vanish, then the left-hand side of the equation would depend on the angle variables. Moreover, because of the orthogonality properties of the S_n's, no combination of the S_n's could have the same effect as one of them. Hence, it is not possible to arrange the coefficients in such a way that the variation of one S_n is covered up by the others. The only way to make the whole left-hand side of (11) independent of the angle variables is to make the coefficient of each S_n equal to zero. When we do so, we get Equation (12) after dividing through by $4\pi f$:

$$-\frac{\epsilon_n}{r_1}\int_0^{r_1}\rho' a'^2\,da' + \frac{1}{2n+1}\left\{\frac{1}{r_1^{n+1}}\int_0^{r_1}\rho' d(a'^{n+3}\epsilon_n) + r_1^n\int_{r_1}^{a}\rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right)\right\} = 0, \tag{12}$$

except in the case when S_n is $(1/3 - \sin^2\phi)$, when we get an extra term, $-\omega^2 r_1^2/8\pi f$, on the right-hand side. The right side is therefore written $(0, -\omega^2 r_1^2/8\pi f)$. We next multiply (12) through by r_1^{n+1} and replace r_1 by r:

$$-r^n\epsilon_n\int_0^{r}\rho' a'^2\,da' + \frac{1}{2n+1}\left\{\int_0^{r}\rho' d(a'^{n+3}\epsilon_n) + r^{2n+1}\int_{r}^{a}\rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right)\right\} = 0. \tag{12a}$$

We now consider the variation of the potential with distance from the center of the earth, so that we regard r as a variable. In differentiating the integrals, it is important to remember that the integral for a general function $f(a')$ is to be determined by

$$\frac{d}{dr}\int_a^r f(a')\,da' = f(r).$$

With this in mind, Equation (12a) is differentiated as follows:

$$\left\{-nr^{n-1}\epsilon_n - r^n\frac{d\epsilon_n}{dr}\right\}\int_0^r \rho' a'^2\,da' - r^n\epsilon_n\rho r^2$$
$$+\frac{1}{2n+1}\left\{\rho(n+3)r^{n+2}\epsilon_n + r^{n+3}\rho\frac{d\epsilon_n}{dr}\right.$$
$$+(2n+1)r^{2n}\int_{r_1}^{a}\rho' d\left(\frac{\epsilon_n}{a'^{n-2}}\right) - r^{2n+1}$$
$$\left.\cdot\left[\rho\frac{d\epsilon_n}{dr}\cdot\frac{1}{r^{n-2}} - \rho\epsilon_n\cdot(-n+2)\frac{1}{r^{n-1}}\right]\right\} = -\frac{5\omega^2 r^4}{8\pi f}.$$

In writing this equation, we must keep in mind that ρ is the value of ρ' when $r = a$. This equation simplifies to Jeffreys' Equation (13) when we combine the two terms in the second bracket which depend on $d\epsilon_n/dr$, and note that the sum of three terms in $\epsilon_n \rho r^{n+2}$ is zero.

Making these substitutions, we arrive at Jeffreys' Equation (13), which includes both integrals and derivatives:

$$
\begin{aligned}
(13)\qquad & -\left(r^n \frac{d\epsilon_n}{dr} + nr^{n-1}\epsilon_n\right) \int_0^r \rho' a'^2 \, da' + r^{2n} \int_r^a \rho' \frac{d}{da'}\left(\frac{\epsilon_n}{a'^{n-2}}\right) da' \\
& \qquad = \left(0, -\frac{5\omega^2 r^4}{8\pi f}\right).
\end{aligned}
$$

We now divide by r^{2n} and get Equation (13a):

$$
\begin{aligned}
(13a)\qquad & -\left(\frac{1}{r^n}\frac{d\epsilon_n}{dr} + \frac{n}{r^{n+1}}\epsilon_n\right) \int_0^r \rho' a'^2 \, da' + \int_r^a \rho' \frac{d}{da'}\left(\frac{\epsilon_n}{a'^{n-2}}\right) da' \\
& \qquad = \left(0, -\frac{5\omega^2}{8\pi f}\right).
\end{aligned}
$$

We differentiate with respect to r and note, as before, the effect of variable limits of integration. We further note that ρ is the value of ρ' at $a' = r$. This gives Equation (13b):

$$
\begin{aligned}
(13b)\qquad & -\left(-\frac{n}{r^{n+1}}\frac{d\epsilon_n}{dr} + \frac{1}{r^n}\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^{n+2}}\epsilon_n + \frac{n}{r^{n+1}}\frac{d\epsilon_n}{dr}\right) \int_0^r \rho' a'^2 da' \\
& \qquad\qquad n \\
& -\left(\frac{1}{r^n}\frac{d\epsilon_n}{dr} + \frac{n}{r^{n+1}}\epsilon_n\right) \rho r^2 - \rho\left[\frac{d\epsilon_n}{dr}\cdot\frac{1}{r^{n-2}} + \epsilon_n \frac{(-n+2)}{r^{n-1}}\right] = 0.
\end{aligned}
$$

In constructing this equation, we did not differentiate under the integral sign in the first term because all quantities there are regarded as functions of a'. We multiply through by $-r^n$, and this gives (13c),

$$
\begin{aligned}
(13c)\qquad & \left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right) \int_0^r \rho' a'^2 da' + \left(\frac{d\epsilon_n}{dr} + \frac{n\epsilon_n}{r}\right) \rho r^2 \\
& \qquad + \rho\left[r^2 \frac{d\epsilon_n}{dr} - r\epsilon_n(n-2)\right] = 0.
\end{aligned}
$$

which simplifies into Jeffreys' Equation (14):

(14) $$\left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right)\int_0^r \rho' a'^2 da' + 2\left(\frac{d\epsilon_n}{dr} + \frac{\epsilon_n}{r}\right)\rho r^2 = 0.$$

Now, from Equation (10), it is easy to see that

$$\int_0^r \rho' a'^2 da' = \frac{1}{3} r^3 \rho_0.$$

Substituting for the integral and dividing through by $r^3/3$, we have Jeffreys' Equation (15), which is the famous equation of Clairaut:

(15) $$\rho_0\left(\frac{d^2\epsilon_n}{dr^2} - \frac{n(n+1)}{r^2}\epsilon_n\right) + \frac{6\rho}{r}\left(\frac{d\epsilon_n}{dr} + \frac{\epsilon_n}{r}\right) = 0.$$

The equation of Clairaut was obtained in 1743. In the intervening two centuries, a great deal has been found out about the possible solutions of this equation subject to the restriction that the density decreases steadily downward. There are two reasons to think that this will happen: First, the denser materials would tend to sink in fluid equilibrium; second, materials which are at a lower level are under high pressure and, therefore, will be somewhat compressed. It follows that the mean density ρ_0 within a given surface will also be greater than the local density ρ, except at the center where $\rho_0 - \rho \rightarrow$ zero.

We suppose that for small values of r, ϵ_n varies like r^p. Then, substituting in Clairaut's equation, we have

(15a) $$\rho_0\left[p(p-1)r^{p-2} - \frac{n(n+1)}{r^2}r^p\right] + \frac{6\rho}{r}\left(pr^{p-1} + \frac{r^p}{r}\right) = 0.$$

Dividing by r^{p-2} and also by ρ, which equals ρ_0 at the center of the earth, we have a quadratic equation in p:

(16) $$p(p-1) - n(n+1) + 6p + 6 = 0.$$

This equation is solved by the usual processes, giving either

(17) $$p = n - 2 \quad \text{or} \quad p = -n - 3.$$

Of the two solutions, we can discard $p = -n - 3$, since in this case the solution would be proportional to $r^{-n-2}S_n$. As n goes from $+1$ to ∞, the exponent on r would be negative. Such a solution would go to ∞ at the center of the earth, and is therefore impossible. If, therefore, for $p = n - 2$, we take $n = 1$, then

$$\epsilon_n = kr^{-1},$$

$$\frac{d\epsilon_n}{dr} = -\frac{k}{r^2}, \tag{17a}$$

$$\frac{d^2\epsilon_n}{dr^2} = \frac{2k}{r^3}.$$

Substituting, we find that, for this case, Clairaut's Equation (15) holds identically for arbitrary density functions ρ and ρ_0. The radial displacement is proportional to S_1 regardless of the distance from the center, and this, in turn, implies a rigid body displacement which need not be further considered.

If $n = 2$, then ϵ_n is neither infinite nor zero near the center. For this border line case, a special treatment is needed because $n - 2$ vanishes, and hence the previous treatment leads to constant ellipticity. We let

$$1 - \frac{\rho}{\rho_0} = Hr^k$$

hold for small r. In this equation, H must be positive so that the density may increase as r increases, and k must be positive to avoid an infinite value of the density at the center. We further suppose

$$\epsilon_2 = A + Br^s. \tag{18}$$

We substitute in Equation (15), and find (18a):

$$\rho_0\left[Bs(s-1)r^{s-2} - \frac{6(A+Br^s)}{r^2}\right] + \frac{6\rho}{r}\left(Bsr^{s-1} + \frac{A+Br^s}{r}\right) = 0. \tag{18a}$$

In this equation, we note that

$$\frac{6A}{r^2}(\rho - \rho_0) = -\frac{6A\rho_0}{r^2}\left(1 - \frac{\rho}{\rho_0}\right) = -6A\rho_0 Hr^{k-2}. \tag{18b}$$

We also can transform the terms whose coefficient is $6\rho B$:

$$\begin{aligned}\rho(6Bs + 6B)r^{s-2} &= \left[\rho_0 - \rho_0\left(1 - \frac{\rho}{\rho_0}\right)\right](6Bs + 6B)r^{s-2} \\ &= \rho_0(6Bs + 6B)r^{s-2} - \rho_0 Hr^{k+s-2}(6Bs + 6B).\end{aligned} \tag{18c}$$

The second term in (18c) disappears because it is of an order higher than r^{s-2}. The remaining terms of (18a) are all multiplied by ρ_0, so that we find

$$(19)\qquad Bs(s+5)r^{s-2} - 6AHr^{k-2} = 0.$$

Equation (19) can be true only if $s = k$. In this case, (19a) will hold:

$$(19a)\qquad Bk(k+5) = 6AH.$$

Since k is positive, B must have the sign of AH. H, however, is positive, so that B has the sign of A. Therefore, ϵ_2 must increase numerically with r.

Finally, if n is greater than 2, then ϵ_n behaves like r^{n-2} for a small r. We thus say that ϵ_n increases numerically with r in all nontrivial cases for points near the center of the earth.

If the ϵ_n's should not continue to increase all the way to the surface, then we would come to a place where

$$\frac{d\epsilon_n}{dr} = 0.$$

Then the following would hold (Jeffreys' Equation (20)):

$$(20)\qquad \frac{d^2\epsilon_n}{dr^2} = \left\{ n(n+1) - \frac{6\rho}{\rho_0} \right\} \frac{\epsilon_n}{r^2}.$$

Since $n(n+1)$ is positive and is at least 6, it follows that the right-hand side of (20) is at least $6(1 - \rho/\rho_0)$, which is positive, since ρ is always less than ρ_0. Hence, the second derivative of ϵ_n will have the sign of ϵ_n and, therefore, ϵ_n would immediately increase again in absolute value.

Our next step is to show that the ϵ_n's should be zero except for $n = 1$ and $n = 2$. In Equation (12), if we put $r_1 = a$, then the integral from r_1 to a vanishes. We also substitute from Equation (9) for

$$(20a)\qquad \int_0^a \rho' a'^2 da' = \frac{1}{3} a^3 \bar{\rho},$$

and Equation (12) becomes

$$(21)\qquad -\epsilon_{na} \cdot \frac{1}{3} a^2 \bar{\rho} + \frac{1}{2n+1} \cdot \frac{1}{a^{n+1}} \int_0^a \rho' d(a'^{n+3}\epsilon_n) = (0, -\omega^2 a^2/8\pi f).$$

We denote the integral in Equation (21) by I. We assume that

ϵ_n is positive; then, integrating by parts, we get

$$I = \rho_a a \epsilon_{na} a^{n+3} - \int_{a'=0}^{a} a'^{n+3} \epsilon_n \, d\rho'. \tag{22}$$

Here the subscript "a" indicates values taken at the surface. Since ρ' is a decreasing function of a', it follows that $d\rho'$ is negative; the integral in Equation (22) is therefore negative:

$$I > \rho_a \epsilon_{na} a^{n+3}. \tag{23}$$

On the other hand, since ϵ_n is a positive, increasing function of a', it is always less than the boundary value ϵ_{na} unless $n = 1$. Here Jeffreys says that ϵ_n does not change. Actually, it has been pointed out to me that it must increase without limit near the center, but this case is trivial.

$$- \int_{a'=0}^{a} a'^{n+3} \epsilon_n \, d\rho' < - \epsilon_{na} \int_{a'=0}^{a} a'^{n+3} \, d\rho'. \tag{23a}$$

Substituting (23a) in (22), we have

$$I < \epsilon_{na} \left(\rho_a a^{n+3} - \int_{a'=0}^{a} a'^{n+3} \, d\rho' \right) . \tag{23b}$$

The right-hand side of (23b) represents the result of integrating by parts the expression

$$\epsilon_{na} \int_{a'=0}^{a} \rho' \, da'^{n+3}.$$

We replace ρ' by $\bar{\rho} + (\rho' - \bar{\rho})$:

$$\epsilon_{na} \int_{a'=0}^{a} \rho' \, da'^{n+3} = \epsilon_{na} \left[\bar{\rho} a^{n+3} + \int_{a'=0}^{a} (\rho' - \bar{\rho}) \, da'^{n+3} \right] . \tag{23c}$$

To evaluate the integral, note that

$$d(a'^{n+3}) = (n+3) a'^{n+2} da' = \frac{n+3}{3} a'^{n} d(a'^{3}). \tag{23d}$$

Therefore,

$$\epsilon_{na} \int_{a'=0}^{a} \rho' \, da'^{n+3} = \epsilon_{na} \left[\bar{\rho}^{n+3} + \frac{n+3}{3} \int_{a'=0}^{a} (\rho' - \bar{\rho}) a'^{n} da'^{3} \right] . \tag{24}$$

For $n = 0$, the last integral vanishes because the differential $d(a'^{3})$

weights the integral in proportion to the volume. In this case, the integral of $\rho' - \bar{\rho}$ must vanish by the definition of mean density.

In general, because of the fact that $\bar{\rho}$ is a volume average of ρ', it will be true that the integral of $(\rho' - \bar{\rho})$ multiplied by any constant and taken from 0 to a will be 0. In particular, if we choose a_0 for the level where $\rho' = \bar{\rho}$, then, since ρ' is a decreasing function $(\rho' - \bar{\rho}) > 0$ under this level, i.e., for $a' < a_0$, and $(\rho' - \bar{\rho}) > 0$ above this level. Then the product

$$(\rho' - \bar{\rho})(a'^n - a_0^n)$$

will be negative for any power of n greater than 0, since, for all such powers, the power of the greater number is greater. Hence,

$$\int_{a'=0}^{a} (\rho' - \bar{\rho}) a'^n \, da'^3 = \int_{a'=0}^{a} (\rho' - \bar{\rho})(a'^n - a_0^n)\, da'^3 < 0.$$

Therefore, the integral in (24) is negative. Since the remaining term is necessarily positive, the integral can only decrease the whole expression, so that

$$I < \epsilon_{na} \bar{\rho} a^{n+3}.$$

Using (23), we see that the quantity I can, in fact, be bracketed between the limits

$$\epsilon_{na} \rho_a a^{n+3} < I < \epsilon_{na} \bar{\rho} a^{n+3}.$$

All the above assumes that ϵ is positive. If it is negative, the inequalities are reversed, and hence, whether ϵ is positive or negative,

$$I = \theta \epsilon_{na} \bar{\rho} a^{n+3},$$

where $0 < \theta < 1$. Going back to (21), therefore,

$$\epsilon_{na} a^2 \bar{\rho} \left(-\frac{1}{3} + \frac{\theta}{2n+1} \right) = \left(0, -\frac{\omega^2 a^2}{8\pi f} \right). \tag{25}$$

If the right-hand side is 0, this equation cannot be satisfied for $n > 1$, since, in that case, the parenthesis on the left must be less than 0. Its coefficient is composed of quantities which also cannot vanish except at the center of the earth. Hence, for all n except $n = 2$, the ϵ_n must be 0 (to the first order) throughout the earth. No harmonics except the second degree zonal harmonics will exist.

With respect to the second degree zonal harmonic, for which the right-hand side is negative, the value of ϵ_{na} must be positive. This, however, implies that ϵ_{na} is positive everywhere, since we have found that the ϵ_n's must increase steadily from the center. Jeffreys summarizes these results as follows:

"*On the hydrostatic theory the radius of a surface of constant density contains no harmonics other than that representing the ellipticity; the ellipticities increase all the way from the centre to the surface, and the surface is oblate.*"

Returning to Clairaut's Equation (15), for $n = 2$, we set

$$\epsilon_2 \equiv \epsilon = r^3\lambda. \tag{26}$$

Its derivatives are:

$$\frac{d\epsilon}{dr} = 3r^2\lambda + r^3\frac{d\lambda}{dr},$$

$$\frac{d^2\epsilon}{dr^2} = 6r\lambda + 6r^2\frac{d\lambda}{dr} + r^3\frac{d^2\lambda}{dr^2}.$$

Substituting these in Clairaut's Equation (15),

$$\rho_0\left(6r\lambda + 6r^2\frac{d\lambda}{dr} + r^3\frac{d^2\lambda}{dr^2} - 6r\lambda\right) + \frac{6\rho}{r}\left(3r^2\lambda + r^3\frac{d\lambda}{dr} + r^2\lambda\right) = 0.$$

Dividing through by $\rho_0 r^3$, we get

$$\frac{d^2\lambda}{dr^2} + 6\left(\frac{\rho}{\rho_0} + 1\right)\frac{1}{r}\frac{d\lambda}{dr} + \frac{24\rho}{\rho_0}\frac{\lambda}{r^2} = 0. \tag{27}$$

We note that for small r, ϵ_n behaves like r^p, where $p = n - 2$. For $n = 2$, this means that ϵ behaves like a constant and hence, from (26), λ must behave like r^{-3}. It follows that λ initially decreases. It cannot afterwards increase, since at the minimum,

$$\frac{d\lambda}{dr} = 0,$$

and we would also have

$$\frac{d^2\lambda}{dr^2} = -\frac{24\rho}{\rho_0 r^2}\lambda,$$

and thus the second derivative would necessarily have the opposite sign from λ. But λ is positive. Hence,

$$\frac{d^2\lambda}{dr^2}$$

would necessarily be negative, and thus λ must decrease all the way from the center to the surface.

In (13), we put $n = 2$; then $S_2 = 1/3 - \sin^2\phi'$. We consider conditions at the surface where $r = a$; then the second term disappears because of the coincidence of the limits of integration, and the integral in the first term is, from Equation (9), replaced by $(1/3)\,\bar{\rho}a^3$. Then

$$\text{(28)} \qquad -\frac{1}{3}\bar{\rho}a^3\left[a^2\left(\frac{d\epsilon}{dr}\right)_{r=a} + 2a\epsilon_a\right] = -\frac{5\omega^2a^4}{8\pi f}.$$

To the first order, we can say that

$$m = \frac{\omega^2a^3}{fM} = \frac{\omega^2}{(4/3)\,\pi f\bar{\rho}},$$

i.e., very roughly, the centrifugal force at the equator divided by the intensity of gravity, and then the right-hand side of (28) becomes

$$\frac{5}{6}\,ma^4\bar{\rho}.$$

We multiply through by $-3/a^3\bar{\rho}$, and get

$$\text{(30)} \qquad a\left(\frac{d\epsilon}{dr}\right)_a + 2\epsilon_a = \frac{5}{2}\,m.$$

At this point, it is advantageous to introduce a new dependent variable η, which is defined by

$$\text{(31)} \qquad \eta = \frac{d\log\epsilon}{d\log r} = \frac{r}{\epsilon}\frac{d\epsilon}{dr}.$$

The derivatives of ϵ are

$$\text{(32)} \qquad \frac{d\epsilon}{dr} = \frac{\eta\epsilon}{r}; \qquad \frac{d^2\epsilon}{dr^2} = \left(\frac{1}{r}\frac{d\eta}{dr} + \frac{\eta^2 - \eta}{r^2}\right)\epsilon.$$

When these are substituted in Equation (15), we get

$$\text{(32a)} \qquad \rho_0\epsilon\left(\frac{1}{r}\frac{d\eta}{dr} + \frac{\eta^2 - \eta}{r^2} - \frac{6}{r^2}\right) + 6\rho\epsilon\left(\frac{\eta}{r} + \frac{1}{r}\right) = 0.$$

We multiply this through by $r^2/\epsilon\rho_0$, and obtain

$$r\frac{d\eta}{dr} + \eta^2 - \eta - 6 + (\eta + 1)\frac{6\rho}{\rho_0} = 0. \tag{33}$$

In order to eliminate ρ in Equation (33), we start from Equation (10):

$$\rho_0 = \frac{3}{r^3}\int_0^r \rho' a'^2 \, da', \tag{33a}$$

which, yields

$$\frac{1}{3}\frac{d}{dr}(\rho_0 r^3) = \rho r^2$$

and

$$\frac{1}{3}\frac{d}{dr}(\rho_0 r^3) = \frac{1}{3}\frac{d\rho_0}{dr}\cdot r^3 + \rho_0 r^2 = \rho r^2. \tag{33b}$$

Dividing by $\rho_0 r^2$, we find

$$\frac{1}{3}\frac{r}{\rho_0}\frac{d\rho_0}{dr} + 1 = \frac{\rho}{\rho_0}. \tag{34}$$

When this is substituted in (33), we get

$$r\frac{d\eta}{dr} + \eta^2 + 5\eta + \frac{2r}{\rho_0}\frac{d\rho_0}{dr}(1 + \eta) = 0. \tag{35}$$

Now it turns out that the expression $\rho_0 r^5 \cdot \sqrt{(1 + \eta)}$ is of great importance in this theory. We shall transform the equation so as to put it in these terms. Our first step is to differentiate this expression logarithmically, which gives

$$\frac{\frac{d}{dr}\{\rho_0 r^5 \sqrt{(1 + \eta)}\}}{\rho_0 r^5 \sqrt{(1 + \eta)}} = \frac{1}{\rho_0}\frac{d\rho_0}{dr} + \frac{5}{r} + \frac{1}{2(1 + \eta)}\frac{d\eta}{dr}. \tag{36}$$

In terms of this logarithmic derivative, we evaluate $d\eta/dr$ and get

$$\frac{d\eta}{dr} = 2(1 + \eta)\frac{\frac{d}{dr}\{\rho_0 r^5 \sqrt{(1 + \eta)}\}}{\{\rho_0 r^5 \sqrt{(1 + \eta)}\}} - \frac{10(1 + \eta)}{r} - \frac{1}{\rho_0}\cdot 2(1 + \eta)\frac{d\rho_0}{dr}. \tag{36a}$$

When this is substituted in (35),

$$2r(1+\eta)\frac{\dfrac{d}{dr}\{\rho_0 r^5\surd(1+\eta)\}}{\rho_0 r^5\surd(1+\eta)} - 10(1+\eta) - \frac{2r(1+\eta)}{\rho_0}\frac{d\rho_0}{dr} + \eta^2 + 5\eta + \frac{2r}{\rho_0}\frac{d\rho_0}{dr}(1+\eta) = 0. \tag{36b}$$

When this equation is simplified, it gives

$$\frac{2\surd(1+\eta)}{\rho_0 r^4}\frac{d}{dr}\{\rho_0 r^5\surd(1+\eta)\} = 10\left(1+\eta-\frac{1}{2}\eta-\frac{1}{10}\eta^2\right) \tag{37}$$

or

$$\frac{d}{dr}\{\rho_0 r^5\surd(1+\eta)\} = \frac{10\left(1+\dfrac{1}{2}\eta-\dfrac{1}{10}\eta^2\right)}{\surd(1+\eta)}\frac{\rho_0 r^4}{2}. \tag{37a}$$

If we set

$$\psi(\eta) = \frac{1+\dfrac{1}{2}\eta-\dfrac{1}{10}\eta^2}{\surd(1+\eta)}, \tag{39}$$

then

$$\frac{d}{dr}\{\rho_0 r^5\surd(1+\eta)\} = 5\rho_0 r^4\psi(\eta). \tag{38}$$

Jeffreys notes that this equation is due to Radau (1885). The point of introducing ψ is that it is effectively a constant within the earth. By logarithmic differentiation, we can obtain from ψ the expression

$$\begin{aligned}\frac{1}{\psi}\frac{d\psi}{d\eta} &= \frac{\dfrac{1}{2}-\dfrac{2}{10}\eta}{1+\dfrac{1}{2}\eta-\dfrac{1}{10}\eta^2} - \frac{1}{2}\frac{1}{1+\eta}\\ &= \frac{1}{20}\frac{\eta(1-3\eta)}{\left(1+\dfrac{1}{2}\eta-\dfrac{1}{10}\eta^2\right)(1+\eta)}.\end{aligned} \tag{40}$$

Clearly, ψ has a maximum or minimum at $\eta = 0$ and at $\eta = 1/3$. Near $\eta = 0$, the logarithmic derivative of ψ is increasing with η, since the numerator is nearly η and the denominator is nearly 1. Hence, at this point, we have a minimum of ψ. At $\eta = 1/3$, on the other hand, we must have a maximum, since this point is a simple 0 of $d\psi/d\eta$, and since there is no discontinuity of the function or its derivative is this interval.

If we return for a moment to the quantity ϵ, we find that, since ϵ/r^3 is a decreasing function, its logarithmic derivative $(1/\epsilon)(d\epsilon/dr) - 3/r$ will be less than 0. Therefore, $\eta > 3$. If we substitute the conventional values at the surface of the earth, namely $M = 1/288$ and $\epsilon_a = 1/297$, we find that $\eta_a = 0.58$. (Jeffreys incorrectly gives 0.57.) Values of η are as in the following table due to Jeffreys, with slight modifications:

$\eta =$	0	1/3	0.57	3
$\psi(\eta) =$	1.00000	1.00074	0.99961	0.8.

Note that Jeffreys has 0.99928 for $\eta = 0.57$; this is another mistake. For $r = 0$, $\eta = 0$. We see that ψ is very nearly constant. Its maximum value exceeds unity by less than 1 part in 1,000 and, at the surface, it is sunk below unity by less than 1 part in 1,000. We have not entirely excluded the possibility that η may make a wide excursion beyond the values that it reaches at the center and the surface of the earth. This is, however, very improbable and, unless this happens, we can say to an accuracy of about 1 part in 1,000 that

$$\frac{d}{dr}\{\rho_0 r^5 \sqrt{(1+\eta)}\} = 5\rho_0 r^4, \tag{42}$$

which is clearly an enormous simplification of Equation (37). Now we would like to express these results in terms of the moment of inertia. For a homogeneous sphere, the moment of inertia is known to be $(2/5)Ma^2$, or

$$\frac{8}{15}\pi\rho a^5.$$

Differentiating, the moment of inertia of a thin spherical shell is

$$\frac{8}{3}\pi\rho r^4 \Delta r,$$

and that for a nonhomogeneous sphere is therefore

$$C = \frac{8}{3}\pi \int_0^a \rho r^4\, dr. \tag{43}$$

To bring this in terms of ρ_0 and its derivative, we first note that the derivative of ρ_0 in (33a) is

$$\frac{d\rho_0}{dr} = -9r^{-4}\int_0^r \rho a'^2 da' + \frac{3}{r^3}\rho r^2 = -\frac{3\rho_0}{r} + \frac{3\rho}{r}.$$

Then, multiplying by r^5, we find

$$r^5\frac{d\rho_0}{dr} = -3r^4\rho_0 + 3r^4\rho.$$

We can now replace ρ by saying

$$\text{(43a)} \quad \frac{8}{3}\pi\int_0^a \rho r^4 dr = \frac{8}{9}\pi\int_0^a 3r^4\rho\, dr = \frac{8}{9}\pi\int_0^a \left(3r^4\rho_0 + r^5\frac{d\rho_0}{dr}\right) dr,$$

which follows Jeffreys' Equation (43).

We now integrate the second term of (43a) by parts:

$$\text{(43b)} \quad \int_0^a r^5\frac{d\rho_0}{dr}dr = r^5\rho_0 \Big|_0^a - 5\int_0^a r^4\rho_0\, dr = a^5\bar{\rho} - 5\int_0^a r^4\rho_0\, dr.$$

We combine the second term of (43b) with the first term in the bracket of (43a) to get Jeffreys' Equation (44):

$$\text{(44)} \qquad C = \frac{8}{9}\pi\left\{\bar{\rho}a^5 - 2\int_0^a r^4\rho_0\, dr\right\}.$$

But, integrating (42), we have (45):

$$\text{(45)} \qquad \int_0^a \bar{\rho}_0 r^4 dr = \frac{1}{5}\bar{\rho}a^5\sqrt{(1+\eta_a)}.$$

And when (45) is substituted into (44), we get (46):

$$\text{(46)} \qquad C = \frac{8}{9}\pi\bar{\rho}a^5\left\{1 - \frac{2}{5}\sqrt{(1+\eta_a)}\right\},$$

or, in terms of the mass,

$$\text{(47)} \qquad \frac{C}{Ma^2} = \frac{2}{3}\left\{1 - \frac{2}{5}\sqrt{(1+\eta_a)}\right\}.$$

In view of (30), the Equation (31) can be rewritten in the form

$$\text{(50)} \qquad \eta_a = \frac{5m}{2\epsilon_a} - 2.$$

When (50) is substituted into (47), we get a direct relation between the moment of inertia of the earth and the hydrostatic value of the flattening:

$$\epsilon_a = \frac{10m}{4 + 25\left(1 - \frac{3}{2}\frac{C}{Ma^2}\right)^2}.$$

Numerical evaluation of this equation, or the equivalent pair of equations from Jeffreys, yields approximately 1/300 for the hydrostatic value of the flattening of the earth. If account is taken of some second order corrections whose theory has been discussed by George Darwin, and which are summarized in the chapter by H. S. Jones in Chapter 1 of [4], it is found that the hydrostatic value of the flattening is near 1/299.8. Figure 5A and Figure 5B show gravity anomalies referred to this flattening.

It is worthwhile to insist on the subtleties which are involved here, because they mean that the hydrostatic flattening is less than the actual flattening. The value which has previously been spoken of as the hydrostatic flattening, namely, 1/297.3, is greater than the actual flattening. If it were really true that the hydrostatic flattening were greater than the actual flattening, it would be very difficult to furnish an explanation. In the actual case when it is less, there is an equally embarrassing superfluity of explanations. Conceivably, the difference is due to the melting of the polar ice caps and some lag in the restoration of isostasy especially, perhaps, in Antarctica. Again, it is conceivable that the discrepancy is a consequence, in some way, of the fact that the polar caps are colder than the equator. It turns out that the temperature difference continues to exist for a surprisingly great distance into the earth. Since we are dealing with quantities of the order of 1 part in 100,000, it is clear that even a very moderate temperature difference may seriously affect the earth's flattening. Again, because of the fact that the laws of heat transport by conduction are irreconcilable with the kind of thermal stratification which is implied by the theory of hydrostatic equilibrium, there will be some necessary distortions of hydrostatic equilibrium in a rotating body, as was first pointed out by von Zeipel (details are given in [5]). Finally, and in my opinion most plausible, there is the explanation of G. J. F. MacDonald (personal communication, 1960) to the effect that the excess bulge around the equator is the result of a retardation in the earth's rotation over the past millions of years.

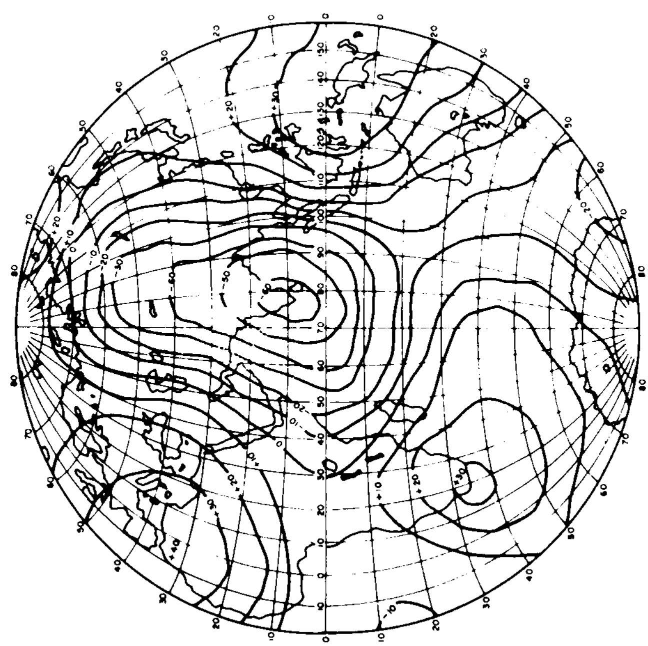

FIGURE 3. Free Air Geoid (Eastern Hemisphere)
Reffered to an ellipsoid with a flattening of 1/298.3

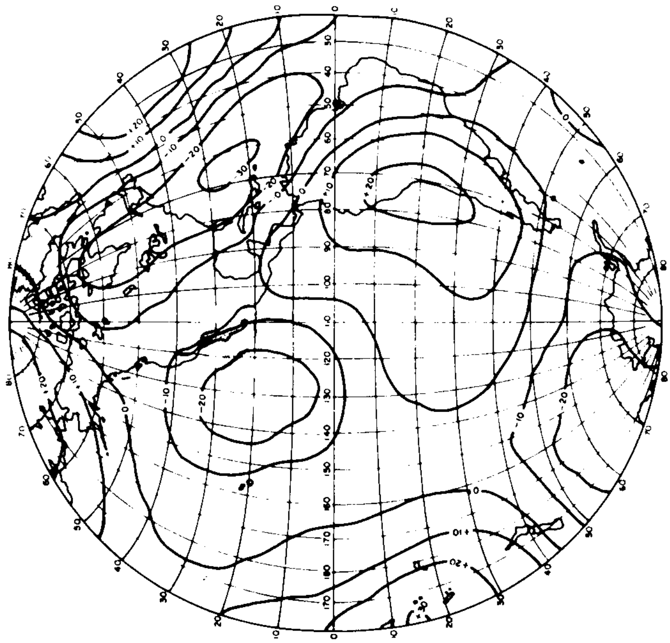

FIGURE 4. Free Air Geoid (Western Hemisphere).
Referred to an ellipsoid with a flattening of 1/298.3

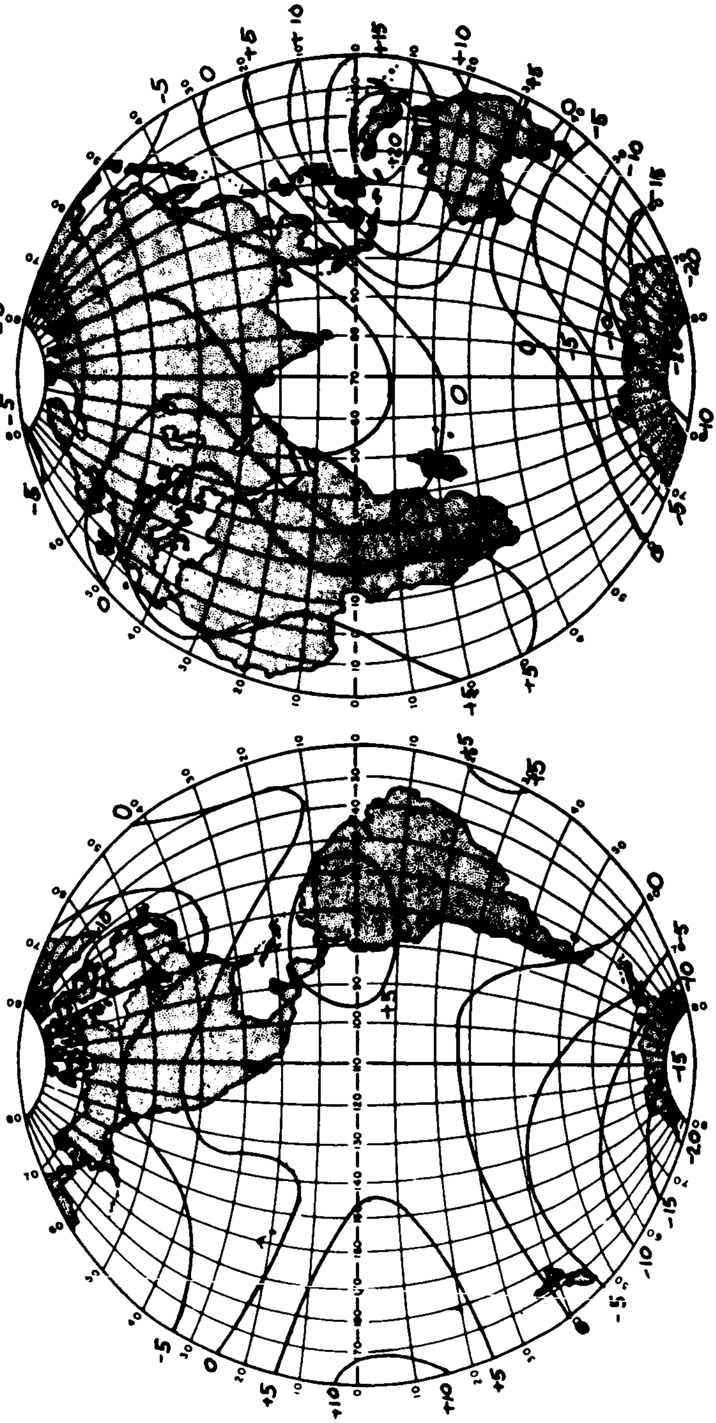

FIGURE 5A and B. Gravity anomalies in milligals derived from satellite perturbations and referred to an ellipsoid with a flattening of 1/299.8

I do not think that any of these explanations can be excluded in a satisfactory way, with the possible exception of the melting of the polar ice caps. Kaula has made some computations along this line which indicate that it is numerically inadequate. I am inclined to think that the most plausible explanation, if we must choose one, is the retardation of the earth's rotation, for which there exists independent evidence.

In any case, it is important to notice that the flattening is a direct function of the polar moment of inertia. If we are given another functional relationship between these two quantities, such as that provided by the luni-solar precession which yields the quantity $(C - A)/C$, then we are able to solve for the hydrostatic flattening. The solution does not depend in any way on what the actual value of the flattening is. If we know C within 1 part in 10,000, then we can calculate the value of the hydrostatic flattening to approximately the same accuracy. On the other hand, an error of 1 part in 10,000 in the actual value of C would upset the observed value of the flattening by the totally unacceptable amount of 10 units in the reciprocal of the flattening. Thus, the presently observed values of the actual flattening are better than are needed to make a satisfactory calculation of the hydrostatic flattening.

References

1. Cassini de Thury, *La meridienne de l'Observatoire Royale de Paris*, H. L. Guerin and J. Guerin, Paris, 1744.

2. H. Jeffreys, *The Earth*, Cambridge Univ. Press, Cambridge, England, 1952.

3. W. D. MacMillan, *Theory of the potential*, Dover, New York, 1958.

4. G. P. Kuiper (ed.), *The earth as a planet of the solar system*, Vol. 2, Univ. Chicago Press, Chicago, Ill., 1954.

5. A. S. Eddington, *Internal constitution of the stars*, Cambridge Univ. Press, Cambridge, England, 1928.

6. Pierre Bouger, Mem. Acad. R. Sci. Paris, 1744, 249-297.

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The Stability of a Rotating Liquid Mass

The problem of the fission of a rotating liquid mass is one which draws on investigations going back some 200 years. The problem has been most extensively treated on the basis of the assumption that the mass is a homogeneous fluid. It is quite clear that the earth is not now a homogeneous fluid; it is even conceivable that the earth never was a homogeneous fluid. Even if it never was, it is worthwhile to discuss the case of the homogeneous fluid because it gives us the best-explored road into the problem. Starting from this road we can make such changes as are required to account for the actual heterogeneity of the earth. We follow the treatment of Jeans 1919, and our equations are numbered like his, in his Chapter III. New equations which we have inserted are followed by small letters.

We begin by asking about the forms which would be taken by a rotating fluid body which is constrained to be an ellipsoid. We shall show that certain ellipsoids are in fact equilibrium configurations. Here again we have simplified the problem and we must later justify the choice of an ellipsoid by showing that it is, in fact, the stable configuration for certain velocity ranges. Note that we are here interested in an exact solution to the approximate problem, rather than, as heretofore, in an approximate solution of the real problem.

In preparation for our problem we note that the equation of the boundary of an ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{51}$$

where the semiaxes of the ellipsoid are a, b, c. If we wish to consider a range of possible ellipsoids then it is useful in many cases, and in particular in the present problem, to consider the family of confocal ellipsoids given by the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1, \tag{52}$$

where λ ranges from 0 to ∞. Following Jeans, we put

$$\begin{gathered} a^2+\lambda=A;\ b^2+\lambda=B;\ c^2+\lambda=C \\ \sqrt{((a^2+\lambda)(b^2+\lambda)(c^2+\lambda))}=(ABC)^{1/2}=\Delta. \end{gathered} \tag{53}$$

We take the quantity $abc=r_0^3$ and the mass of the ellipsoid as given by

$$M=\frac{4}{3}\pi\rho abc=\frac{4}{3}\pi\rho r_0^3.$$

Now the potential of this mass at an internal point with coordinates x, y, z is given by ([5])

$$V_i=-\pi\rho abc\int_0^\infty\left(\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}-1\right)\frac{d\lambda}{\Delta} \tag{55}$$

if we take the units such that the absolute constant of gravitation F is 1. For practical use, we should multiply ρ by F wherever it appears. Notice that the integration is over λ; thus the potential can be considered as composed of a part which increases proportionally to x^2, another which increases with y^2 and a third which increases with z^2 as we move about in the interior of the ellipsoid.

For an exterior point the famous theorem of Ivory asserts that the potential is the same as that which would have been obtained for an ellipsoid whose surface passed through this exterior point and which had the same mass. This result is summed up in Jeans' equation

$$V_0=-\pi\rho abc\int_\lambda^\infty\left(\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}-1\right)\frac{d\lambda}{\Delta}, \tag{54}$$

where λ is the parameter of the ellipsoid which passes through the given external point. Fuller discussions of this problem are to be found in [3] and in standard treatises on potential theory.

Now Jeans introduces a set of abbreviated notations. He writes

$$\int_0^\infty \frac{d\lambda}{\Delta} = J$$

and also

$$\int_0^\infty \frac{d\lambda}{A^m B^n C^p \Delta} = J_{A^m B^n C^p}. \tag{56}$$

With these notations the equation for the interior potential assumes the form

$$V_i = -\pi\rho abc(x^2 J_A + y^2 J_B + z^2 J_C - J). \tag{57}$$

In this form it is easy to see that the potential is the sum of a constant term and terms dependent on x^2, y^2, and z^2 as previously mentioned. In addition, we find that $J_A + J_B + J_C = 2/abc$ because

$$\nabla^2 V_i = -4\pi\rho. \tag{63a}$$

We can also verify by a fairly simple manipulation the formula that

$$J_B - J_A = (a^2 - b^2) J_{AB} \tag{59}$$

and similarly his equation

$$J_{A^m B^{n+1} C^p} - J_{A^{m+1} B^n C^p} = (a^2 - b^2) J_{A^{m+1} B^{n+1} C^p}. \tag{60}$$

With these preliminaries we remark that on a rotating body the potential referred to the rotating axes is given by

$$V_i + \frac{1}{2}\omega^2(x^2 + y^2). \tag{62}$$

On a figure of equilibrium the above potential must be constant over a whole boundary. If we also require that the boundary shall be an ellipsoid then we have an equation of the form (51). The normal way of combining these two equations is to multiply one of them by undetermined multiplier, say θ, and add to form a new function, M, as follows:

$$M = V_i + \frac{1}{2}\omega^2(x^2 + y^2) + \theta\pi\rho abc\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right).$$

When this is done we can regard x and y, for example, as independent variables on the surface, so that we can legitimately ask that the partial derivative of M with respect to x and y following the surface shall be 0. When we perform the necessary differentiations we must include z as a function of x and y. We shall have, therefore,

$$\frac{\partial M(x,y)}{\partial x} = \frac{\partial M(x,y,z)}{\partial x} + \frac{\partial M(x,y,z)}{\partial z}\frac{\partial z}{\partial x},$$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial M(x,y,z)}{\partial y} + \frac{\partial M(x,y,z)}{\partial z}\frac{\partial z}{\partial y}.$$

The second terms on the right are rather ugly, and since we have not yet decided what we are going to do with θ it is permitted, since the equations are linear, to say that we will choose θ in such a way that

$$\frac{\partial M(x,y,z)}{\partial z} = 0.$$

When we do so we have three similar equations in x, y, and z, since the ugly terms on the right-hand side have now been disposed of:

$$J_A - \frac{\omega^2}{2\pi\rho abc} = \frac{\theta}{a^2}; \tag{65}$$

$$J_B - \frac{\omega^2}{2\pi\rho abc} = \frac{\theta}{b^2}; \tag{66}$$

$$J_C = \frac{\theta}{c^2}. \tag{67}$$

Two of them simply express the condition that M is constant over the surface; but the third equation in effect defines θ. Naturally it makes no difference which of the equations we consider to be the one which defines θ. If we add all three equations we obtain:

$$J_A + J_B + J_C - \frac{2\omega^2}{2\pi\rho abc} = \theta\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \tag{64}$$

$$\left(\frac{2}{abc} - \frac{2\omega^2}{2\pi\rho abc}\right)\frac{1}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} = \theta = \frac{2\left(1 - \frac{\omega^2}{2\pi\rho}\right)}{abc\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)}.$$

Jeans gets the same result by taking advantage of a special property of the combined equation. He obtains the divergence of M and notes that if the divergence vanishes the function is a spherical harmonic. He can find a value for θ which will make the divergence vanish. The function is now a spherical harmonic and constant over the boundary of the ellipsoid, hence it must also be constant throughout the mass of the ellipsoid. Under these circumstances he can obtain the three important equations simply by equating to zero the coefficients of x^2, y^2 and z^2 since the function must be independent of the coordinates.

From these equations Jeans proceeds to obtain the conditions for the existence of rotating homogeneous ellipsoids. He first subtracts corresponding sides of (65) and (66) and obtains:

$$J_B - J_A = (a^2 - b^2)J_{AB} = \frac{\theta}{b^2} - \frac{\theta}{a^2} = (a^2 - b^2)\frac{\theta}{a^2b^2}. \tag{67a}$$

Theta is then eliminated between this equation and (67), which gives us

$$(a^2 - b^2)\left[a^2b^2J_{AB} - c^2J_C\right] = 0. \tag{68}$$

Now it will be clear that it is possible to satisfy the three fundamental equations either by taking

$$a^2 = b^2 \tag{69}$$

or

$$a^2b^2J_{AB} = c^2J_C. \tag{70}$$

These two cases correspond respectively to the Maclaurin ellipsoids and the Jacobi ellipsoids. The Maclaurin ellipsoids, it will be shown later on, are stable for small values of the angular velocity of rotation. All known planets are in the region of stability of the Maclaurin ellipsoids. They are oblate ellipsoids of revolution. The Jacobi ellipsoids are produced only, it turns out, when the velocity of rotation is such that a breakup is being approached. We, therefore, begin by discussing the Maclaurin ellipsoids. Clearly these include the case of the sphere for which $a = b = c$ and the angular velocity of rotation is 0. It is important to see that we have shown that these ellipsoids are *equilibrium* figures, whether or not they are figures of stable equilibrium.

For the Maclaurin ellipsoids we can omit equation (66) which is identical to (65) and we can eliminate θ between equation (65) and equation (67) which gives at once

$$a^2 J_A - c^2 J_C = \frac{\omega^2 a^2}{2\pi\rho abc}. \tag{70a}$$

We next substitute in equation (70a) for J_A and J_C and get

$$\text{(70b)}\qquad \begin{aligned} &a^2 \int_0^\infty \frac{d\lambda}{(a^2+\lambda)\Delta} - c^2 \int_0^\infty \frac{d\lambda}{(c^2+\lambda)\Delta} = \frac{\omega^2 a^2}{2\pi\rho abc}, \\ &\int_0^\infty \{a^2(c^2+\lambda) - c^2(a^2+\lambda)\} \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)\Delta} = \frac{\omega^2 a^2}{2\pi\rho abc} \end{aligned}$$

which is easily transformed into

$$\frac{(a^2 - c^2)}{a^2} \int_0^\infty \frac{\lambda d\lambda}{AC\Delta} = \frac{\omega^2}{2\pi\rho abc}. \tag{71}$$

The integration of (71) offers some difficulties. See [**5**, Vol. II, p. 71]. According to [**3**, p. 131], we have that X', the force component in the x-direction, is, in Jeans' notation,

$$\frac{X'}{x} = 2\pi\rho abc \int_\eta^\infty \frac{d\lambda}{A\Delta}.$$

Now Moulton tells us [**3**, p. 134] that when the lower limit of integration, which he calls κ, is 0, then in the case of an oblate spheroid we have

$$\frac{X'}{x} = -2\pi\rho \frac{\sqrt{(1-e^2)}}{e^3} [-e\sqrt{(1-e^2)} + \sin^{-1} e]$$

which must equal

$$-2\pi\rho a^2 c \int_0^\infty \frac{d\lambda}{A\Delta} = -2\pi\rho a^2 c J_A$$

and from this it follows that

$$a^2 c J_A = \frac{\sqrt{(1-e^2)}}{e^3} [-e\sqrt{(1-e^2)} + \sin^{-1} e].$$

In the same way we can use the z coordinate data of Moulton

$$\frac{Z'}{z} = -2\pi\rho abc \int_\eta^\infty \frac{d\lambda}{C\Delta}.$$

For $\eta = 0$

$$\frac{Z'}{z} = -\frac{4\pi\rho}{e^3}\left[e - \sqrt{(1-e^2)}\tan^{-1}\frac{e}{\sqrt{(1-e^2)}}\right]$$

so that

$$ca^2 J_C = \frac{2}{e^3}[e - \sqrt{(1-e^2)}\sin^{-1}e].$$

Combining these two we form the equation

$$\frac{J_A}{c} - \frac{cJ_C}{a^2} = \frac{1}{a^2}\left\{\frac{1}{c^2}\frac{\sqrt{(1-e^2)}}{e^3}[-e\sqrt{(1-e^2)} + \sin^{-1}e] - \frac{2}{a^2}\frac{\sqrt{(1-e^2)}}{e^3}\left[\frac{e}{\sqrt{(1-e^2)}} - \sin^{-1}e\right]\right\}$$

which reduces, after some trouble, using (70a), to the result

$$\frac{\omega^2}{2\pi\rho} = \frac{1}{e^3}(3-2e^2)(1-e^2)^{1/2}\sin^{-1}e - 3\left(\frac{1}{e^2} - 1\right) \tag{72}$$

where e is the eccentricity defined by $e^2 = (a^2 - c^2)/a^2$. From this equation it is possible to calculate values of the quantity $\omega^2/2\pi\rho$ as a function of e. These values are tabulated on page 39 of Jeans. The critical value is 0.81267 for e which is the value at which the Maclaurin spheroids cease to be stable and make the transition to the Jacobi ellipsoids.

A calculation of the Jacobi ellipsoids is considerably more difficult. Numerical values have been obtained by the use of elliptic integrals by Darwin. Although the Jacobi ellipsoids and the Maclaurin ellipsoids can be calculated past the point of junction the Maclaurin spheroids will be unstable if they are more oblate than this critical value. The situation with the Jacobi ellipsoids is different. They form a continuous sequence which goes from ellipsoids with a large value of a through those where $a = b$, to ellipsoids with large value of b relative to a. The Jacobi ellipsoid for which $a = b$ coincides with one of the Maclaurin ellipsoids and represents the junction between the Maclaurin ellipsoid and the Jacobi ellipsoids. The series is entirely symmetrical so that those with increasing a and those with increasing b are effectually identical.

The situation which has arisen here is typical of that in the study of rotating liquid mases. A sequence of configurations, in this case the Maclaurin ellipsoids, can be traced up to its intersection with another series. Beyond this point the first series becomes unstable and the stability is transferred to the second series.

When we pursue these studies by considering a further addition of angular momentum we find that the Jacobi ellipsoid becomes elongated. When the long axis comes to be something like $1.9 \times r_0$ a new deformation begins. In place of the Jacobi ellipsoid we have an asymmetrical figure which is generally called the pear-shaped figure of equilibrium because one end is narrower than the other. The calculated forms of the pear-shaped figure show, however, that it is more like the shape of a tenpin, that is to say relatively long as compared with a pear.

A series of pear-shaped configurations can be calculated going to higher and higher values of the angular momentum. These configurations, however, unlike the Jacobi ellipsoids, cannot represent the actual path of evolution of a rotating liquid mass. It turns out that the pear-shaped configurations are unstable. They are unstable not only in the sense that the effects of tidal friction will gradually tend to modify the body but in the more drastic sense that as soon as the Jacobi ellipsoid has received enough angular momentum to begin the formation of the pear-shaped body then it must continue catastrophically to change in some way which it has not yet been possible to follow mathematically. Although the pear-shaped configurations do not give us the actual path over which the body moves as it breaks up yet we may be sure that the breakup begins at the point where the pear-shaped configurations begin to be possible and we can further be sure that the path of evolution is tangent to the path of the series of pear-shaped bodies at the moment when breakup begins. This can probably be interpreted as meaning that the breakup begins with the formation of a neck around one end of the body. It is reasonable to suppose that further evolution proceeds by the deepening of this constriction until one end of the body is separated. In order to validate the above chain of reasoning for actual application to the problem of the earth it is necessary first of all to show that the ellipsoidal configurations are stable not only if we introduce the constraint that only ellipsoid configurations will be possible but also if this constraint is removed. This point has been discussed by Poincaré.

The fact that we are able with a single value of θ to satisfy these equations means that the ellipsoid is actually an equilibrium figure in the problem of a self gravitating liquid. We notice that θ is not a function of the coordinates but only of the angular velocity ω. Tracing this fortunate fact backwards we see that it is a consequence of the fact that the potential can be expressed in the very simple form shown in Equation (57) or perhaps we might equally well say that it is a consequence of the fact that the Laplacian ∇^2 takes a very simple form shown in Equation (63a). Suppose for instance that the equilibrium figure had not been an exact ellipsoid but something near it. In this case, when we went to solve for θ we would not have been able to find a single numerical constant but instead some kind of a function.

Poincaré showed that there is a method of investigating the stability of a series of bodies like the Maclaurin ellipsoids which greatly diminishes the effort involved. Poincaré begins by considering the general problem of equilibrium. Stability in a static system implies that the potential energy W is a minimum for a particular configuration as compared to all adjacent configurations. In a rotating system it can be shown that the same is true if we add a term as in (62).* We might think of a space of many dimensions, each dimension representing one of the parameters θ_1, θ_2, etc. which describe the configuration. We think of one of these, the angular momentum μ, as increasing vertically upward. In this space of many dimensions, we consider a set of surfaces of constant potential energy. Each of these surfaces must form a hill whose top is at the stable configuration. We can plot μ against one of these variables which describes the configuration, say θ. We draw the curve $W =$ constant; this curve must be concave downward. The value of θ which corresponds to equilibrium will be the value at the top of the bulge since W increases as μ increases.

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*We must note, however, that the convention is to take the potential energy W as increasing *outward* from a gravitating body, while the potential V increases *inward*. If the volume is Ω, and an element of Ω is $d\Omega$,

$$W = -\iiint_{\Omega} V\rho \, d\Omega.$$

Now if we consider a series of configurations of equilibrium then we are in effect considering the series of points which are at the peaks of the curves $W =$ constant. Let us suppose that one of these values is stable. Then we cannot reach an unstable configuration as we follow along this sequence of states unless in one of the parameters, θ, these curves become concave upwards instead of concave down. When this happens it may be true that the curves when extended outwards continue to curl up. Or it may be true that when extended outwards they turn down again after having gone a sufficient distance. In the latter case it is clear that we can trace out a new set of crests (or rather two new sets of crests) which start out at the point where the first sequence becomes unstable and spread out from it in both directions through the new set of peaks. In the opposite case, when the curves beyond the point of stability turn up then we shall ordinarily expect that for values of μ under the last stable value there existed, in the curves $W =$ constant, dips on either side of the set of humps which formed our original linear sequence. These configurations can also be represented by a line which passes through the last stable value of our original linear sequence. The third possibility is of course the limiting case where the point of instability is represented by a flat surface extending indefinitely in all directions and corresponding to neutral equilibrium. Setting this case aside for the moment, as trivial and as included in the other cases if minor changes of wording are made, we say that a linear sequence of configurations can only pass from stable to unstable when it encounters another linear sequence. This is a result of the continuity properties of W in these parameters. It is not in any way a consequence of the special properties of rotating ellipsoids.

In our particular case the sequence of Maclaurin ellipsoids must surely be considered stable at its initial point, where we are dealing with a sphere and zero rotation. As the angular momentum of this sphere increases we will be passing along a series of stable configurations until this is intersected by another set. It has been shown, by methods which I am not giving here, that the first sequence of forms which intersects the sequence of Maclaurin spheroids is the sequence of Jacobi ellipsoids. From this it follows that the Maclaurin spheroids will be stable up to the point where they encounter the series of Jacobi ellipsoids.

We can also see that the question whether the Jacobi ellipsoids are stable or not in this sequence depends on whether the curve which represents the sequence of Jacobi ellipsoids turns up or turns down in these diagrams. That is to say it depends on whether the Jacobi ellipsoids with higher values of the angular momentum are also ellipsoids with higher values of energy or not. Numerical computations have shown that in fact the Jacobi ellipsoids with higher energy are also those with higher angular momentum so that the curve does in fact turn upwards and the Jacobi ellipsoids are stable. From this it follows that a sequence of bodies of progressively increasing angular momentum will pass through a series of Maclaurin ellipsoids and then through a series of Jacobi ellipsoids. The stability of the Jacobi ellipsoids is terminated by a set of nonellipsoidal pear-shaped figures, which has been found to be unstable. This second intersection takes place not far beyond the point at which the Jacobi ellipsoids begin to form. As a consequence in most discussions of stability, the appearance of the Jacobi ellipsoids is taken as an indication of the approaching catastrophe.

In this discussion we have spoken as if the angular momentum could increase steadily. This is, of course, unrealistic; the angular momentum is constant. It turns out, however, that the quotient of the angular momentum divided by the density is the parameter which enters this discussion. Hence we may treat problems which are really those of increasing density as though they were problems of increasing angular momentum. The problems of increasing density, however, are exactly those which would be expected in a liquid mass which has newly condensed and is in the process of cooling. We may expect that in the early days of the earth the density increased as the heat was lost. It is against this background that the above discussions of stability become relevant. Up to this point we have been considering a mass of liquid of constant density. We have done so because this is the only case in which it is possible to follow the mathematics very well. We have chosen to make an exact treatment of a problem which is something like the real problem rather than to do the usual thing, which is to make a rough treatment of the actual problem.

In order to apply our results to the actual case of the earth itself we must consider inhomogeneous masses. Jeans attacked the problem in two ways. His first method was to consider a model which consisted of a nucleus of finite density surrounded by an atmosphere of zero density. Clearly this is the limiting case of the kind of a two-fluid system which Wiechert worked with. The problem is quite tractable mathematically once the study has been made on the homogeneous mass. It is simply a matter of defining one of the geopotential surfaces above the nucleus as the true surface. The volume enclosed between this surface and the nucleus is called the atmosphere; it is referred to as V_a, compared with V_n of the nucleus. The results which have already been derived for the behavior of the homogeneous mass can now be applied at once to this theoretical inhomogeneous planet.

In particular, Jeans found that if the ratio of the volume of the atmosphere to the volume of the nucleus exceeded about 1/3, then it would turn out that the fission would not take place along the sequence of the Jacobi ellipsoids. The rapidly rotating Maclaurin spheroid would develop a fissure around its equatorial zone through which matter would be ejected. This could also be expressed by saying that the contours of the geopotential no longer close around the earth.

He finds that there are two possible sequences of configurations: for a body in which the nucleus is small and very dense compared to the rest of its structure we have equatorial ejection of matter; on the other hand, if the nucleus is sufficiently large compared to the whole mass, then the behavior is qualitatively like that of a homogeneous mass, which we have been discussing.

It is true that the model does not really resemble the earth, but let us do the best we can to fit the earth to it. The polar moment of inertia C of the earth it is known to be given by:

$$\frac{C}{Ma^2} = 0.3307.$$

If the earth were homogeneous, we would have 0.4 instead of 0.3307. Thus, the earth has approximately 5/6 as much angular momentum as a homogeneous sphere of the same size. The question is, how big a homogeneous sphere would we need in order to have the same angular momentum as the earth, assuming that the total mass were the same? The answer is that the ratio of the radii should be the square root of 5/6 or 0.91. The ratio of the volumes is then just about 3/4. Hence, if we had an object consisting of the homogeneous sphere in the interior and a weightless shell outside so arranged that the space V_a between the shells was about 1/3 the volume of the inner shell, then this composite object would have approximately the same angular momentum and approximately the same value of C/Ma^2 as the earth. Jeans shows that this configuration is just on the borderline of the cases when fission takes place by the formation of a Jacobi ellipsoid. For more homogeneous bodies, fission is sure to take place by the development of the Jacobi ellipsoid; for less homogeneous bodies, that is bodies with a similar nucleus, breakup is sure to take place by the spreading away of a portion of the atmosphere around the equator. From this treatment it appears that the earth is near the limiting case.

Jeans' second, and more realistic model, involves the assumption of a polytropic distribution of density. Polytropic density distributions have been extensively studied in the theory of the internal constitution of the stars, largely because Emden (1907) made a series of numerical integrations of them. The terminology of these spheres goes back to Emden's assumption that stars are in convective equilibrium. For convective equilibrium, the ratio γ of the specific heat at constant pressure to the specific heat at constant volume is of decisive importance. Emden took as his parameter the quantity n given by the equation

$$\gamma = 1 + \frac{1}{n}.$$

The relation of n to any of the physically significant parameters of the distribution can only be reached through some detailed numerical integrations; as a consequence, n is for many purposes, and in particular for this one, merely a parameter which defines the density distribution. For $n = 0$, the density is uniform. For $n = 1$, it turns out that it is proportional to the function $(a/r)\sin(r/a)$. For $n = 3$, we have the kind of distributions with a strong concentration to the center which are believed to be typical of stars like the sun. For $n = 5$, the star lacks an outer boundary, and for $n = \infty$ we have the distribution which would characterize an isothermal atmosphere and would extend to infinity. Jeans has calculated the behavior of polytropic gas spheres rotating with sufficient rapidity to break up. He finds that if the polytropic index is less than about 0.8 the star will be sufficiently homogeneous so that it will break up via the formation of Jacobi ellipsoids. If, however, the polytropic index exceeds this quantity, it will break up by the formation of an equatorial ring somewhat like Saturn's rings. Recently Roberts has restudied this problem; he finds that the critical value of the polytropic index is near 1.0.

A numerical integration of the Emden table for the polytrope $n = 0.5$ shows that the value of C/Ma^2 will be 0.32. For the earth the same ratio is 0.33; it follows that the earth is slightly more homogeneous than the Emden polytrope $n = 0.5$. On this model the earth would break up through the formation of a Jacobi ellipsoid rather than by the equatorial ejection of matter.

The actual situation inside the earth may well be intermediate between these two extreme models. Hence the actual earth would probably break up via the Jacobi ellipsoid.

A second point on which Jeans made important numerical investigations is the question of the effect of the internal density distribution on the limiting value of the angular momentum required for break up. For the case of the homogeneous ellipsoid and the somewhat similar case of nearly homogeneous ellipsoids, Jeans has sought the value of the angular velocity ω at which the transition would take place from a Maclaurin spheroid to a Jacobi ellipsoid. He finds the following general formula

$$\frac{\omega^2}{2\pi\rho} = 0.18712 + 0.06827\frac{\rho_0 - \sigma}{\rho_0} + [0.01602 + 0.07098(\gamma - 2)]\left(\frac{\rho_0 - \sigma}{\rho_0}\right)^2 \tag{499}$$

which is applicable really only to relatively small deviations from a homogeneous mass. In (499), $\bar{\rho}$ is the mean density, ρ_0 is the density at the center of the earth, and σ is the density at the boundary.

When this series is applied to the earth, we find that the critical period of rotation is 1^h58^m. For a homogeneous body of the earth's mass, it is 2^h40^m; and if a homogeneous body rotating at this speed is transformed, without change of angular momentum, into an inhomogeneous body for which

$$\frac{C}{Ma^2} = 0.33$$

the period of rotation is 2^h11^m. It would seem to follow that the earth could not have broken up as a result of the formation of the core since it would wtill be rotating too slowly.

The result is, however, very doubtful, as Jeans would have been the first to say; the series does not converge well, and in fact the last term is larger than the one which precedes it, in the case of the earth. Jeans applied the series only to the case in which γ is near 2, which improves the convergence.

I have made some calculations based on later work by Roberts, which suggests that in fact the critical period for the earth is near 2^h18^m, so that the earth can in fact be destabilized by the formation of the core.

References

1. R. Emden, *Gaskugeln*, B. G. Teubner, Berlin, 1907.
2. J. H. Jeans, *Problems of cosmogony and stellar dynamics*, Cambridge Univ. Press, Cambridge, 1919.
3. F. R. Moulton, *An introduction to celestial mechanics*, Macmillan, New York, 1916.
4. P. H. Roberts, *On highly rotating polytropes*. I, Astrophys. J. **137** (1963), 1129-1141.
5. W. Thomson and P. G. Tait, *Principles of mechanics and dynamics*, reprinted by Dover Press, New York, 1962, from *Treatise on natural philosophy*, Cambridge Univ. Press, Cambridge, 1912.
6. E. Wiechert, *Ueber die Massenvertheilung im innern der Erde*, Göttingen, Gesellschaft der Wissenschaften, *Nachrichten*, (1897), 221-243.

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Geodetic Problems and Satellite Orbits

I. **Introduction.** Clearly, when tracking satellites, our only real knowledge that certain problems exist in the area of geodesy is through a study of the satellite tracking data, noting that present knowledge of geodesy is inadequate to theoretically describe and / or predict the detailed time dependence of the received tracking data. For this reason, the principal topic to be discussed in this chapter is the effect of geodetic errors on the time dependence of satellite tracking data as received by a tracking station located on the surface of the Earth from a near-earth satellite. These geodetic errors fall into two categories, geodetic errors which effect the location of the tracking station on the surface of the Earth and geodetic errors which effect the motion of the satellite (and therefore its position at some given value of the time). Consequently, subsidiary topics which shall be discussed are:

1. Methods for specifying the motion of a tracking station in inertial space, given the usual geodetic measurements available for a point on the earth's surface,
2. The motion of a near-earth satellite when influenced by the various harmonics of the earth's gravity field (geopotential), and
3. The functional dependence of various types of tracking data upon the trajectories of the station and satellite in inertial space.

These topics do not cover many problem areas relating to satellite motion and the accurate reception and tabulation of tracking data. Such problem areas, while important from the standpoint of achieving accurate prediction of the trajectories of satellites, can reasonably well be divorced from the geodetic problem areas. Consequently, this series of lectures will assume a rather narrow definition of the word geodetic problems — namely problems associated with the science of determining the shape and size of the Earth and its gravity field.

Fundamentally, the procedure for determining the orbit of a satellite can be considered as the process of assuming the satellite to be under the influence of a known force field and then using the tracking data to determine which solution to the equations of motion one should choose. By this I mean the following. Assuming for the moment that the forces acting on the satellite are known, an infinity of solutions to the differential equations of motion exist until boundary conditions are imposed — such as values for the initial position and velocity of the satellite at some chosen epoch. The tracking data is used to determine as accurately as possible these initial conditions. Consequently, errors in satellite orbits can arise from errors in the forces that act on the satellite and errors in the computed boundary conditions. Within the area of interest of these lectures, the geopotential is considered as the sole source of error in the satellite forces, and tracking station location errors the sole source of error in obtaining errored boundary conditions.

In principle, errors in the location of tracking stations can be discussed entirely separately from errors in the satellite forces. However, in practice, complete separation of the two sources of errors cannot be made. The primary reason is that the accurate determination of the station location depends in practice upon a knowledge of the geopotential (near the earth's surface) and consequently errors in the geopotential introduce errors in both the station and satellite trajectories in inertial space. Another important reason is because, to zeroth order, satellite tracking data provides information on the position and/or velocity of the satellite *relative* to that of the station. Consequently, it is frequently difficult to separate orbit errors accurately into those directly related to the station position and those directly related to the satellite motion.

It can be seen from the above discussion that central to the determination of station positions and satellite orbits is an accurate specification of the earth's gravitational force field, and I shall now briefly discuss a representation for the gravity field of the Earth. We chose the sign convention such that the force is given by $+\operatorname{grad} U$, where U is the gravitational potential of the Earth. It is common to express this potential as an expansion in surface harmonics so that:

$$U(R,\phi,\lambda) = \frac{K}{R}\left\{1 + \sum_{n=2}^{\infty}\left(\frac{R_0}{R}\right)^n \left[J_n P_n(\sin\phi) + \sum_{m=1}^{\infty} P_n^m(\sin\phi)\,(C_n^m \cos m\lambda + S_n^m \sin m\lambda)\right]\right\}$$

where

$K =$ gravity force constant ($\mathrm{km}^3/\mathrm{sec}^2$),

$R_0 =$ mean equatorial radius of Earth (km),

$R =$ geocentric radius (km),

$\phi =$ geocentric latitude (rad),

$\lambda =$ geocentric longitude (rad),

and where

$$P_n^m(Z) = (1 - Z^2)^{m/2}\frac{d^m}{dZ^m}\,P_n(Z).$$

The geocentric coordinates R, ϕ, and λ have their origin located at the center of gravity of the Earth. The geocentric latitude is measured from a plane which passes through the earth's C.G. and is normal to the earth's spin axis. The geocentric longitude is measured positive eastward from the plane containing the spin axis and a special marker at the observatory in Greenwich, England — the so called Greenwich meridian. Since the origin of this coordinate system is at the center of gravity of the Earth, it follows that $J_1 = C_1^1 = S_1^1 = 0$. To the accuracy that we will consider in these lectures we may assume that there is sufficient energy dissipation that the earth's spin axis is the principal axis of the largest moment of inertia of the Earth and therefore we may assume that the spin axis passes through the earth's C.G. Consequently, in the above expansion for the geopotential we also may take $C_2^1 = S_2^1 = 0$. Finally, to the accuracy which we shall consider, we may assume that the earth's gravitational field is time independent and that the spin axis, equatorial plane, and Greenwich meridian are fixed with respect to the crust or surface of the Earth. Except for some relatively minor considerations when discussing the geoid, we shall not be interested in the gravitational field below the physical surface of the Earth.

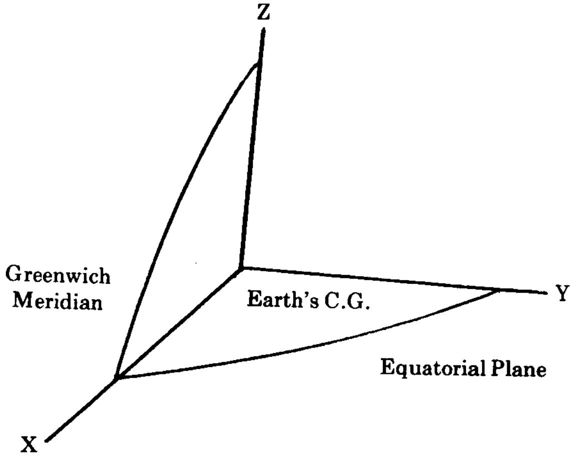


FIGURE 1. Right-handed coordinates.

Corresponding to the geocentric coordinates R, ϕ, and λ there is a natural right-handed cartesian coordinate system fixed with respect to the Earth. This is shown in Figure 1. The Greenwich meridian is the $X - Z$ plane and the equatorial plane coincides with the $X - Y$ plane.

Because of the earth's rotation it is not convenient to describe the satellite motion in a coordinate system which is fixed with respect to the earth's crust. A very natural coordinate system for the satellite motion is one which has its Z-axis coinciding with the earth's spin axis and its X and Y axis approximately fixed relative to inertial space (fixed relative to the celestial sphere). This inertial coordinate system and its relationship with the earth fixed cartesian system is shown in Figure 2. Very briefly, the inertial system is defined in the following way (see [1]). The apparent motion of the sun around the Earth approximately lies in a plane called the ecliptic plane. The intersection of this plane with the earth's equatorial plane defines a line which is approximately fixed in inertial space. We take the positive X-axis of the inertial system as the direction of this line of intersection going from the C.G. of the Earth in that direction where the sun crosses the equatorial plane going from south to north. This direction is known to the astronomer as the First Line of Aries. This coordinate system is called the True Equatorial System of Date to denote that it is defined by the direction of the instantaneous spin axis of the Earth and the intersection of the instantaneous equatorial and ecliptic planes. This system experiences small accelerations due to the fact that the earth's spin axis precesses and nutates relative to inertial space and the apparent motion of the sun around the Earth does not lie exactly in a fixed plane. However, for our purposes this coordinate system is a sufficient approximation to an inertial system and for coordinate systems which are more accurately inertial one may refer to [1].

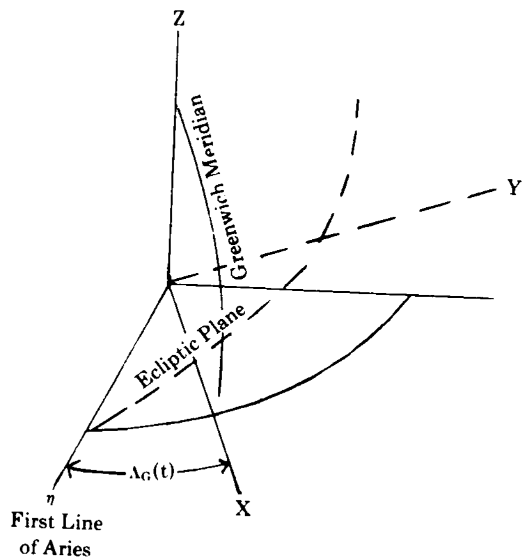


FIGURE 2. Inertial and Earth Fixed Coordinates.

It is inevitable that other coordinate systems must be introduced when discussing the location of a tracking station on the surface of the Earth. This is because all surveying is done on the surface of the Earth and it is most natural to define coordinate systems which are surface coordinate systems. I shall now briefly discuss the various geodetic coordinates required to locate a tracking station (see [2] and [3] for details).

A surface from which a natural surface coordinate system can be developed is one of the equipotential surfaces for the Earth. If this equipotential surface is chosen to coincide with mean sea level (average height of the sea surface when corrected for tides, weather effects, etc.) the surface is known as the geoid. This surface, by definition, is everywhere normal to the direction of the force of gravity, and all measurements of relative height are most naturally referenced to the geoid. When over land the geoid is not measurable in as straightforward a manner as one might think. Clearly many areas will have the geoid located below the physical surface of the Earth. When this is the case it is necessary to correct for the gravitating mass that is above the geoid when using gravity measurements to determine the geoid. Correcting for this mass inevitably involves assumptions as to the density, inhomogeneities, etc., of the crustal mass, and for clarity one refers to the co-geoid (see [2] and [3]) rather than the geoid when discussing the determination of an equipotential surface over land masses. To the accuracy required for these lectures however we may assume that the geoid and co-geoid are coincident and, consistent with the previous assumptions, we may assume that the geoid is time independent.

The shape of the geoid is sufficiently complex that it is inconvenient to use in computations. For this reason it is common to use an oblate spheroid (ellipse of revolution) which approximately follows the geoid in specifying the geodetic coordinates of a station. Figure 3 shows a meridianal section of a spheroid with the pertinent quantities used to define the spheroid and the coordinates of a point on the surface of the spheroid. A spheroid, being an ellipse of revolution, has its surface defined when its semi-major axis and eccentricity are defined. In practice the flattening, f, is given instead of the eccentricity and is related to the eccentricity by the formula: $f=1-\sqrt{(1-\epsilon^2)}$. The latitude and longitude of a station are always referenced to the spheroid. The geodetic latitude, ϕ_G, is defined by dropping a perpendicular to the surface of the spheroid and noting

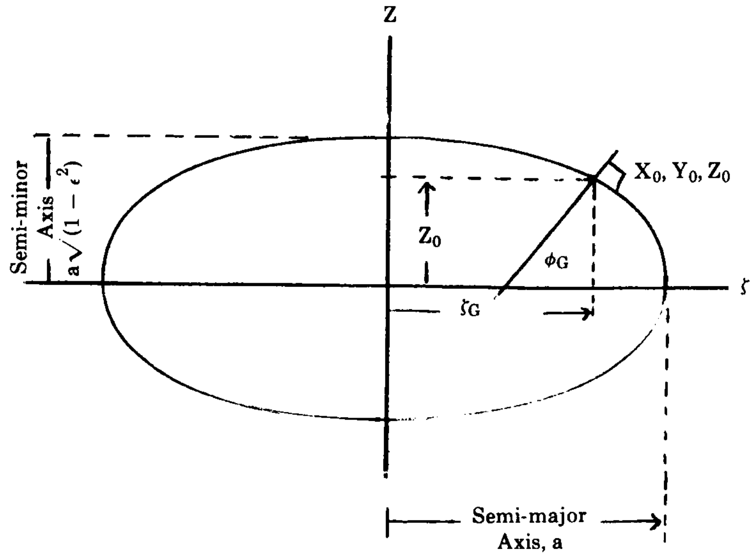


FIGURE 3. Ellipse Defining an Approximate Geoid.

the angle of intersection of this normal with the equatorial plane. Consequently, the cartesian coordinates ζ_G, Z_0 in the meridian containing the station are (see Figure 3).

$$\zeta_G = \frac{a}{\sqrt{(1 + (1-f)^2 \tan^2 \phi_G)}} = \sqrt{(X_0^2 + Y_0^2)},$$

$$Z_0 = (1-f)^2 \zeta_G \tan \phi_G.$$

The longitude is, of course, related to the cartesian coordinates X_0, Y_0 by $\lambda_G = \tan^{-1} Y_0/X_0$.

In specifying the orientation of a spheroid with respect to the spin axis and center of gravity of the Earth the intent is normally to have the semi-minor axis coincide with the spin axis and the semi-major axis lying in the equatorial plane with the center of the spheroid at the center of gravity of the Earth. In practice the specification of this orientation is done at the surface of the Earth at a point which is denoted as the datum point. This implies that the spheroid is oriented to the geoid at a point on the surface of the Earth which does not coincide with either the spheroid or the geoid. Such a connection is subject to measurement errors such that any given spheroid associated with a major surveyed area does not in fact have its center at the center of gravity of the Earth or on the earth's spin axis.

With the advent of satellites and their use for improving the knowledge of the force field of the Earth it is becoming common practice to define a world wide survey system or datum which has its spheroid, by definition, oriented correctly with respect to the center of gravity of the Earth and its spin axis. For example, the current NASA World Datum has as its semi-major axis and flattening

$$R_0 = 6378.166 \text{ kilometers},$$

$$f_0 = 1/298.24.$$

With such a definition for the orientation of the spheroid it then becomes a straightforward procedure to state the coordinates of the geoid and the various geodetic coordinates of the tracking station relative to this spheroid and to give transformation formulas for obtaining the geocentric coordinates of a station. Of course when using such a world wide datum it is necessary to obtain transformation formulas from the datum of a major surveyed network such as the North American Datum to the World Datum. Such transformations normally assume that the spheroid for the local datum has its axes parallel to the axes of the world datum spheroid so that a translation only is needed to transform from one spheroid to the other.

Before proceeding further, I shall now briefly show that to first order in the flattening, f, a spheroid approximates an equipotential surface for the Earth. This proof depends upon the experimental fact that

$$J_2 = O(f),$$

$$J_n, C_n^m, S_n^m = O(f^2), \qquad n > 2.$$

The proof proceeds in the following manner. For any point on the spheroid

$$X_0, Y_0, Z_0, R_0 = \sqrt{(X_0^2 + Y_0^2 + Z_0^2)},$$

let

$$\sin\phi = \frac{Z_0}{R_0}, \quad \cos\phi = \frac{\sqrt{(X_0^2 + Y_0^2)}}{R_0},$$

a = semi-major axis of spheriod,

f = flattening.

Then

$$\frac{R_0^2}{a^2}\left[\cos^2\phi + \frac{\sin^2\phi}{(1-f)^2}\right] = 1.$$

For any point rigidly connected to the Earth, the measured gravitational potential will be the sum of the gravitational potential, U, as measured in inertial space and a potential whose gradient yields the centrifugal force arising from the earth's rotation. Letting this earth-fixed potential be ψ and noting that all coefficients in the expansion for U are $O(f^2)$ except J_2:

$$\psi = \frac{K}{R}\left[1 + \frac{J_2}{2}\left(3\frac{Z^2}{R^2} - 1\right) + \frac{\omega_E^2 R(X^2+Y^2)}{2K} + O(f^2)\right],$$

where ω_E = angular rotation rate of Earth (rad/sec). We consider now the potential, ψ_0, for any point X_0, Y_0, Z_0 on the spheroid. From the above equations:

$$\psi_0 = \frac{K}{a}\left\{1 - \frac{J_2}{2} + \frac{\omega_E^2 a^3}{2K} + \sin^2\phi\left[f + \frac{3}{2}J_2 - \frac{\omega_E^2 a^3}{2K}\right] + O(f^2)\right\},$$

where it has been noted that:

$$\frac{\omega_E^2 a^3}{2K} = O(f).$$

Thus, letting

$$f = -\frac{3}{2}J_2 + \frac{\omega_E^2 a^3}{2K} + O(J_2^2),$$

we have

$$\psi_0 = \frac{K}{a}\left\{1 - \frac{J_2}{2} + \frac{\omega_E^2 a^3}{2K} + O(f^2)\right\},$$

which is a constant to $O(f)$.

The above proof indicates that the geoid (more properly the co-geoid) will not differ markedly from a properly defined spheroid. Consequently, the spheroid provides a convenient base for specifying quantitatively the geoid. This is done by specifying the geoidal height, $H(\phi_G, \lambda_G)$ for any given geodetic latitude, ϕ_G, and longitude, λ_G, as defined on the spheroid. This relationship is shown in Figure 4A where it can be seen that any point X_G, Y_G, Z_G on the geoid is related to the geodetic latitude and longitude by the formulas:

$$X_G = (\zeta_G + H\cos\phi_G)\cos\lambda_G,$$

$$Y_G = (\zeta_G + H\cos\phi_G)\sin\lambda_G,$$

$$Z_G = (1-f)^2\zeta_G\tan\phi_G + H\sin\phi_G.$$

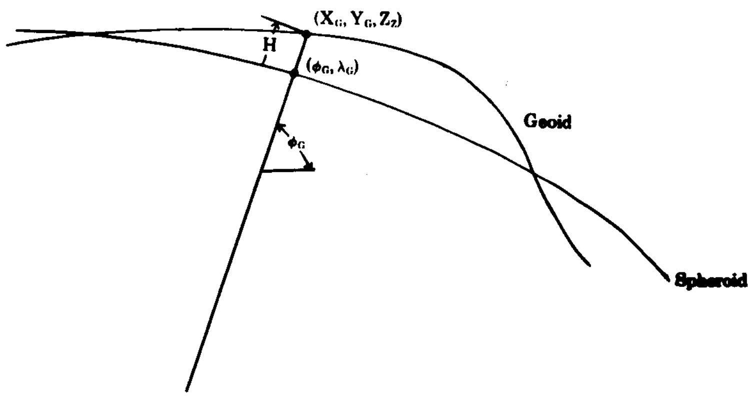


FIGURE 4A.

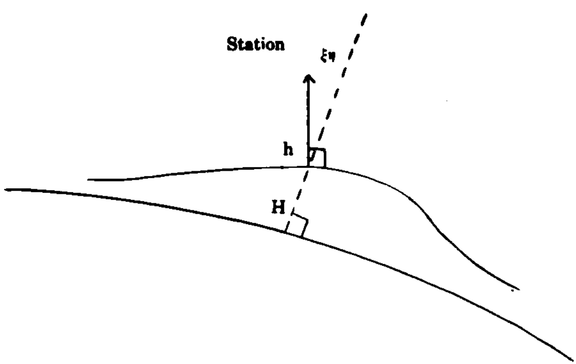


FIGURE 4B.

We are now ready to include the remaining geodetic quantities needed to specify the geocentric location of a tracking station. Those quantities which have not yet been discussed are (in order of importance):

h = elevation of station above geoid (measured normal to geoid),

ξ = deflection of local vertical in meridian (positive north),

ζ = deflection of local vertical in prime meridian (positive east),

$\delta X, \delta Y, \delta Z$ = position of center of spheroid associated with local survey relative to center of world-wide (NASA) spheroid.

Figure 4B shows schematically the first of these three quantities in relation to the geoid and spheroid. The last three are self explanatory.

Without further discussion I shall now give the final computational procedure for determining a station's geocentric cartesian coordinates given the geodetic quantities that I have just previously discussed. For further details see [2] and [3].

$$\zeta_L = \frac{a}{\sqrt{(1 + (1 - f)\tan^2\phi_G)}}, \qquad a, f = \text{semi-major axis and flattening for local spheroid.}$$

$$\begin{aligned} X_R = {} & [\zeta_L + (H + h)\cos\phi_G]\cos\lambda_G \\ & - h[\xi\sin\phi_G\cos\lambda_G + \eta\cos\phi_G\sin\lambda_G] \\ & + \delta X + \text{second order in } \xi \text{ and } \eta, \\ Y_R = {} & [\zeta_L + (H + h)\cos\phi_G]\sin\lambda_G \\ & - h[\xi\sin\phi_G\sin\lambda_G - \eta\cos\phi_G\cos\lambda_G] \\ & + \delta Y + \text{second order in } \xi \text{ and } \eta, \\ Z_R = {} & [(1 - 1/f)^2\zeta_L + (H + h)\cos\phi_G]\tan\phi_G + h\xi\cos\phi_G \\ & + \delta Z + \text{second order in } \xi \text{ and } \eta. \end{aligned}$$

II. **Discussion of orbits.** In §I we briefly considered a suitable representation for the geopotential and its relation to methods for locating a tracking station on the surface of the Earth. I now wish to turn our attention to the motion of a satellite under the influence of the geopotential and to present some working formulas relating the geometry of the satellite relative to such a station, which will be needed in the future sections when we consider in more detail the effect of errors in the location of the tracking station and in the satellite motion.

Generally when we speak of a satellite orbit we imply the ability to compute (to some acceptable accuracy) the position of the satellite as a function of time in inertial space (for example the True Equatorial System of Date). The computation of such a satellite ephemeris clearly implies that a well-defined force field has been assumed to be acting on the satellite, and satellite tracking data has been used to determine the orbit parameters (initial boundary conditions) for the solution of the differential equations of motion for the satellite.

Since we are primarily interested in the geodetic aspects of satellites and their motion I shall make the following restrictive assumptions to simplify the analysis which will be presented in the following sections.

A. Assumptions concerning satellite orbits.

1. Satellite motion

a. nonrelativistic approximation to equations of motion,

b. near-earth satellites with small eccentricity (satellite altitude not less than about 1000 km and eccentricity $\epsilon \leqq .05$).

2. Satellite forces not considered (see discussion in [4])

a. nongravitational in origin,

(1) air drag,

(2) radiation pressure,

(3) electromagnetic,

b. nonstatic and extra-terrestrial gravitational forces,

(1) Sun, Moon, other planets, etc.

(2) earth's body and sea tides.

In addition to these assumptions we presume that we have at our disposal a world-wide network of tracking stations together with the necessary data links and computer programs to establish (or track) the satellite to an accuracy limited by the accuracy of the geopotential and station locations assumed and the accuracy of the experimental tracking data. To further simplify our considerations I shall assume that there are negligible errors in the experimental tracking data. In particular I assume:

B. Assumptions concerning experimental tracking data.

1. Signal propagation errors due to atmosphere are not considered,

a. ionospheric and tropospheric refraction (scintillation if optical data),

b. ducting, skip propagation, etc.

2. Experimental instrumentation errors are negligible,

a. misalignment and poorly calibrated tracking instruments,

b. "front-end" receiver (detector) noise,

c. errors in transmission and formatting of data.

There are four fundamental measurements that are commonly made during the time that a satellite is above the horizon of a tracking station. These are:

1. Vector slant range

$$\boldsymbol{\rho}(t) \equiv \mathbf{r}_s(t) - \mathbf{r}_R(t).$$

2. Scalar slant range

$$\rho(t) = |\boldsymbol{\rho}(t)|.$$

3. Slant range unit vector

$$\hat{\rho}(t) = \boldsymbol{\rho}(t)/\rho(t).$$

4. Scalar slant range rate

$$\dot{\rho}(t) = (d/dt)\,\rho(t) = \hat{\rho}(t)\cdot\dot{\boldsymbol{\rho}}(t),$$

where:

$\mathbf{r}_s(t),\ \dot{\mathbf{r}}_s(t) =$ satellite position and velocity in True Equatorial System of Date,

$\mathbf{r}_R(t),\ \dot{\mathbf{r}}_R(t) =$ tracking station position and velocity in True Equatorial System of Date.

The slant range vector is typically the type of data taken by a tracking radar using the narrow beam pattern of the antenna to measure the slant range unit vector and its range (time of flight) instrumentation to measure the scalar slant range. Some radar tracking systems measure only the scalar slant range recognizing that the operating frequency is too low to accurately define angles. Optical tracking, of course, measures the slant range unit vector, that is, right ascension and declination or azimuth and elevation. Finally tracking systems exist which use the measurement of the radio Doppler shift to make direct measurement of the scalar slant range rate. Some installations measure the slant range vector as well as the scalar slant range.

Clearly, the above types of data involve various combinations of quantities directly related to the relative geometry between the satellite and station during the time that the satellite is above the station's horizon. The remainder of this section will be devoted to presenting notation, convenient coordinate systems, and expressions relating the various quantities associated with the relative geometry between the satellite and station.

Let

$$t_c = \text{time of closest approach of satellite to station,}$$

$$t_R = \text{time of satellite rise above station's horizon,}$$

$$t_s = \text{time of satellite set below station's horizon,}$$

$$\beta(t) = \text{satellite argument of latitude,}$$

$$\Delta\beta_0 = \beta(t_s) - \beta(t_c) \cong \beta(t_c) - \beta(t_R),$$

$$E_l, A_z = \text{elevation and azimuth of satellite at } t_c.$$

Figures 5, 6, and 7 show the geometry of the pass and present a convenient coordinate system in which to consider the motion of the satellite relative to the station. This coordinate system is fixed in the satellite inertial space and has its coordinate axes defined at the time of closest approach, t_c. The Z-axis is defined to be the direction of the instantaneous angular momentum vector of the satellite at t_c. In Figure 5, the X-axis is defined as that line of intersection between the equatorial plane and the plane normal to the Z-axis and which contains the satellite position at t_c. The Y-axis is chosen such that the X, Y, Z coordinate system is a right-handed system. Clearly, the $X - Y$ plane is the osculating plane of the orbit at the time of closest approach.

Figure 6 presents in more detail the pass geometry at the time of closest approach where the H-axis passes through the position of the satellite at t_c. Figure 7 presents the geometry of the pass

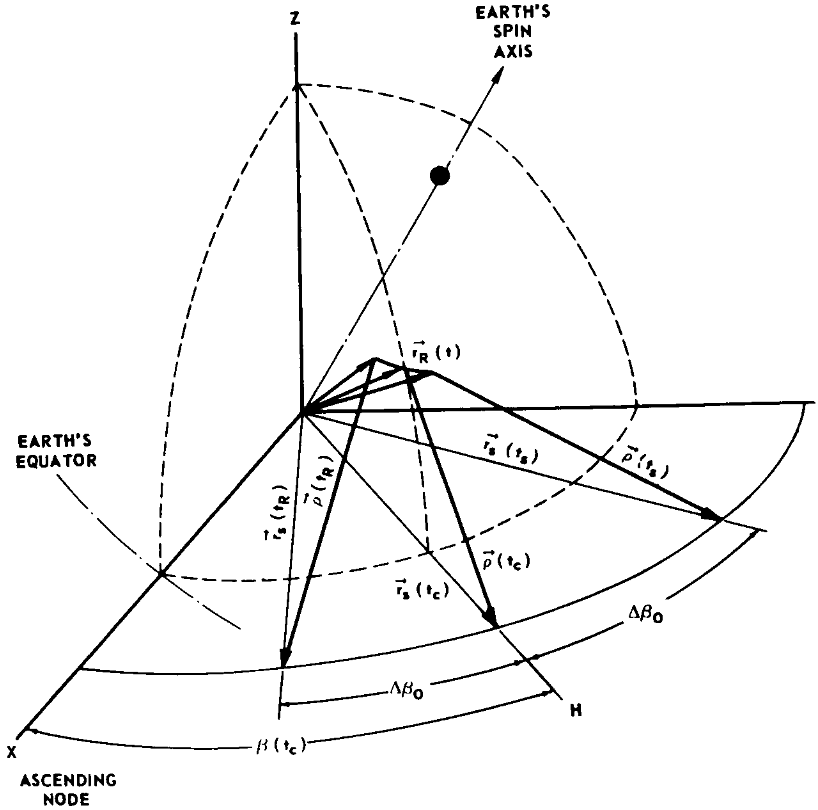


FIGURE 5. Geometry During Satellite Pass
($x - y$ plane = Orbital Plane)

projected on the $X - Y$ plane and where the new coordinate axis, L, has been introduced to make the H, L, Z coordinate system a right-handed system. In Figure 7, the satellite position relative to its position at the time of closest approach is approximately shown with the change in the argument of latitude being denoted by $\Delta\beta$. (For simplicity the motion of the station during the time of the pass has been approximated as zero for clarity.) The coordinate system which will be of primary interest to us in the following sections is the H, L, Z coordinate system presented in these three figures.

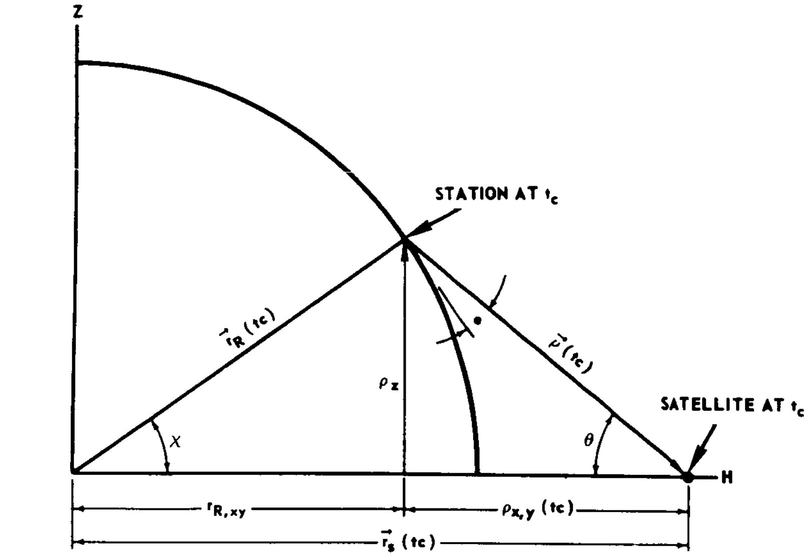


FIGURE 6. Geometry at Time of Minimum Slant Range
($H - Z$ Plane, Satellite motion into page)

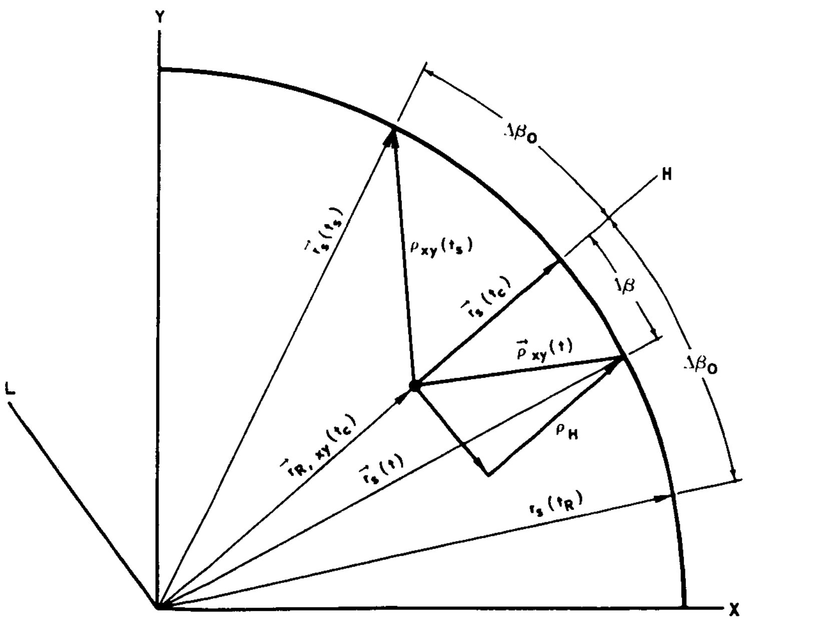
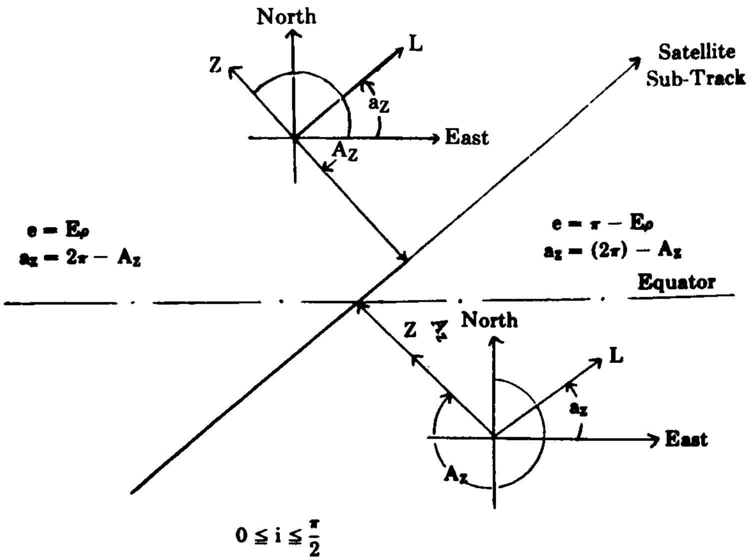
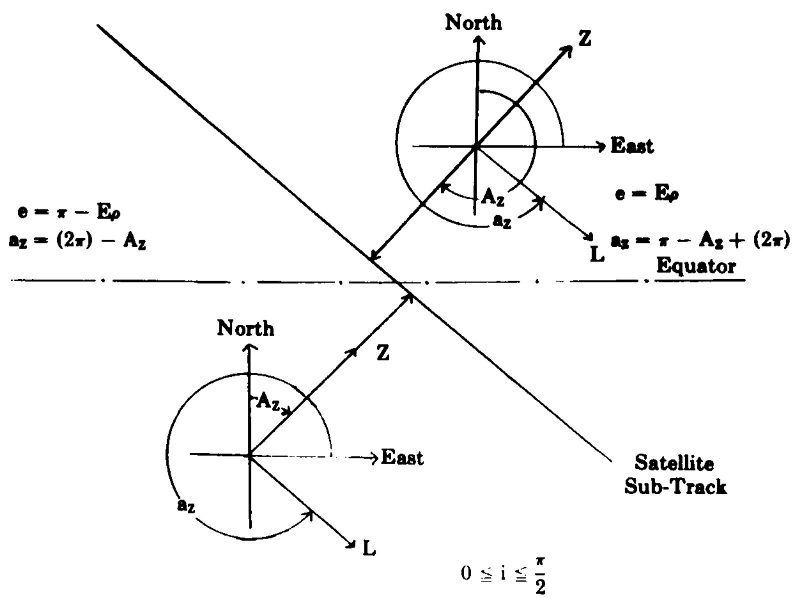


FIGURE 7. Geometry of Pass (Orbital Plane)

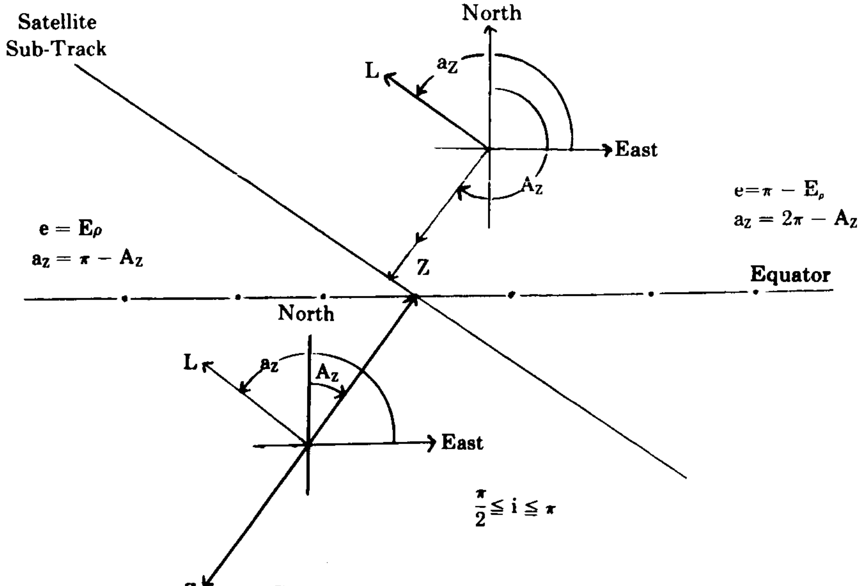


Satellite on Ascending Node

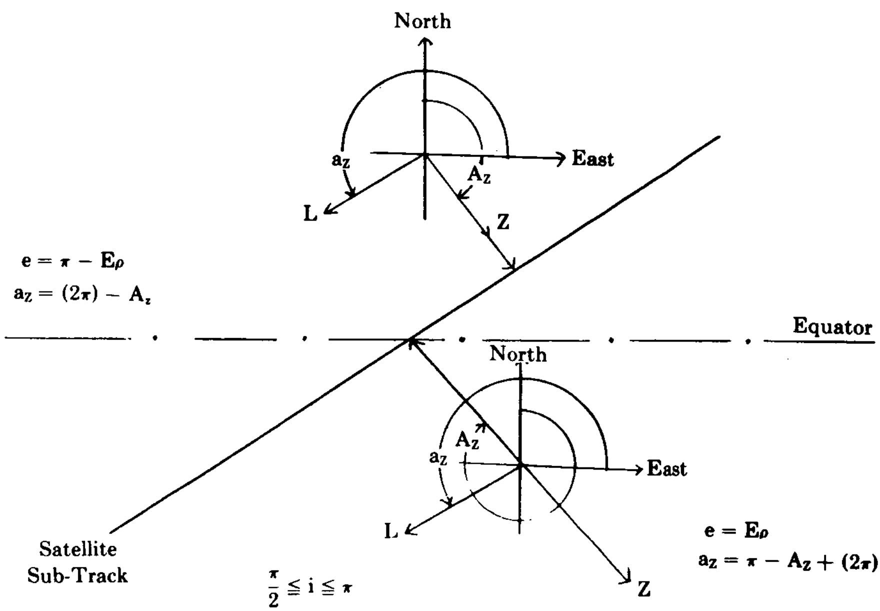


Satellite on Descending Node

FIGURE 8A. Pseudo Elevation and Azimuth
(Advance Satellite Motion)



Satellite on Ascending Node



Satellite on decending node

FIGURE 8B. Pseudo Elevation and Azimuth
(Retrograde Satellite Motion)

FIGURE 8C. Pseudo Elevation and Azimuth

Azimuth	Satellite Inclination $0 \leqq i \leqq \pi/2$		$\pi/2 < i \leqq \pi$	
$0 \leqq A_z < \pi/2$	$e = \pi - E_l$	$a_z = -A_z$	$e = E_l$	$a_z = \pi - A_z$
$\pi/2 \leqq A_z < \pi$	$e = E_l$	$a_z = \pi - A_z$	$e = \pi - E_l$	$a_z = -A_z$
$\pi \leqq A_z < 3\pi/2$	$e = E_l$	$a_z = \pi - A_z$	$e = \pi - E_l$	$a_z = -A_z$
$3\pi/2 \leqq A_z < 2\pi$	$e = \pi - E_l$	$a_z = -A_z$	$e = E_l$	$a_z = \pi - A_z$

$E_l =$ Elevation
$A_z =$ Azimuth

$e =$ Pseudo Elevation
$a_z =$ Pseudo Azimuth

The usual definitions for the elevation, E_l, and azimuth, A_z, are inconvenient when deriving general formulas valid for all possible paths of satellites past a given tracking station. For example, if a satellite passes through the zenith of the station the azimuth makes a discontinuous change of 180°. Two quantities directly related to the azimuth and elevation are much more conveniently used in such derivations. These have been denoted as the "pseudo azimuth", a_z, and "pseudo elevation", e. Figures 8A, 8B, and 8C show the relationships between the normally defined azimuth and elevation and the pseudo azimuth and elevation. It can be seen that the pseudo azimuth and elevation are obtained by altering the quadrants in which the azimuth and elevation lie so that there is continuity in changing from one type of pass geometry to another. For example, referring to Figure 6, the pseudo elevation is indicated and (for the case shown) can be seen to be identical with the normally defined elevation. This pseudo elevation will remain continuous as the vector ρ_z decreases through zero and goes negative, at which time the pseudo elevation increases beyond 90°. From Figures 8A and 8B it can also be seen that as ρ_z goes negative there is no discontinuity in the value for the pseudo azimuth.

In the sections to follow the effects of the errors will be considered to first order. Consequently, the coefficients multiplying these errors need be derived only to a crude accuracy. For example, to sufficient accuracy the change in the station position during the time of the pass can be neglected in the expression for the slant range when it is involved in expressions which have been expanded to first order in the errors. Those relations which will be needed in the following lectures are now briefly summarized to the required. accuracy. For details, see [**5**].

Let

$$r_R = |\mathbf{r}_R(t_c)|, \; r_s = |\mathbf{r}_s(t_c)|,$$

$$r_{R,s} = r_R/r_s, \; \rho_s = \rho(t_c)/r_s.$$

Then, from Figure 7,

$$r_{R,s}^2 = 1 + \rho_s^2 - 2\rho_s \cos\theta,$$

and

$$\sin\theta = r_{R,s}\sin(\pi/2 + e) = r_{R,s}\cos e.$$

These two formulas may be rearranged to yield:

$$\rho_s = \cos\theta - \sqrt{(r_{R,s}^2 - \sin^2\theta)},$$

$$= \frac{1 - r_{R,s}^2}{\sqrt{(1 - r_{R,s}^2\cos^2 e)}} + r_{R,s}\sin e.$$

Neglecting the station motion in inertial space, to zeroth order the slant range vector in the H, L, Z coordinate system becomes

$$\boldsymbol{\rho}(t) = \begin{pmatrix} \rho_H(t) \\ \rho_L(t) \\ \rho_Z(t) \end{pmatrix}$$

$$= r_s \begin{pmatrix} \rho_s\cos\theta - 1 + \cos\Delta\beta(t) \\ \sin\Delta\beta(t) \\ -\rho_s\sin\theta \end{pmatrix} + \text{first order},$$

where,

$$\Delta\beta(t) = \dot{\beta}(t_c)(t - t_c) + O(\epsilon).$$

Finally, defining the quantities

$$\alpha_s = 1 - \rho_s\cos\theta,$$

$$C(t) = 1 - \cos\Delta\beta(t),$$

$$\boldsymbol{\rho}(t) = r_s \begin{pmatrix} 1 - \alpha_s - C(t) \\ \sin\Delta\beta(t) \\ -\rho_s\sin\theta \end{pmatrix} + \text{first order},$$

with

$$\rho(t) = \sqrt{(\boldsymbol{\rho}(t)\cdot\boldsymbol{\rho}(t))} = r_s\sqrt{(\rho_s^2 + 2\alpha_s C(t))} + \text{first order}.$$

III. **Effects of geodetic errors.** With this section we shall begin the discussion of the effects of the geodetic errors. I begin by considering the station location errors. In the first section, we considered the Earth fixed cartesian coordinates of the tracking station. Let its corresponding spherical coordinates be:

$$\phi_R = \text{geocentric latitude,}$$
$$= \sin^{-1}(Z_R/r_R);$$
$$\lambda_R = \text{geocentric longitude,}$$
$$= \tan^{-1}(Y_R/X_R);$$
$$r_R = \text{geocentric radius,}$$
$$= \sqrt{(X_R^2 + Y_R^2 + Z_R^2)}.$$

Let the errors in these coordinates be $\delta\phi_R$, $\delta\lambda_R$, δr_R respectively. Then, a representation of these errors in distance units to first order in the errors are:

$$E_{r_R} = \delta r_R,$$
$$E_{\phi_R} = r_R \delta\phi_R,$$
$$E_{\lambda_R} = r_R \cos\phi_R \delta\lambda_R.$$

I now wish to rotate these errors into the H, L, Z coordinate system defined in §II.

Rotating first about the station radius vector by the pseudo-azimuth, a_z, (Figures 8A and 8B):

E_{r_R} is unchanged,
$$E_{L_R} = E_{\phi_R} \sin a_z + E_{\lambda_R} \cos a_z,$$
$$E'_{Z_T} = E_{\phi_R} \cos a_z - E_{\lambda_T} \sin a_z,$$

where E'_{Z_T} is perpendicular to $\mathbf{r}_R$ and lies in the H-Z plane and is frequently referred to as the station cross-track error. Making now a rotation about the L-axis by an angle χ (Figure 6),

$$E_{H_R} = E_{r_R} \cos\chi - E'_{Z_R} \sin\chi,$$
E_{L_R} is unchanged,
$$E_{Z_R} = E_{r_R} \sin\chi + E'_{Z_R} \cos\chi.$$

From Figure 6, it can be seen that

$$\sin\chi = \rho_s \cos e,$$
$$\cos\chi = \rho_s \sin e + r_{R,s}.$$

Successive substitutions for $\sin\chi$, $\cos\chi$ and then E'_{Z_R} yield:

$$E_{H_R} = r_{R,s} E_{r_R} + \rho_s [\sin e\, E_{r_R} - \cos e \cos a_z E_{\phi R} + \cos e \sin a_z E_{\lambda_R}],$$

$$E_{L_R} = \sin a_z E_{\phi R} + E_{\lambda_R} \cos a_z,$$

$$E_{Z_R} = r_{R,s}[\cos a_z E_{\phi R} - \sin a_z E_{\lambda_R}] + \rho_s[\cos e\, E_{r_R} + \sin e \cos a_z E_{\phi R} - \sin e \sin a_z E_{\lambda_R}].$$

These are the expressions for the station error which we shall eventually use in computing the effect of station error on tracking data residuals. From here on we shall assume that these errors are scaled by the mean equatorial radius, R_0.

I now want to direct our attention to the more involved task of obtaining similar expressions for errors in the satellite motion during the time the satellite is above the station's horizon. We assume that the satellite has been tracked so that satellite position errors may be considered only to first order. We denote the coordinates of the satellite by r_s, ϕ_s, λ_s in inertial space. These are related (see [6] and [7]) to the osculating kepler elements by the relations:

$$r_s = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\beta-\omega)} \quad \text{(units of } R_0)$$

$$\sin\phi_s = \sin i \sin\beta,$$

$$\cos\phi_s \cos(\lambda_s - \Omega) = \cos i \sin\beta,$$

$$\cos\phi_s \cos(\lambda_s - \Omega) = \cos\beta,$$

$$\tan(\lambda_s - \Omega) = \cos i \tan\beta,$$

where:

a = semi-major axis (units of R_0),

ϵ = eccentricity,

i = inclination,

ω = argument of perigee

Ω = right ascension (longitude) of ascending node,

M = mean anomaly,

$\dot{M} = n_0$ = anomalistic mean motion,

β = argument of latitude,

$f = \beta - \omega =$ true anomaly,

$\Phi = M + \omega.$

When a change in the geopotential is made of the form:

$$\Delta U = \frac{K}{R_0 r_s}\left\{\frac{\Delta K}{K} + \sum_{n=2}^{\infty}\frac{1}{r_s^n}\left[\Delta J_n P_n(\sin\phi_s) + \sum_{m=1}^{\infty} P_n^m(\sin\phi_s)(\Delta C_n^m \cos m\lambda_s + \Delta S_n^m \sin m\lambda_s)\right]\right\},$$

the equations of motion for the *changes* in the osculating elements to first order are:

$$\delta\dot{a} = \frac{2}{n_0 a}\frac{\partial\Delta U}{\partial\beta} + O(\epsilon),$$

$$\delta\dot{\epsilon} = \frac{1}{n_0 a}\left[\sin(\beta-\omega)\frac{\partial\Delta U}{\partial a} + \frac{2}{a}\cos(\beta-\omega)\frac{\partial\Delta U}{\partial\beta}\right] + O(\epsilon),$$

$$\frac{d\delta i}{dt} = \frac{1}{n_0 a^2}\cot\beta\frac{\partial\Delta U}{\partial i} + O(\epsilon),$$

$$\sin i\,\delta\dot{\Omega} = \frac{1}{n_0 a^2}\frac{\partial\Delta U}{\partial i} + O(\epsilon),$$

$$\epsilon\delta\dot{\omega} = \frac{1}{n_0 a}\left[-\cos(\beta-\omega)\frac{\partial\Delta U}{\partial a} + \frac{2}{a}\sin(\beta-\omega)\frac{\partial\Delta U}{\partial\beta}\right] + O(\epsilon),$$

$$\delta\dot{\Phi} = -\frac{3}{2}\frac{\delta a}{a}n_0 - \frac{2}{n_0 a}\frac{\partial\Delta U}{\partial a} - \cos i\,\delta\dot{\Omega} + O(\epsilon).$$

In the above formulas, quantities such as $\partial\Delta U/\partial r_s$ have been approximated by:

$$\frac{\partial\Delta U}{\partial r_s} = \frac{\partial\Delta U}{\partial a} + O(\epsilon),$$

and $\delta\Phi = \delta M + \delta\omega$ has been used to avoid terms $O(1/\epsilon)$.

Integration of the above differential equations of motion with the appropriate boundary conditions will provide one description of the effect of errors in the geopotential on the satellite trajectory. We shall transform these changes in the osculating elements into the H, L, Z coordinate system in order to discuss these effects on the time dependence of the tracking residuals. However, I first want to give two examples of solutions to these equations to provide a better intuitive feel for the kinds of effects that arise from errors in the geopotential.

Let us first consider the effect of changing the boundary conditions. The general solution of these equations of motion can always be considered as being composed of a particular solution of the inhomogeneous equations (including terms explicitly dependent upon ΔU) and a general solution of the homogeneous part of the equations ($\Delta U \equiv 0$). Considering the solution of the homogeneous equations first, we set $\Delta U \equiv 0$ and obtain the following constants.

$$
\begin{aligned}
\delta a_0 &= \text{change in semi-major axis,} \\
\delta \epsilon_0 &= \text{change in eccentricity,} \\
\delta i_0 &= \text{change in inclination,} \\
\delta \Omega_0 &= \text{change in right ascension of ascending node,} \\
\epsilon_0 \delta \omega_0 &= \text{change in argument of perigee,} \\
\delta M_0 &= \text{change in mean anomaly,}
\end{aligned}
$$

with

$$\delta \Phi_0(t) = \delta M_0 - 3/2 \frac{\delta a_0}{a_0} n_0(t - t_0) + \text{higher orders,}$$

$t_0 =$ some epoch, conveniently chosen to be the epoch of the original orbit.

It can be seen that when t_0 is chosen as the time of the initial orbit epoch the constants δa_0, $\delta \epsilon_0$, δi_0, $\delta \Omega_0$, $\delta \omega_0$, and δM_0 can be interpreted as changes to the orbit parameters at the orbit epoch.

The above constants, which arise mathematically from a solution of the homogeneous perturbed equations of motion, are not trivial additions to the perturbed satellite motion from a physical point of view. When an orbit is determined from tracking data using erroneous station locations and satellite forces, the resulting orbit parameters will obviously be in error even if there is zero error in the tracking data itself. Consequently, when considering the effect of geodetic errors on the satellite motion, account must be taken of the errors in the orbit parameters themselves. The resulting time dependence of the tracking data residuals due directly to errors in the orbit parameters will be derived using the above solution to the homogeneous equations—keeping in mind that they are not arbitrary but a rather complicated implicit functional of the geodetic errors and amount and distribution of tracking data along the satellite trajectory.

I shall choose one other (relatively simple) example to aid in understanding intuitively the effect of satellite force errors on the satellite motion and eventually on the tracking data residuals. This example allows only an error in the value of J_3, the so-called pear-shaped term. A particular solution of the above equations of motion for $\Delta J_3 \neq 0$ is to first order. ($\Delta J_3/J_2$ is always considered of first order, ΔJ_3 of second order.)

$$\delta a = \text{second order},$$

$$\delta\epsilon = \frac{\Delta J_3}{2J_2}\frac{\sin i}{a}\sin\omega + O\left(\epsilon\frac{\Delta J_3}{J_2}\right),$$

$$\delta i = O\left(\epsilon\frac{\Delta J_3}{J_2}\right),$$

$$\epsilon\delta\omega = \frac{\Delta J_3}{2J_2}\frac{\sin i}{a}\cos\omega + O\left(\epsilon^2\frac{\Delta J_3}{J_2}\right),$$

$$\delta\Omega = O\left(\epsilon\frac{\Delta J_3}{J_2}\right),$$

$$\delta\Phi(t) = O\left(\epsilon\frac{\Delta J_3}{J_2}\right).$$

From these equations it can be seen that an error in J_3 gives rise to periodic errors in the eccentricity and argument of perigee, the period being the time of one revolution of perigee.

The example of an erroneous J_3 is directly generalizable to the form of the errors in the satellite motion arising from errors in the odd zonal harmonics ($\Delta J_n \neq 0$, n odd). Without further remarks, the principal effect of geopotential errors are (see [4]):

1. Error in even zonal coefficients ($\Delta J_n \neq 0$, n even):

a. Secular errors in ω, Ω, Φ (increase approximately linear with time),

b. Long period errors in ω, Φ,

c. Short (orbital) period errors in all osculating elements.

2. Error in odd zonal coefficients ($\Delta J_n \neq 0$, n odd):
Long period errors in ϵ and ω.

3. Errors in the nonzonal coefficients ($\Delta C_n^m, \Delta S_n^m \neq 0$)
Periodic errors of angular frequency,

$$\omega_m = m(\omega_E - \dot{\Omega}), \quad 1 < m \leqq n.$$

As a first step in obtaining the errors in the satellite motion in the H, L, Z system, I shall transform the errors to a moving coordinate system which will also display more clearly the nature of the errors. This coordinate system is shown in Figure 9A, where:

$\delta r_s(t)$ = error in satellite radius (satellite altitude error),

$\delta l_s(t)$ = error in orbital plane normal to r_s (satellite along-track error),

$\delta Z_s(t)$ = error in direction of satellite angular momentum vector (satellite cross-track error).

From Figures 9A and 9B it can be seen that

$$\delta l_s = r_s[\cos\phi_s \cos I \; \delta\Omega + \delta\beta] + \text{second order},$$

$$\delta Z_s = -r_s[\cos\phi_s \sin I \, \delta\Omega - \sin\beta\delta i] + \text{second order}.$$

Note from these figures that the local inclination, I, obeys the relations:

$$\cos\phi_s \cos I = \cos i,$$

$$\cos\phi_s \sin I = \sin i \sin\beta;$$

$$\delta l_s = r_s[\delta\beta + \cos i \, \delta\Omega] + \text{second order},$$

$$\delta Z_s = r_s[\sin\beta\delta i - \cos\beta \sin i \, \delta\Omega] + \text{second order}.$$

Using now the relations between the various kepler elements:

$$\begin{aligned} \delta\beta(t) &= \delta f(t) + \delta\omega(t) \\ &= \delta\Phi(t) + 2[\delta\epsilon \sin(\beta - \omega) - (\epsilon\delta\omega)\cos(\beta - \omega)] + O(\epsilon), \end{aligned}$$

and

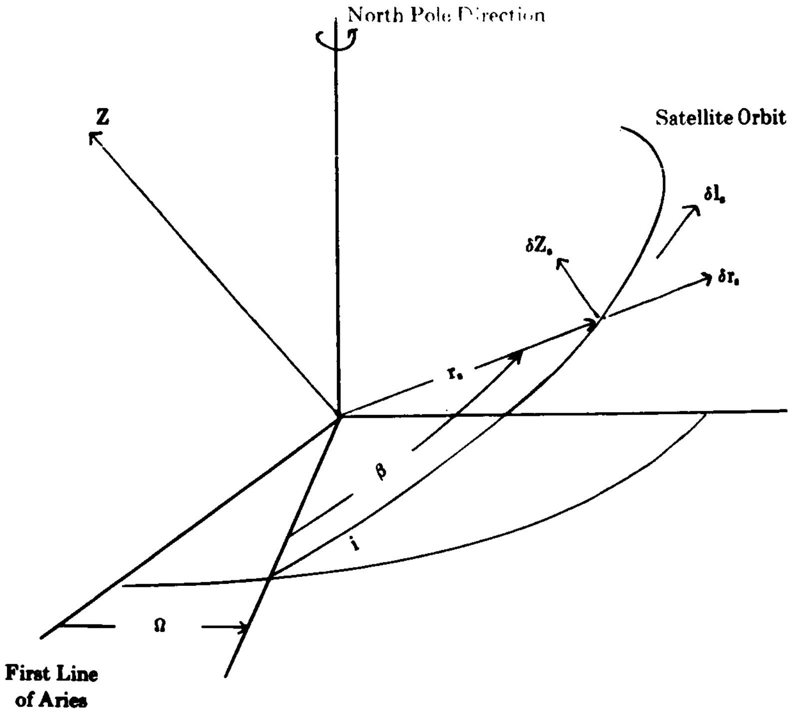


FIGURE 9A.

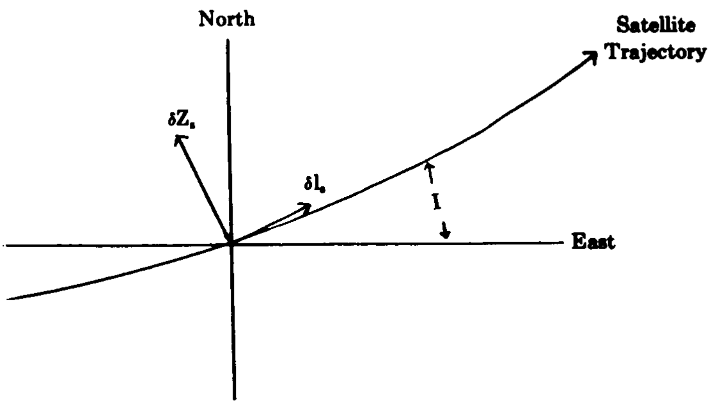


FIGURE 9B.

$$\delta\left[\frac{a(1-\epsilon^2)}{1+\epsilon\cos(\beta-\omega)}\right]$$
$$= \delta a - a\left[\delta\epsilon\cos(\beta-\omega) - (\epsilon\delta\omega)\sin(\beta-\omega)\right] + O(\epsilon),$$

we have:

$$\delta r_s(t) = \delta a - a\left[\delta\epsilon\cos(\beta-\omega) + (\epsilon\delta\omega)\sin(\beta-\omega)\right] + O(\epsilon),$$
$$\delta l_s(t) = a\left[\delta\Phi + 2\delta\epsilon\sin(\beta-\omega) - 2(\epsilon\delta\omega)\cos(\beta-\omega)\right] + O(\epsilon),$$
$$\delta Z_s(t) = a\left[\delta i\sin\beta - \delta\Omega\sin i\cos\beta\right] + O(\epsilon).$$

IV. **Errors in satellite motion.** We apply these results to two examples.

1. *Errors in the orbit parameters at epoch.* The constant orbit parameter errors can be directly substituted into the expressions for the satellite altitude, along-track and cross-track errors. We then have:

$$\delta r_s(t) = \delta a_0 - a\left[\delta\epsilon_0\cos(\beta-\omega) + (\epsilon_0\delta\omega)\sin(\beta-\omega)\right] + O(\epsilon),$$
$$\delta l_s(t) = a\Big[\delta M_0 + \delta\omega_0 + \cos i\,\delta\Omega_0 - \frac{3}{2}\frac{\delta a_0}{a} n_0(t-t_0)$$
$$+ 2(\delta\epsilon_0\sin(\beta-\omega) - (\epsilon_0\delta\omega_0)\cos(\beta-\omega))\Big] + O(\epsilon),$$
$$\delta Z_s(t) = a\left[\delta i_0\sin\beta - \sin i\,\delta\Omega_0\cos\beta\right] + O(\epsilon).$$

Recognizing that the argument of perigee, ω, is a slowly varying function of time, the above expressions can be rewritten in a more transparent form by letting

$$\delta A_0(t) = -a\left[\delta\epsilon_0\cos\omega(t) - (\epsilon_0\delta\omega_0)\sin\omega(t)\right],$$
$$\delta B_0(t) = -a\left[\delta\epsilon_0\sin\omega(t) + (\epsilon_0\delta\omega_0)\cos\omega(t)\right],$$
$$\delta l_0 = a\left[\delta M_0 + \delta\omega_0 + \delta\Omega_0\cos i\right],$$
$$\delta l_1 = -\frac{3}{2}\delta a_0,$$
$$\delta l_2 = 2\delta B_0(t),$$
$$\delta l_3 = -2\delta A_0(t),$$
$$\delta Z_1 = -a\sin i\,\delta\Omega_0,$$
$$\delta Z_2 = a\,\delta i_0,$$

so that when errors exist only in the orbit parameters,

$$\delta r_s(t) = -\frac{2}{3}\delta l_1 - \frac{\delta l_3}{2}\cos\beta + \frac{\delta l_2}{2}\sin\beta + O(\epsilon),$$

$$\delta l_s(t) = \delta l_0 + \delta l_1(\beta - \beta_0) + \delta l_2 \cos\beta + \delta l_3 \sin\beta + O(\epsilon),$$

$$\delta Z_s(t) = \delta Z_1 \cos\beta + \delta Z_2 \sin\beta + O(\epsilon).$$

The above equations display the principal time dependence of the errors in the satellite motion when errors exist only in the orbit parameters at the orbit epoch. However, do not overlook the slow time dependence occurring through the motion of perigee and therefore δl_2 and δl_3, and the small time dependence occurring due to the use of the osculating elements for a and i. As is to be expected, if there is an error in the period of the satellite motion, the satellite along-track error grows linearly with time and the satellite altitude exhibits an altitude error δa_0 which will not average to zero. δl_0 is the position error in the along-track direction at the epoch. It can be seen that the remaining terms in the error equations are oscillatory at the orbital period.

2. *Error in the third zonal coefficient*, J_3. Substituting the errors for the kepler elements corresponding to ΔJ_3 into the expressions for δr_s, δl_s, δZ_s we have:

$$\delta r_s(t) = \frac{1}{2}\frac{\Delta J_3}{J_2}\sin i \sin\beta + O\left(\epsilon\frac{\Delta J_3}{J_2}\right) + O(\Delta J_3),$$

$$\delta l_s(t) = -\frac{\Delta J_3}{J_2}\sin i \cos\beta + O\left(\epsilon\frac{\Delta J_3}{J_2}\right) + O(\Delta J_3),$$

$$\delta Z_s(t) = O\left(\epsilon\frac{\Delta J_3}{J_2}\right) + O(\Delta J_3).$$

A very interesting point can be seen from these equations. We had previously noted that the errors in the kepler elements due to an error in J_3 were long period to first order—that is order $\Delta J_3/J_2$. However, once transformed to a coordinate system that is closer to giving a direct measure of the satellite position error, the effects (to this same order) become short period. Because the dominant effect is now short period the resulting satellite errors exhibit a similar time dependence to the errors caused by orbit parameter errors along Example 1. This means that over short intervals of time, say a few days it is possible to "soak up" most of the error due to this geopotential error by appropriate adjustment of the satellite orbit parameters.

To exhibit this effect clearly, we combine the two previous examples assuming that no errors exist except in the value for J_3 and allow an error in the orbit parameters which will minimize the effect of J_3 being in error. From the previous results, we have:

$$\begin{aligned}
\delta A(t) &= \delta A_0(t) \\
&= -a\left[\delta\epsilon_0 \cos\omega(t) - (\epsilon_0\,\delta\omega_0)\sin\omega(t)\right], \\
\delta B(t) &= \delta B_0(t) - a\frac{\Delta J_3}{2J_2}\frac{\sin i}{a} \\
&= -a\left[\frac{\Delta J_3}{2J_2}\frac{\sin i}{a} + \delta\epsilon_0 \sin\omega(t) + (\epsilon_0\,\delta\omega_0)\cos\omega(t)\right], \\
\delta l_0 &= a\left[\delta M_0 + \delta\omega_0 + \cos i\,\delta\Omega_0\right], \\
\delta l_1 &= -(3/2)\,\delta a_0, \\
\delta L_2(t) &= 2\,\delta B(t) = \delta l_2(t) - a\frac{\Delta J_3}{J_2}\frac{\sin i}{a}, \\
\delta L_3(t) &= -2\,\delta A(t) = \delta l_3(t), \\
\delta Z_1 &= -a\sin i\,\delta\Omega_0, \\
\delta Z_2 &= a\,\delta i_0,
\end{aligned}$$

and:

$$\begin{aligned}
\delta r_s(t) &= -(2/3)\,l_1 - \frac{\delta l_3(t)}{2}\cos\beta + \frac{\delta L_2(t)}{2}\sin\beta + \text{higher orders}, \\
\delta l_s(t) &= \delta l_0 + \delta l_1(\beta - \beta_0) + \delta L_2(t)\cos\beta + \delta l_3(t)\sin\beta + \text{higher orders}, \\
\delta Z_s(t) &= \delta Z_1\cos\beta + \delta Z_2\sin\beta + \text{higher orders}.
\end{aligned}$$

These equations have intentionally been written to look formally like those which represented only orbit parameter errors. The only difference that occurs when ΔJ_3 is not zero to the order considered here is:

$$\delta L_2(t) - \delta l_2(t) = -\frac{\Delta J_3}{J_2}\sin i + \text{higher orders}.$$

Since $\delta l_2(t)$, and therefore $\delta L_2(t)$, are varying with time very slowly, it becomes difficult to separate an orbit parameter error from this type of geodetic error. This tendency for orbit parameter adjustment

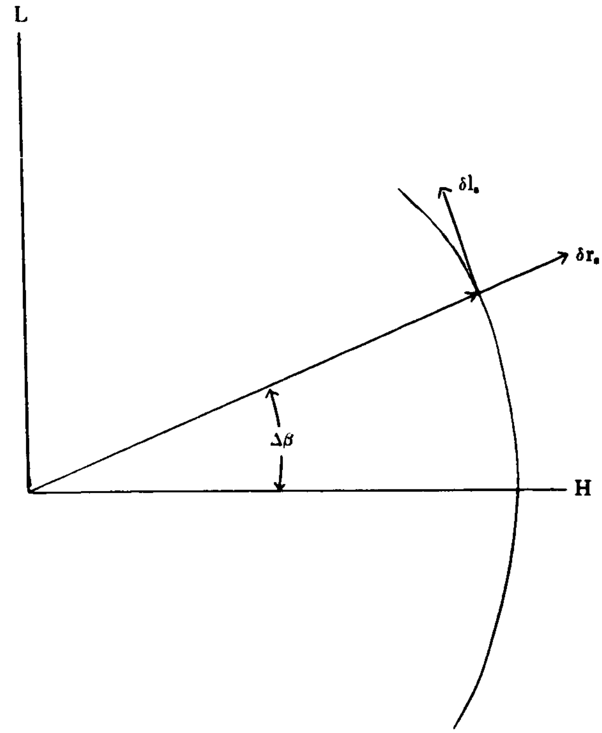


FIGURE 10.

to hide geodetic errors, exhibited in this example, is a general result for many types of geodetic errors, particularly errors in the zonal harmonic coefficients of the geopotential. It is for this reason that long satellite trajectories are usually required to determine accurately the zonal harmonic coefficients in the presence of other errors such as station location errors and experimental data errors (see [**8**], [**9**] and [**10**]).

We have considered the general character of the errors in the satellite motion over long spans of time through two examples. I will next consider in more detail the effect of these errors on the tracking data for a specific pass of the satellite above a specific station's horizon. To do this we transform the satellite motion errors to the H, L, Z coordinate system. For some given pass, the H-axis, passes through the satellite position at closest approach and is fixed in inertial space. Figure 10 gives the geometry of the errors in the

δr_s, δl_s moving coordinate system relative to the fixed coordinate system of H and L. From Figure 10 it can be seen that:

$$\begin{aligned}
\delta H_s &= \delta r_s \cos\Delta\beta - \delta l_s \sin\Delta\beta,\\
\delta L_s &= \delta r_s \sin\Delta\beta + \delta l_s \cos\Delta\beta,\\
\delta Z_s &\text{ unchanged},\\
\Delta\beta &= \beta(t) - \beta(t_c),\\
t_c &= \text{time of closest approach.}
\end{aligned}$$

Letting

$$C(\Delta\beta) = 1 - \cos\Delta\beta,$$

it can be seen that during the pass $|C(t)| \ll 1$ for near-earth satellites. Rewriting the above equations:

$$\begin{aligned}
\delta H_s &= \delta r_s - \delta l_s \sin\Delta\beta - \delta r_s C(\Delta\beta),\\
\delta L_s &= \delta l_s + \delta r_s \sin\Delta\beta - \delta l_s C(\Delta\beta),\\
\delta Z_s &\text{ unchanged.}
\end{aligned}$$

The procedure from here on involves expanding $\delta r_s(t)$, $\delta l_s(t)$, and $\delta Z_s(t)$ in the functions $\sin\Delta\beta$, $C(\Delta\beta) = 1 - \cos\Delta\beta$, etc. and then by substitution into the above equations for δH_s and δL_s, express the time dependence of the satellite errors in the H, L, Z coordinate system in functions of the form $\sin\Delta\beta$, $C(\Delta\beta)\sin\Delta\beta\, C(\Delta\beta)$, etc. This procedure can be done in general but is not too useful to the developement of a physical understanding of the effects of the errors. Consequently, I shall make this transformation using the two examples discussed previously; one may consult [5] for consideration of the general case.

I use a subscript c to denote a time dependent quantity evaluated at $t = t_c$. The result then becomes:

$$\begin{aligned}
\delta H_s(\beta_c, \Delta\beta(t)) = \delta r_c &- [\delta l_c + \delta A_c \sin\beta_c - \delta B_c \cos\beta_c]\sin\Delta\beta\\
&- [\delta r_c - 3\delta A_c \cos\beta_c - 3\delta B_c \sin\beta_c]\, C(\Delta\beta)\\
&- [\delta A_c \sin\beta_c - \delta B_c \cos\beta_c]\sin\Delta\beta\, C(\Delta\beta)\\
&+ O(C^2) + \text{higher orders,}
\end{aligned}$$

$$\delta L_s(\beta_c, \Delta\beta(t)) = \delta l_c + [\delta r_c - 2\delta A_c \cos\beta_c - 2\delta B_c \sin\beta_c]\sin\Delta\beta$$
$$- \delta l_c C(\Delta\beta) + [\delta A_c \cos\beta_c + \delta B_c \sin\beta_c]\sin\Delta\beta\, C(\Delta\beta)$$
$$+ O(C^2) + \text{higher orders},$$
$$\delta Z_s(\beta_c, \Delta\beta(t)) = \delta Z_c + [\delta\Omega_0 \sin i \sin\beta_c + \delta i_0 \cos\beta_c]\sin\Delta\beta$$
$$- \delta Z_c C(\Delta\beta) + \text{higher orders},$$

where:

$$\delta r_c = \delta a_0 + \delta A_c \cos\beta_c + \delta B_c \sin\beta_c,$$
$$\delta l_c = a(t_c)\,[\delta M_0 + \delta\omega_0 + \delta\Omega_0 \cos i_c] - (2/3)\,\delta a_0(\beta_c - \beta_0)$$
$$+ 2\delta B_c \cos\beta_c - 2\delta A_c \sin\beta_c,$$
$$\delta Z_c = -\,\delta\Omega_0 \sin i(t_c) \cos\beta_c + \delta i_0 \sin\beta_c,$$
$$\delta A_c = -\,a(t_c)\,[\delta\epsilon_0 \cos\omega(t_c) - (\epsilon_0\delta\omega_0)\sin\omega(t_c)],$$
$$\delta B_c = -\,a(t_c)\left[\frac{\Delta J_3}{2J_2}\,\frac{\sin i(t_c)}{a(t_c)} + \delta\epsilon_0 \sin\omega(t_c) + (\epsilon_0\delta\omega_0)\cos\omega(t_c)\right].$$

In developing these formulas we have used the relations:

$$\cos(\beta_c + \Delta\beta) = \cos\beta_c - \sin\beta_c \sin\Delta\beta - \cos\beta_c\, C(\Delta\beta),$$
$$\sin(\beta_c + \Delta\beta) = \sin\beta_c + \cos\beta_c \sin\Delta\beta - \sin\beta_c\, C(\Delta\beta),$$
$$\sin^2\Delta\beta = 1 - \cos^2\Delta\beta = 2C(\Delta\beta) + O(C^2),$$

and where $-\,(2/3)\,\delta a_0 \Delta\beta$ has been considered negligible by virtue of our assumption that the orbit has been "tracked" to reasonable accuracy so that $\delta a_0 \beta(t_c)$ is not large.

V. **Data residuals.** I shall use the previous results to consider the effect of station and geopotential errors on tracking data residuals. By data residuals I mean:

Data residual = Experimental data point − theory at time of data point, where, as stated in §II, we neglect experimental noise and instrumentation contributions to the residuals.

Clearly, the error in the slant range vector is:

$$\delta\boldsymbol{\rho} = \delta\mathbf{r}_s(t) - \delta\mathbf{r}_R(t),$$

which, in the H, L, Z coordinate system is:

$$\delta\boldsymbol{\rho} = \begin{pmatrix} \delta\rho_H \\ \delta\rho_L \\ \delta\rho_C \end{pmatrix} = \begin{pmatrix} H_s(\beta_C, \Delta\beta) - E_{H_T} \\ L_s(\beta_C, \Delta\beta) - E_{L_T} \\ Z_s(\beta_C, \Delta\beta) - E_{Z_T} \end{pmatrix} + \text{second order;}$$

we discussed the station error E_{H_T}, E_{L_T}, E_{Z_T} in §III and discussed the nature of H_s, L_s, Z_s in §IV.

Corresponding to this error, the error in the scalar slant range, i.e., the slant range data residuals are given by:

$$\delta\rho \equiv |\boldsymbol{\rho} + \delta\boldsymbol{\rho}| - \boldsymbol{\rho} = \frac{1}{\rho}\,\boldsymbol{\rho}\cdot\delta\boldsymbol{\rho} + \text{second order}$$

$$= \hat{\rho}\cdot\delta\boldsymbol{\rho} + \text{second order,}$$

where, from §II:

$$\boldsymbol{\rho}(t) = r_s \begin{pmatrix} \rho_s\cos\theta - C(t) \\ \sin\Delta\beta(t) \\ -\rho_s\sin\theta \end{pmatrix} + \text{first order,}$$

$$\rho(t) = r_s[\rho_s^2 + 2\alpha_s C(t)]^{1/2} + \text{first order,}$$

$$\alpha_s = 1 - \rho_s\cos\theta,$$

$$C(t) = 1 - \cos\Delta\beta(t).$$

The errors in the slant range unit vector, i.e., the angular data residuals, are given by:

$$\delta\hat{\rho} = \delta\left(\frac{\boldsymbol{\rho}}{\rho}\right) = \frac{\delta\boldsymbol{\rho}}{\rho} - \hat{\rho}\frac{\delta\rho}{\rho}.$$

That is, the angular error scaled to distance error is

$$\rho\delta\hat{\rho} = \delta\boldsymbol{\rho} - \hat{\rho}(\hat{\rho}\cdot\delta\boldsymbol{\rho}).$$

Finally, the error in the scalar slant range rates, i.e., doppler data residuals, are given by:

$$\frac{d}{dt}\delta\rho = \frac{d}{dt}\left[\frac{1}{\rho}(\rho\delta\rho)\right]$$

$$= \frac{1}{\rho^3}\left[-\frac{1}{2}(\rho\delta\rho)\frac{d\rho^2}{dt} + \rho^2\frac{d}{dt}(\rho\delta\rho)\right].$$

Each of the above types of residuals can be computed by substituting in the appropriate expressions for the error in the vector slant range.

Using now the two examples in §IV as a guide, we can write

$$\delta\boldsymbol{\rho}(t) = \delta\boldsymbol{\rho}_C + \delta\boldsymbol{\rho}_1 \sin\Delta\beta(t) + \delta\boldsymbol{\rho}_2\, C(t) + \text{higher orders}$$

where

$$\delta\boldsymbol{\rho}_C = \delta\mathbf{r}_s(t_c) - \delta\mathbf{r}_R(t_c).$$

(The proof of this form for general geopotential errors is lengthy and is given in [**5**].) Substituting this form into the above expressions for slant range residuals:

$$\begin{aligned}\frac{\rho(t)}{r_s}\,\delta\rho(t) &= \rho_s[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] \\ &\quad + [\delta\rho_{L_C} + \rho_s(\cos\theta\,\delta\rho_{H_1} - \sin\theta\,\delta\rho_{Z_1})]\sin\Delta\beta(t) \\ &\quad + [2\,\delta\rho_{L_1} - \delta\rho_{H_C} + \rho_s(\cos\theta\,\delta\rho_{H_2} - \sin\theta\,\delta\rho_{Z_2})]\,C(t) \\ &\quad + O[\sin\Delta\beta\, C(t)] + \text{higher orders.}\end{aligned}$$

For satellites whose altitude is of the order of 1000 km,

$$\rho_s \leqq .25, \qquad C(t) \leqq .15.$$

Therefore, to a fair approximation:

A. Scalar slant range residuals:

$$\begin{aligned}\frac{\rho(t)}{r_s}\,\delta\rho(t) &= \rho_s[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] + [\delta\rho_{L_C} + O(\rho_s)]\sin\Delta\beta(t) \\ &\quad + [2\,\delta\rho_{L_1} - \delta\rho_{H_C} + O(\rho_s)]\,C(\Delta\beta) + O[\sin\Delta\beta\cdot C(t)]\end{aligned}$$

Similarly, by substitution into the expressions for the other types of data:

B. Angular residuals:

$$\left(\frac{\rho(t)}{r_s}\right)^3 r_s\,\delta\hat{\rho} = \delta\hat{\rho}_C + \delta\hat{\rho}_1 \sin\Delta\beta(t) + \delta\hat{\rho}_2\, C(t) + O[\sin\Delta\beta\cdot C(t)],$$

$$\delta\hat{\rho}_C = \rho_s^2 \begin{pmatrix} \delta\rho_{H_C} - \cos\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] \\ \delta\rho_{L_C} \\ \delta\rho_{Z_C} + \sin\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] \end{pmatrix},$$

$$\delta\hat{\rho}_1 = \rho_s \begin{pmatrix} -\cos\theta\,\delta\rho_{L_C} + O(\rho_s) \\ -[\cos\theta\,\delta\rho_{H_C} - \sin\theta\delta\rho_{Z_C}] + O(\rho_s) \\ \sin\theta\,\delta\rho_{L_C} + O(\rho_s) \end{pmatrix},$$

$$\delta\hat{\rho}_2 = \begin{pmatrix} 2\,\delta\rho_{H_C} + O(\rho_s) \\ O(\rho_s) \\ 2\,\delta\rho_{Z_C} + O(\rho_s) \end{pmatrix}.$$

C. Range rate residuals:

$$-\frac{1}{n_0}\left(\frac{\rho(t)}{r_s}\right)^3 \frac{d}{dt}\delta\rho(t) = \rho_s^2[\delta\rho_{L_C} + O(\rho_s)]$$

$$- \rho_s[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C} + O(\rho_s)]\sin\Delta\beta(t) + O(\rho_s^2 C(t)),$$

where $n_0 = \dot{\beta}(t_c)$.

These results are summarized in Table 1 for purposes of comparison, where they have been scaled to like functions of time. It should be noted that in the above expressions and Table 1 the angular residuals have been written as a three-dimensional vector in the H, L, Z coordinate system. However, in reality, the residuals are only a two-dimensional vector since

$$\boldsymbol{\rho}(t)\cdot\delta\hat{\rho}(t) = 0.$$

This table summarizes the largest contributions to the expressions for data residuals when experimental errors are neglected. Clearly, the errors $\delta\rho_{H_C}$, $\delta\rho_{L_C}$, and $\delta\rho_{Z_C}$, can be expressed in terms of the station location errors, orbit parameters errors, and geopotential errors following the procedure outlined in §III and §IV. A rough sketch of the time dependence of the various terms are given in Figures 11 so that for any given geodetic error its effect on the time dependence of the data residuals can be found.

Several interesting conclusions can be drawn from Table 1. First, it can be seen that for comparable signal-to-noise ratios, range and range rate data yield roughly the same information. This, at first glance, is surprising since one would suspect that range data, being the time derivative of the range, would loose some information (roughly analogous to the constant of integration if one attempted to integrate the range rate data to obtain range). Clearly, this is not true except to note that it has been assumed that the transmitter frequency of the satellite which generates the doppler data is known exactly so that the incoming signal can "zero-beat" out the satellite transmitter frequency. (To the extent that this is not true, a term which is constant with

TABLE 1

Summary of Range, Range Rate, and Angle Data Residuals to $O(\rho_s C(t))$

TYPE OF DATA / TIME DEPENDENCE	RANGE $\delta\rho(t)$	RANGE RATE $-\dfrac{1}{n_0}\dfrac{d}{dt}\delta\rho(t)$	ANGLE $r_s\delta\hat{\rho}(t) = r_s\begin{pmatrix}\delta\hat{\rho}_H(t)\\ \delta\hat{\rho}_L(t)\\ \delta\hat{\rho}_Z(t)\end{pmatrix}$
$\left(\dfrac{r_s}{\rho(t)}\right)^3 \rho_s^3$ (symmetric in time)	$\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}$	$\dfrac{\delta\rho_{L_C}}{\rho_s} + O(1)$	$\dfrac{1}{\rho_s}\begin{pmatrix}\rho_{H_C} - \cos\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}]\\ \delta\rho_{L_C}\\ \delta\rho_{Z_C} + \sin\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}]\end{pmatrix}$
$\left(\dfrac{r_s}{\rho(t)}\right)^3 \rho_s^2 \sin\Delta\beta(t)$ (anti-symmetric in time)	$\delta\rho_{L_C} + O(\rho_s)$	$-\dfrac{1}{\rho_s}[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] + O(1)$	$\dfrac{1}{\rho_s}\begin{pmatrix}-\cos\theta\,\delta\rho_{L_C} + O(\rho_s)\\ -[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] + O(\rho_s)\\ \sin\theta\delta\rho_{L_C} + O(\rho_s)\end{pmatrix}$
$\left(\dfrac{r_s}{\rho(t)}\right)^3 \rho_s C(t)$ (symmetric in time)	$2[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] + O(\rho_s)$	$O(\rho_s)$	$\dfrac{2}{\rho_s}\begin{pmatrix}\delta\rho_{H_C} + O(\rho_s)\\ O(\rho_s)\\ \delta\rho_{Z_C} + O(\rho_s)\end{pmatrix}$

$$\left(\frac{r_s}{\rho(t)}\right)^2 = \frac{1}{\rho_s^2 + 2\alpha_s C(t)}, \quad C(t) = 1 - \cos\Delta\beta(t), \quad \alpha_s = 1 - \rho_s\cos\theta$$

n_0 = satellite mean motion.

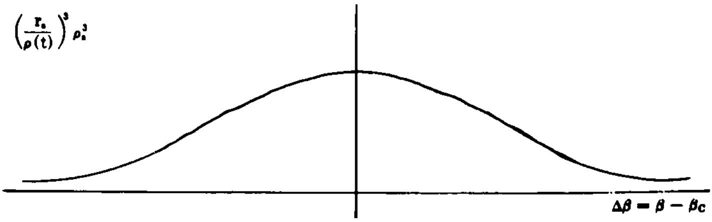


FIGURE 11A. First Symmetric Time Dependence

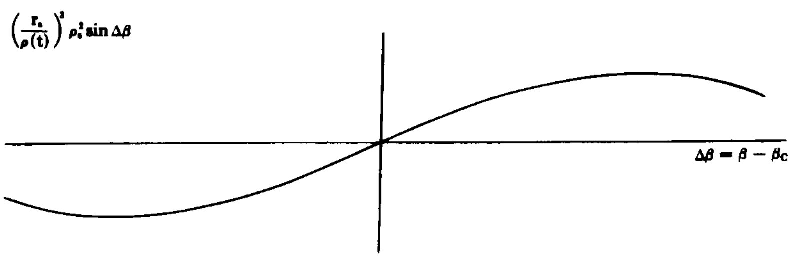


FIGURE 11B. First Anti-Symmetric Time Dependence

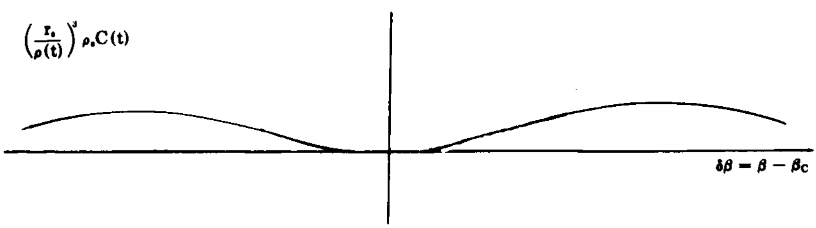


FIGURE 11C. Second Symmetric Time Dependence time should be added which can easily be separated out from the time dependence noted in the table.) The second conclusion is that when range and/or range rate is measured, the following measurements of the relative error between satellite and station can be made from a single pass.

$$\delta\rho_{L_C},$$

$$\delta\rho_{H_C}\cos\theta - \sin\theta\,\delta\rho_{Z_C}.$$

Considering now the parameters that can be determined with angular residuals from a single pass, we have:

$$\delta\rho_{L_C},$$

$$\delta\rho_{H_C} - \cos\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] = \sin\theta\,[\sin\theta\,\delta\rho_{H_C} + \cos\theta\,\delta\rho_{Z_C}],$$

$$\delta\rho_{Z_C} + \sin\theta\,[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}] = \cos\theta\,[\sin\theta\,\delta\rho_{H_C} + \cos\theta\,\delta\rho_{Z_C}],$$

and

$$[\cos\theta\,\delta\rho_{H_C} - \sin\theta\,\delta\rho_{Z_C}],$$

so that more information is available in optical data than range or range rate data for equivalent signal to noise ratios and data rates.

Touching, for the moment on the relative merits of different types of data, the following should be noted. Range and range rate systems are usually radio tracking systems and consequently have all weather capabilities and are designed to yield very high data rates. I believe most people agree that no radio tracking system significantly exceeds the data point accuracy of a good optical (angle) tracking instrument. However, optical tracking systems are not all weather and as a maximum can only take data at night. Including the tedious job of reducing the optical photographs, we can see that range and range rate systems yield high data rates in all weather conditions but per satellite pass may yield less information than a high quality set of optical data. Consequently, it would appear that a high quality radio range or range rate system and a high quality optical tracking system are complementary to each other. For example, optical data provides an excellent means for monitoring the accuracy of radio tracking systems. This fact has been recognized in the ANNA geodetic satellite (see [**11**] and [**12**]) in which was flown an active flashing light to aid in obtaining increased optical data rates together with instrumentation for two radio tracking systems.

So far we have been concerned only with the data residuals for a single satellite pass. Clearly, when considering many such sets of data residuals, one has the capability of measuring the time dependence of the orbit error over long time spans to gain information on geopotential terms which produce secular and long-period effects. When using such data to make a significant improvement in current values for station position parameters and coefficients of the expansion of the geopotential, a sufficiently large number of parameters must be inferred from the data that it is essential to have very large amounts of tracking data. In fact experience has shown that one really needs many satellites at differing inclinations, to determine accurately the nonzonal coefficients of the geopotential.

The techniques and associated computer programs which are used to perform such determinations of geodetic parameters are outside the scope of this set of notes. It is sufficient to note that one must have available high quality tracking data from many satellites and extensive computer programs before such an attempt is capable of improving on current accuracies.

References

1. R. R. Newton, *Astronomy for the non-astronomer,* IRE Trans. on Space Electronics and Telemetry, Vol. Set-6, No. 1, March, 1960.

2. B. G. Bomford, *Geodesy,* Clarendon Press, Oxford, 1952.

3. W. A. Heiskanen and F. A. Vening Meinesz, *The earth and its gravity field,* McGraw-Hill, New York, 1958.

4. W. M. Kaula, *Celestial geodesy,* National Aeronautics and Space Adminstration, Report TN D-1155, March 1962.

5. W. H. Guier, *Studies on Doppler residuals. I: Dependence on satellite orbit error and station position error,* Applied Physics Laboratory, The Johns Hopkins University, Report TG-503, June 1963.

6. D. Brouwer and G. Clemence, *Methods on celestial mechanics,* Academic Press, Princeton, N. J., 1961.

7. H. C. Plummer, *An introductory treatise on dynamical astronomy,* Dover, New York, (1960 edition).

8. J. A. O'Keefe, A. Eckels and R. K. Squires, *The gravitational field of the earth,* Astronom. J., **64** (1959), 245-253.

9. R. R. Newton , H. S. Hopfield and R. C. Kline, *Odd harmonics of the earth's gravitational field,* Nature, **190** (1961), 617.

10. Y. Kozai, *Numerical results from orbits*, SAO Special Report No. 101, July 31, 1962.

11. M. M. Macomber, *Project ANNA*, Proceedings of the First International Symposium on the Use of Artificial Satellites for Geodesy, Washington, D. C., 1962.

12. W. H. Guier, *Navigation using artificial satellites - The transit system*, Proceedings of the First International Symposium on the Use of Artificial Satellites for Geodesy, Washington, D. C., 1962.

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Elements of Calculus of Variations and Optimum Control Theory

I. **Introduction.** The calculus of variations and optimum control theory, along with certain associated computational methods, will be presented in parallel format to show the basic similarities in spite of what may superficially seem to be glaring differences. The two theories together form one theory, with separate vocabularies arising from usage current to its era of development.

Consider the following problem in classical calculus of variations, namely a bead on a frictionless wire falling under the influence of gravity, commonly called the brachistochrone problem (see Figure 1). Find the path of least time between points 1 and 2 for a bead of mass m sliding along the wire under the influence of gravity alone. The time required for descent is

$$T = \int_1^2 \frac{ds}{(2gy)^{1/2}} = \int_{x_1}^{x_2} \frac{(1 + (y')^2)^{1/2}}{(2gy)^{1/2}}\, dx, \tag{I.1}$$

where the last integral is written for a curve $y = y(x)$, $x_1 \leqq x \leqq x_2$. Restated: Of all arcs joining the points 1 and 2, find the arc for which $T =$ minimum.

*The manuscript was prepared by Richard H. Lance of Cornell University.

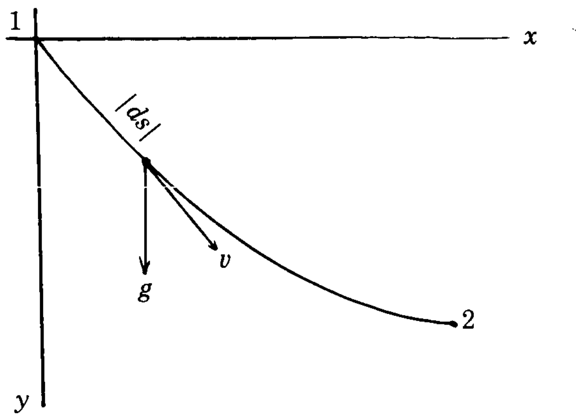

FIGURE 1. The Brachistochrone Problem

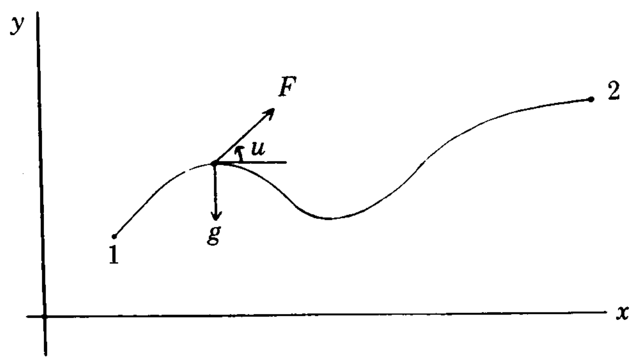

FIGURE 2. Rocket Trajectory of Minimum Time

The modern theory of the calculus of variations has its beginning in the study of the brachistochrone by the Bernoulli brothers. If they had been living today, they probably would have formulated a modern brachistochrone problem as follows: to find the path of least time between two points for a rocket under the influence of gravity and a thrust force with variable direction but with constant magnitude. An additional constraint is imposed: The slope of the optimal path is to have fixed values at 1 and 2. This is a problem in optimum control theory. Mathematically formulated in terms of the variables shown in Figure 2 for a rocket of mass unity, the differential equations are:

$$\ddot{x} = F\cos u, \qquad \ddot{y} = F\sin u - g. \tag{I.2}$$

The end values $x(0)$, $y(0)$, $\dot{x}(0)$, $\dot{y}(0)$, $x(T)$, $y(T)$, $\dot{x}(T)$, $\dot{y}(T)$ are fixed, and the problem is to make T a minimum.

This control problem is, in fact, the classically formulated Problem of Mayer. One speaks of the variables $x, \dot{x}, y, \dot{y}$ as the state variables, and of the function $u(t)$ as the control variable. We wish to choose $u(t)$ so that we go from point 1 to point 2 in the least time.

Let us rewrite the last problem in a more convenient form. Let

$$x^1 = x, \qquad x^2 = y, \qquad x^3 = \dot{x}, \qquad x^4 = \dot{y}. \tag{I.3}$$

Then the problem is to solve the differential equations

$$\dot{x}^1 = x^3, \qquad \dot{x}^2 = x^4, \qquad \dot{x}^3 = F\cos u, \qquad \dot{x}^4 = F\sin u - g, \tag{I.4}$$

with $x^i(0)$ fixed, $x^i(T)$ fixed $(i = 1, 2, 3, 4)$; $T = \min$.

This type of problem can also be written in the form of the general Problem of Bolza: Given

$$\dot{x}^i = f^i(t, x, u) \qquad (i = 1, \cdots, n),$$

a set of differential equations, find among the class of arcs satisfying some end point conditions — say $x^i(0)$ fixed, and perhaps $x^i(T)$ on a line or surface in x^i space — the functions $x^i(t)$ and the control $u(t)$, $0 \leqq t \leqq T$, for which

$$g(T) + \int_0^T f(t, x, u)\, dt = \min. \tag{I.5}$$

It is to be understood that the symbols x and u represent vectors with, in the case of x, n components.

Among the topics we could consider are these:

1. Properties of solutions,
2. Construction of solutions,
3. Existence of solutions,
4. Sufficiency conditions.

In this chapter we will consider only Topic 1, which includes discussions of the necessary conditions which must be satisfied by solutions of the above-formulated problems.

II. **Minimum of a function of n variables.** Before studying the problem of minimizing a functional such as (I.1), let us consider the problem of minimizing a function of n variables. As an example, consider

$$f(x,y) = \min.$$

The first order necessary conditions that must be satisfied are

$$f_x = 0, \qquad f_y = 0, \tag{II.1}$$

where $f_x = \partial f/\partial x$, etc.; and the second order test is

$$f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 \geqq 0. \tag{II.2}$$

Of course, these conditions with the equality excluded in (II.2) when $(h,k) \neq (0,0)$ guarantee only that a point is a local mimimum. Since there is no global test for the absolute mimimum, we usually must find all the points satisfying (II.1) and (II.2) and then test to ascertain the absolute mimimum.

In the more general case of a function of n variables,

$$f(x^1, x^2, x^3, \cdots, x^n),$$

we write the necessary conditions analogous to (II.1) and (II.2) as

$$f_{x^i}(x_0) = 0;\ i = 1, 2, \cdots, n, \tag{II.3$_1$}$$

$$f_{x^ix^j}(x_0)\, h^ih^j \geqq 0 \qquad \text{for all } h, \tag{II.3$_2$}$$

which must be satisfied for all points x_0 which are minima. In (II.3$_2$) the usual summation convention has been adopted. The equations (II.3$_1$) can be interpreted as the condition that $\operatorname{grad} f = 0$. To see this, observe that for each h we have

$$\phi(t) = f(x_0 + th) \geqq f(x_0) = \phi(0)$$

if x_0 is a minimum point and t is near $t = 0$. Thus $\phi'(0) = 0$ and $\phi''(0) \geqq 0$ for such a point. Thus it follows that

$$0 = \phi'(0) = f'(x_0, h) = f_{x^i}(x_0)\, h^i, \tag{II.4}$$

which is identical to (II.3$_1$). The function $f'(x_0, h)$ in (II.4) is called the differential of f at x_0, the first variation of f at x_0 and the directional derivative of f at x_0 in the direction h. Equation (II.3$_2$) is obtained from the latter condition on $\phi(t)$,

$$0 \leqq \phi''(0) = f''(x_0, h) = \frac{d^2f}{dt^2}(x_0 + th)\,\Bigg|_{t=0} = f_{x^ix^j}(x_0)\, h^ih^j.$$

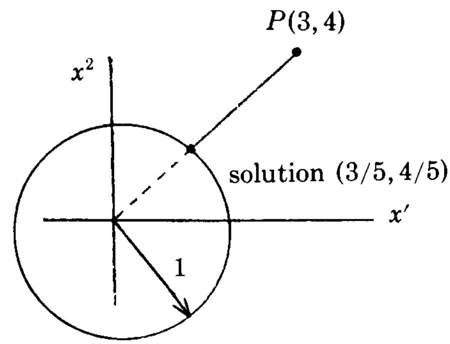

FIGURE 3. Shortest Distance to Circle

As an example, let us find the shortest distance from a point P, say $(3,4)$, to the circle centered at the origin, radius 1. Minimize $[(x^1-3)^2+(x^2-4)^2]^{1/2}$, or simply

$$f_0(x)=\frac{1}{2}\,[(x^1-3)^2+(x^2-4)^2]$$

subject to the constraint

$$f_1(x)=\frac{1}{2}\,[1-(x^1)^2-(x^2)^2]\geqq 0,$$

where the inequality constraint has been imposed for generality. Computing the directional derivative, at $x_0=(3/5,4/5)$, we have

$$f_0'(x_0,h)=-\frac{4}{5}\,(3h^1+4h^2)\equiv k_0,$$

$$f_1'(x_0,h)=-\frac{1}{5}\,(3h^1+4h^2)\equiv k_1;$$

we observe the relation

(II.5) $$k_0-4k_1=0,$$

which is the "multiplier rule" for this very simple case.

We note in this example that inside the circle $f_1>0$, while $f_1<0$ outside the circle. Now (II.5) requires that k_0 and k_1 are both positive, both negative, or both zero. Thus if $f_0(x_0)$ is a minimum, we have $k_0>0$, $k_1>0$, if the vector h at x_0 points towards the interior of the circle. Calling K the class of vectors of this vectors, i.e., all $\bar{k}$ such that $k_0 < 0$, $k_1 \geqq 0$, the multiplier rule (II.5) can be restated in a disguised form:

$$\text{No } \bar{k} \text{ in } \bar{K} \text{ is interior to } K.$$

This is the form of the multiplier rule which forms the basis for the results found in modern texts such as Pontrjagin [1].

One further example is the problem of finding the shortest distance between the circle, Figure 3, and a point P which is constrained to lie on or above the line $3x^3 + 4x^4 - 25 = 0$. The mathematical formulation is

$$f_0(x) = \frac{1}{2}\left[(x^1 - x^3)^2 + (x^2 - x^4)^2\right] = \min$$

subject to

$$f_1(x) = \frac{1}{2}\left[1 - (x^1)^2 - (x^2)^2\right] \geqq 0,$$

$$f_2(x) = 3x^3 + 4x^4 - 25 \geqq 0.$$

To solve this problem, we look at the class K of all vectors $k = (k_0, k_1, k_2)$, where $k_i = f_i'(x, h)$, with h an arbitrary vector. Here $x = (x^1, x^2, x^3, x^4)$; $x_0 = (3/5, 4/5, 3, 4)$ is the known solution. $\bar{K}$ is $\bar{k}_\rho$ such that $\bar{k}_1 \geqq 0$, $\bar{k}_2 \geqq 0$, and $\bar{k}_0 \leqq 0$. This can be seen by considering how the functions f_1 and f_2 change as the point P and the terminal point at the circle move, as in the previous example. For this case,

$$k_0 = f_0'(x_0, h) = -\frac{4}{5}\left[3(h^1 - h^3) + 4(h^2 - h^4)\right],$$

$$k_1 = f_1'(x_0, h) = -\frac{1}{5}(3h^1 + 4h^2),$$

$$k_2 = f_2'(x_0, h) = (3h^3 + 4h^4).$$

Thus the multiplier rule is simply

$$k_0 - 4k_1 - \frac{4}{5}k_2 = 0.$$

If we write $F = f_0 - 4f_1 - (4/5)f_2$, then the multiplier rule is

$$F'(x_0, h) = k_0 - 4k_1 - \frac{4}{5}k_2 = 0,$$

which is equivalent to

$$F_{x^i} = 0 \qquad \text{at } x_0.$$

Let us consider the theory of minima of functions of n variables in more detail, now that we have an idea of what must be observed in view of the simple examples given above. Because every problem that is to be solved numerically must be discretized, i.e., reduced to a problem given in terms of functions of n variables, it is important to have a good grasp of the theory before proceeding to more advanced topics.

For the function $f(x) = f(x^1, x^2, \cdots, x^n)$, the level surfaces are those for which $f(x) =$ constant. As we know, the vector normal to a level surface, i.e., the vector in the direction of greatest rate of change of f, is grad f and has the components $f_{x^j} \equiv (\partial f/\partial x^j)$ $(j = 1, \cdots, n)$. The rate of change in any other direction $h = (h^1, h^2, \cdots, h^n)$ is then

$$(\operatorname{grad} f) \cdot h \equiv f'(x_0, h) = \frac{d}{dt} f(x_0 + th) \Big|_{t=0}$$
$$= \frac{\partial f}{\partial x^1}\Big|_{x_0} h^1 + \frac{\partial f}{\partial x^2}\Big|_{x_0} h^2 + \cdots + \frac{\partial f}{\partial x^n}\Big|_{x_0} h^n,$$

where $x_0 + th \equiv (x_0^1 + th^1,\ x_0^2 + th^2,\ \cdots,\ x_0^n + th^n)$. Thus we write

$$f'(x_0, h) = f_{x^i}(x_0)\, h^i = (\operatorname{grad} f) \cdot h = (\operatorname{grad} f, h),$$

as the directional derivative of f in the direction h. If the level curve is as shown in Figure 4, and assuming grad $f \neq 0$, then for h pointing out (like h_1 in Figure 4), $f'(x_0, h) > 0$; for h pointing in (like h_2 in Figure 4), $f'(x_0, h) < 0$; and for h tangent to the curve (like h_3 in Figure 4), $f'(x_0, h) = 0$, since grad f is normal to the level surface.

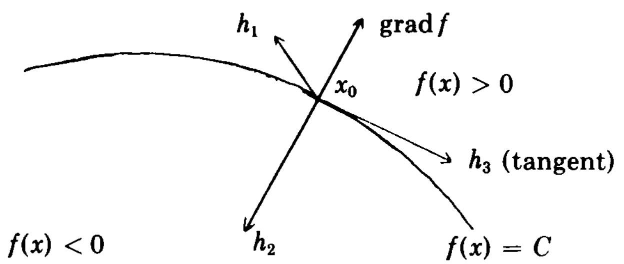

FIGURE 4. Level Curve

The directional derivative can be particularized by specifying h to lie tangent to some curve $x(t)$ that intersects the curve $f(x) = c$, i.e., we require that $x(0) = x_0$, $\dot{x}(0) = h$. Then

$$\begin{aligned} f'(x_0, h) &= \frac{d}{dt} f(x(t)) \Big|_{t=0} \\ &= f_{x^i}(x_0)\, \dot{x}^i(0), \end{aligned}$$

where $\dot{x}^i(0)$ has replaced h^i.

The economy of the notation introduced here enables us to write Taylor's Theorem as follows:

For one variable

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \cdots,$$

and

$$f(x_0 + h) = f(x_0) + f'(x_0)\, h + \frac{1}{2} f''(x_0)\, h^2 + \cdots.$$

For n variables we write Taylor's Theorem as

$$f(x_0 + h) = f(x_0) + f'(x_0, h) + \frac{1}{2} f''(x_0, h) + \cdots,$$

where

$$\begin{aligned} f'(x_0, h) &= f_{x^i} h^i, \\ f''(x_0, h) &= f_{x^i x^j} h^i h^j. \end{aligned}$$

Suppose that x_0 is the solution of the problem $f(x) = \min$. How do the level surfaces look near x_0? From the expansion

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0, x - x_0) + \cdots = \text{constant},$$

since $f'(x_0) = 0$. In two dimensions,

$$\begin{aligned} f(x, y) = f(x_0, y_0) + f_{xx}(x - x_0)^2 + 2f_{x,y}(x - x_0)(y - y_0) \\ + f_{yy}(y - y_0)^2 + \cdots \\ = \text{const.} \end{aligned}$$

Truncation of the series at the second order terms shows that near the minimum point the level surfaces are approximate ellipses.

For a problem with constraints, the classical procedure is to introduce Lagrange multipliers; e.g., in the problem $f(x) = \min$, subject to $g(x) = 0$, form the function $F(x) = f(x) - \lambda g(x)$. We will choose λ such that

$$F_{x^i}(x_0) = \operatorname{grad} F|_{x_0} = 0,$$

where x_0 is the solution point. There is a unique λ provided $\operatorname{grad} g \neq 0$. To see this graphically, consider Figure 5. It is clear that in order for a solution to exist, the curves $f(x) = f(x_0)$ and $g(x) = 0$ must *not* cross but must be tangent at the solution point. Tangency is equivalent to the existence of a λ so that

$$\operatorname{grad} f = \lambda \operatorname{grad} g.$$

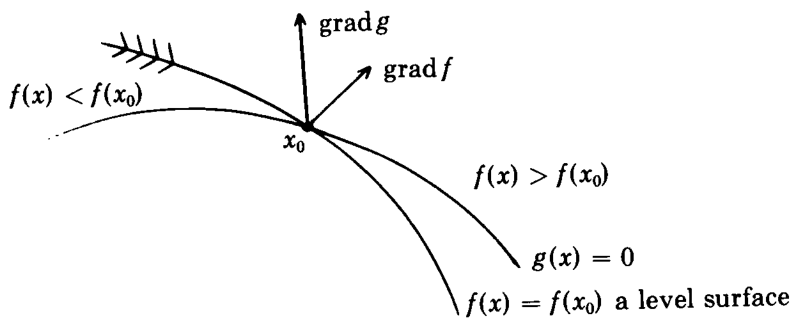

FIGURE 5. Minimum Subject to Constraint

In the above problem we can accept either $\lambda > 0$ or $\lambda < 0$. However, for a problem with an inequality constraint, say,

$$f(x) = \min, \qquad g(x) \geqq 0,$$

with solution x_0, it can be shown by similar graphical arguments that in order for $\operatorname{grad} F = \operatorname{grad} f - \lambda \operatorname{grad} g = 0$, λ must be nonnegative.

To generalize, we state the following without proof:

THEOREM. *For the problem* $f(x) = \min$, *with two constraints*:
Case I: $g_1(x) = 0$, $g_2(x) = 0$;
Case II: $g_1(x) = 0$, $g_2(x) \geqq 0$;
Case III: $g_1(x) \geqq 0$, $g_2(x) \geqq 0$;
if x_0 is a solution, i.e., $f(x_0)$ is a minimum of $f(x)$ subject to the constraints, then there exist multipliers λ_1 and λ_2 such that, when we set

$$F = f - \lambda_1 g_1 - \lambda_2 g_2,$$

we have

$$F_{x^i} = \operatorname{grad} F = 0 \ \ at \ \ x_0;$$

for Case II: $\lambda_2 \geqq 0$; *for Case* III: $\lambda_2 \geqq 0$; $\lambda_3 \geqq 0$.

These results will now be interpreted in terms of the vectors K and $\bar{K}$ introduced earlier. If we write

$$\begin{aligned} k_0 &= f'(x_0, h), \\ k_1 &= g_1'(x_0, h), \\ k_2 &= g_2'(f_0, h), \end{aligned} \tag{II.6}$$

then for $k_1 \geqq 0$ and $k_2 \geqq 0$, we must have $k_0 \geqq 0$.

Equivalent to the above theorem is the following:

THEOREM. *Let K be all vectors $k = (k_0, k_1, k_2)$ defined by* (II.6) *and let $\bar{K}$ be all vectors $\bar{k} = (\bar{k}_0, \bar{k}_1, \bar{k}_2)$ such that $\bar{k}_0 < 0$, $\bar{k}_1 \geqq 0$, and $\bar{k}_2 \geqq 0$. Then no vector $\bar{k}$ in $\bar{K}$ is in K.*

More generally still, for the problem $f(x) = \min$ subject to the constraints

$$\begin{aligned} g_\alpha(x) &= 0 \qquad (\alpha = 1, \cdots, m'), \\ g_\beta(x) &\geqq 0 \qquad (\beta = m' + 1, \cdots, m), \end{aligned}$$

if x_0 is a solution, i.e.,

$$\begin{aligned} g_\alpha(x_0) &= 0 \qquad (\alpha = 1, \cdots, m'), \\ g_{\beta'}(x_0) &= 0 \qquad (\beta' = m' + 1, \cdots, m''), \\ g_{\beta''}(x_0) &> 0 \qquad (\beta'' = m'' + 1, \cdots, m), \end{aligned}$$

then we have the multiplier rule:

There exist multipliers $\lambda_0 \geqq 0$, $\lambda_1, \cdots, \lambda_m$ not all zero such that

(1) $\lambda_{\beta'} \geqq 0$,

(2) $\lambda_{\beta''} = 0$;

(3) the function $F = \lambda_0 f - \lambda_\gamma g_\gamma$ $(\gamma = 1, \cdots, m)$ has the property that $\operatorname{grad} F = 0$ at x_0.

If the matrix

$$\left(\frac{\partial g_\sigma(x_0)}{\partial x^i}\right) \qquad (\sigma = 1, \cdots, m'')$$

has the rank m'', then $\lambda_0 > 0$ and can be chosen so that $\lambda_0 = 1$.

If so chosen, the multipliers are unique.

III. **Classical calculus of variations.** Let us now return to classical theory and derive the necessary conditions for a minimum in a general form. The equation for the brachistochrone problem can be written

$$J(y) = \int_{x_1}^{x_2} \frac{(1 + (y')^2)^{1/2}}{y^{1/2}}\, dx = \min \tag{III.1}$$

where we now write $y = y(x)$ for $x_1 \leqq x \leqq x_2$; $[x_1, y(x_1)]$, and $[x_2, y(x_2)]$ are held fast; and unessential constants have been ignored. Another source of problems is that of the minimization of the area of a surface of revolution, the generator of which passes through any two points 1 and 2, as in Figure 6. The functional to be minimized is

$$J(y) = \int_{x_1}^{x_2} 2\pi y\, ds = \int_{x_1}^{x_2} 2\pi y\, (1 + y'^2)^{1/2} \tag{III.2}$$

where the factor 2π may be dropped, since it is unimportant. Some typical elementary problems are covered by the following forms of functionals:

$$J(y) = \int_{x_1}^{x_2} y^r\, (1 + (y')^2)^{1/2} dx = \min \qquad (r \text{ is real}),$$

$$J(y) = \int_{x_1}^{x_2} ((y')^2 - y^2)\, dx = \min,$$

$$J(y) = \int_{x_1}^{x_2} (1 - (y')^2)^{1/2} dx = \min.$$

In the general form, the fixed end point problem is written: Determine $y = y(x)$ for $x_1 \leqq x \leqq x_2$, with $[x_1, y(x_1)]$, $[x_2, y(x_2)]$ held fast, such that

$$J(y) = \int_{x_1}^{x_2} f[x, y(x), y'(x)]dx = \min.$$

If the minimizing arc is $y_0 = y_0(x)$ for $x_1 \leqq x \leqq x_2$, then we have the

MAIN THEOREM. (1) *$f - y'f_{y'}$ is continuous along y_0 and*

$$\frac{d}{dx}(f - y'f_{y'}) = f_x \text{ on } y_0; \tag{III.3}$$

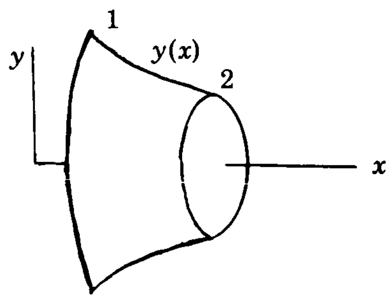

FIGURE 6. Minimum Surface of Revolution

$f_{y'}$ is continuous along y_0 and

(III.4) $$\frac{d}{dx} f_{y'} = f_y \text{ on } y_0;$$

(2) *For $(x, y_0(x), Y')$ in the region R of definition of $f(x,y,y')$*

(III.5) $$E(x, y_0(x), y_0'(x), Y') \geqq 0,$$

where

$$E(x,y,y',Y') = f(x,y,Y') - f(x,y,y') - (Y' - y') f_{y'}(x,y,y').$$

Equations (III.3) and (III.4) are the Euler equations, and Equation (III.5) is the Weierstrass condition.

Before proceeding with the proof of the above theorem, let us consider a few examples.

If $f = (y')^2 - y^2$, then $f_{y'} = 2y'$, i.e., there can be no corners on y_0. Since $f_x = 0$, we have the condition from (III.3) above

$$f - y'f_{y'} = -(y')^2 - y^2 = \text{constant},$$

and since $f_y = -2y$, the Euler equation is

$$y'' + y = 0,$$

which has the solution

$$y = a\cos x + b\sin x.$$

If $f = y^r(1 + (y')^2)^{1/2}$, then

$$f_{y'} = \frac{y^r y'}{(1 + (y')^2)^{1/2}},$$

and

$$f - y'f_{y'} = \frac{y^r}{(1 + (y')^2)^{1/2}}.$$

The first Euler equation is then integrated once to give a conservation principle:

$$\frac{y^r}{(1 + (y')^2)^{1/2}} = \text{constant}.$$

In terms of the variables introduced in §II, the problem in many dimensions is:

$$x = x(t) \quad \text{for} \quad t^0 \leqq t \leqq t^1,$$

$[t^0, x(t^0)]$ and $[t^1, x(t^1)]$ are fixed,

$$J(x) = \int_{t^0}^{t^1} f[t, x(t), \dot{x}(t)]dt = \min.$$

If $x_0 = x_0(t)$ for $t^0 \leqq t \leqq t^1$ is the minimizing arc, then the Main Theorem is:

MAIN THEOREM. (1). *$f - \dot{x}f_{\dot{x}}$ is continuous along x_0 and*

$$\frac{d}{dt}(f - \dot{x}f_{\dot{x}}) = f_t \text{ on } x_0; \tag{III.6}$$

$f_{\dot{x}}$ is continuous along x_0 and

$$\frac{d}{dt} f_{\dot{x}} = f_x \text{ on } x_0; \tag{III.7}$$

(2)

$$E(t, x_0(t), \dot{x}_0(t), \dot{X}) \geqq 0 \tag{III.8}$$

for all $(t, x_0(t), X)$ in R, where

$$E(t, x, \dot{x}, \dot{X}) = f(t, x, \dot{X}) - f(t, x, \dot{x}) - (\dot{X} - \dot{x}) f_{\dot{x}}(t, x, \dot{x}).$$

It is easy to give a graphical interpretation of the Weierstrass condition. Let $z = f(y')$, holding x and y fixed. In the y'-x plane, Figure 7, at the point y_0', $z_0 = f(y_0')$, draw the indicatrix $z - z_0 = f_{y'}(y_0')(Y' - y_0')$, i.e., the tangent to the curve at that point. We see that $f(Y') \geqq f(y_0') + (Y' - y_0') f_{y'}(y_0')$. Thus the Weierstrass condition is interpreted as the condition that the curve $z = f(y')$

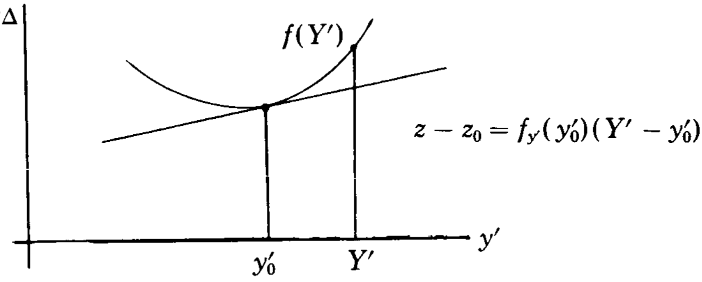

FIGURE 7. The Weierstrass Condition

lies everywhere above the indicatrix in the neighborhood of the minimum y_0.

Before giving the proof of the Main Theorem, we must make some qualitative definitions.

Weak Neighborhood. For a given interval $(x_1 \leqq x \leqq x_2)$, an arc $y = y(x)$ is said to lie in a weak neighborhood of another arc $y = y_0(x)$ if $y(x)$ and $y'(x)$ respectively differ little from $y_0(x)$ and $y_0'(x)$ in the interval.

Strong Neighborhood. For a given interval $(x_1 \leqq x \leqq x_2)$, an arc $y = y(x)$ is said to lie in a strong neighborhood of another arc $y = y_0(x)$ if $y(x)$ differs little from $y_0(x)$.

The Euler equations are derived using the concept of a weak neighborhood; the Weierstrass condition is based on the concept of a strong neighborhood.

Let us prove the Main Theorem in terms of variables used in the second statement of it.

Let the function

$$h = h(t) \qquad \text{for} \quad t^0 \leqq t \leqq t^1$$

be an admissible (weak) variation, i.e., $h(t^0) = 0$, $h(t^1) = 0$, so that the function

$$x_0 + \epsilon h = x_0(t) + \epsilon h(t) \qquad \text{for} \quad t^0 \leqq t \leqq t^1$$

has the same end points as $x_0(t)$. If ϵ is an arbitrary, small number, we write

$$\text{(III.9)} \qquad \phi(\epsilon) = J(x_0 + \epsilon h) = \int_{t^0}^{t^1} f(t, x_0(t) + \epsilon h(t), \dot{x}_0(t) + \epsilon \dot{h}(t))\, dt.$$

In order that x_0 be a minimizing arc,

$$\phi'(0) = 0 \qquad \text{and} \qquad \phi''(0) \geqq 0.$$

Hence,

$$\text{(III.10)} \qquad 0 = \phi'(0) = J'(x_0, h) = \int_{t^0}^{t^1} (f_x h + f_{\dot x} \dot h)\, dt,$$

and

$$\text{(III.11)} \qquad 0 \leqq \phi''(0) = J''(x_0, h) = \int_{t^0}^{t^1} 2\omega(t, h, \dot h)\, dt,$$

where

$$2\omega = f_{xx} hh + 2f_{x\dot x} h\dot h + f_{\dot x\dot x} \dot h\dot h.$$

Equation (III.10) is analogous to the directional derivative introduced in §II. Let us rewrite (III.10) as

$$\text{(III.12)} \qquad J'(x_0, h) = \int_{t^0}^{t^1} [M(t)\, h(t) + N(t)\, \dot h(t)]dt.$$

FUNDAMENTAL LEMMA. *If $M(t)$ and $N(t)$ are piecewise continuous, then $\int_{t^0}^{t^1} (Mh + N\dot h)\, dt = 0$ for all admissible h, if and only if $N(t) = \int_{t^0}^{t} M(s)\, ds +$ constant.*

PROOF. Let $q(t) = \int_{t^0}^{t} M(s)\, ds$, so that $\dot q(t) = M(t)$, and put $h(t) = \int_{t^0}^{t} [N(r) - q(r)]dr - C(t - t^0)$. We clearly have $h(t^0) = 0$, and if we choose C such that $h(t^1) = 0$, then $h(t)$ is admissible. Let $p(s) = q(s) + C$, so that

$$h(t) = \int_{t^0}^{t} [N(s) - p(s)]ds.$$

Then $\dot p(t) = \dot q(t) = M(t)$, and $\dot h(t) = N(t) - p(t)$.

Case 1. Assume that $\int_{t^0}^{t^1} (Mh + N\dot h)\, dt = 0$ for all admissible h. Using the h just defined, we have

$$0 = \int_{t^0}^{t^1} (Mh + N\dot h)\, dt = \int_{t^0}^{t^1} (\dot p h + (\dot h + p)\, \dot h)\, dt$$

$$= \int_{t^0}^{t^1} \dot h^2 dt + [ph]_{t=t^0}^{t=t^1} = \int_{t^0}^{t^1} \dot h^2 dt$$

since $h(t^0) = h(t^1) = 0$. However, if h is ever different from zero for $t^0 \leqq t \leqq t^1$, then $\dot h^2$ is sometimes positive and never negative, and we would have

$$\int_{t^0}^{t^1} \dot h^2 dt > 0.$$

So $\dot{h} = 0$, and we have $N = p$; that is

$$N(t) = \int_{t^0}^{t} M(s)\, ds + C.$$

Case 2. Assume the relation above. Let $h(t)$ be admissible. As $M(t) = \dot{N}(t)$ from our assumption as to the form of N, we get

$$\int_{t^0}^{t^1} (Mh + N\dot{h})\, dt = \int_{t^0}^{t^1} (\dot{N}h + N\dot{h})\, dt = [Nh]_{t=t^0}^{t=t^1} = 0.$$

The proof of (III.7) follows directly from the lemma. To prove (III.6), we have by (III.7)

$$\begin{aligned} f_t &= f_t + f_x \dot{x} - \dot{x}\frac{d}{dt} f_{\dot{x}} \\ &= f_t + f_x \dot{x} + f_{\dot{x}} \ddot{x} - \frac{d}{dt}(\dot{x} f_{\dot{x}}) \\ &= \frac{d}{dt}(f - \dot{x} f_{\dot{x}}). \end{aligned}$$

To prove the Weierstrass condition (III.8) we refer to Figure 8 in which $\epsilon > 0$. We will admit strong modifications of the form shown, calling the modification $Y(t)$ for $t_0 \leqq t \leqq t_0 + \epsilon$ and $Y(t, \epsilon)$ for $t_0 + \epsilon \leqq t \leqq t^1$. Note that the modification is continuous but has corners:

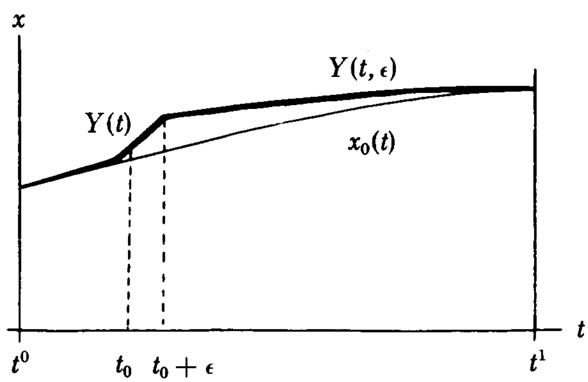

FIGURE 8. Arc with Corners

Specifically, let $X(t)$ be a function with $X(t_0) = x(t_0)$ and $(t_0, x(t_0), \dot{X}(t_0))$ in R. Then take

$$Y(t) = x(t) + (t - t_0)(\dot{X}(t_0 + \epsilon) - \dot{x}(t_0 + \epsilon)),$$
$$Y(t, \epsilon) = x(t) + \epsilon(\dot{X}(t) - \dot{x}(t)).$$

Then

$$\dot{Y}(t) = \dot{x}(t) + X(t_0, + \epsilon) - \dot{x}(t_0 + \epsilon)$$
$$\dot{Y}(t, \epsilon) = \dot{x}(t) + \epsilon(\ddot{X}(t) - \ddot{x}(t)).$$

For the arc with corners,

$$J(x_0) = \phi(0) \leqq \phi(\epsilon) = \int_{t^0}^{t_0} f(t, x_0(t), \dot{x}_0(t))\, dt$$
$$+ \int_{t_0}^{t_0+\epsilon} f(t, Y(t), \dot{Y}(t))\, dt$$
$$+ \int_{t_0+\epsilon}^{t^1} f(t, Y(t, \epsilon), \dot{Y}(t, \epsilon))\, dt.$$

Hence,

$$0 \leqq \dot{\phi}(0) = f(t_0, x_0(t_0), \dot{X}(t_0)) - f(t, x_0(t), \dot{x}_0(t))$$
$$+ \int_{t_0}^{t^1} (f_x x_\epsilon + f_{\dot{x}} \dot{x}_\epsilon)\, dt,$$

where $x_\epsilon = \dot{X} - \dot{x}$. By (III.7), the integral becomes $f_{\dot{x}} x_\epsilon |_{t_0}^{t^1}$. Note that $x_\epsilon(t^1, 0) = 0$. Thus we have the Weierstrass condition

$$0 \leqq \phi'(0) = f(t_0, x_0(t_0), \dot{X}(t_0)) - f(t_0, x_0(t_0), \dot{x}_0(t_0))$$
$$- (\dot{X}(t_0) - \dot{x}_0(t_0))\, f_{\dot{x}}(t, x_0(t_0), \dot{x}_0(t_0)).$$

Transversality conditions arise in variational problems in which one or both end points are not fixed. For example, in finding the shortest distance between a point P in a plane and a curve $y_1(x)$ in the same plane, one end point is fixed at P and the other is variable. It is clear that the minimal curve y_0 will be a straight line which is normal to the curve and passes through P, Figure 9. The tangent will have the direction $(1, y_1')$, and the end point condition is

(III.13) $$(1, y_1') \perp (dx, dy_1).$$

The relation (III.13) is called the transversality condition.

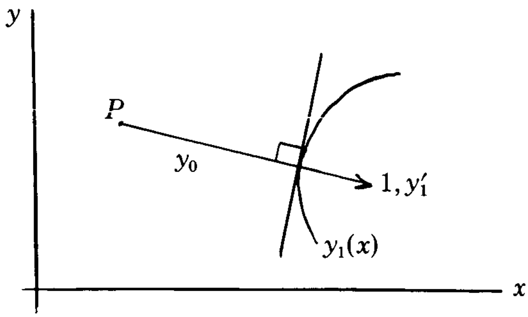

FIGURE 9. Transversality Condition

For the general problem

$$J = \int_{x_0}^{x_1} f(x, y, y')\, dx = \min$$

with x_1 constrained to be on some curve $y(x)$, the transversality condition to be satisfied at the variable end point x_1 is the line with the direction $(f - y'f_{y'}, f_{y'})$ must be perpendicular to (dx, dy). Hence

$$\text{(III.14)} \qquad (f - y'f_{y'})\, dx + f_{y'} dy = 0.$$

Let us prove (III.14) in complete generality in terms of the variables used in the proof of the Weierstrass condition

$$x = x^i(t) \quad \text{for} \quad t^0 \leqq t \leqq T,$$

$[t^0, x^i(t^0)]$ held fast; $T, x^i(T)$ are constrained to lie on a surface S. The general problem is

$$\text{(III.15)} \quad J(x) = g[T, x^i(T)] + \int_{t^0}^{T} f(t, x(t), \dot{x}(t))\, dt = \min.$$

Let $x_0 = x_0(t)$ for $t^0 \leqq t \leqq T_0$ be the solution, and choose a one-parameter family of curves $x(t, \epsilon)$, $t^0 \leqq t \leqq T(\epsilon)$ joining the initial point to a point $(T(\epsilon),\ X(\epsilon))$ on S, where $X(\epsilon) = x(T(\epsilon), \epsilon)$. The family of curves should contain x_0 for $\epsilon = 0$; that is, $x(t, 0) = x_0(t)$, with $T(0) = T_0$. We form the function

$$J(\epsilon) = g[T(\epsilon), X(\epsilon)] + \int_{t_0}^{T(\epsilon)} f(t, x(t, \epsilon), \dot{x}(t, \epsilon))\, dt.$$

Then

$$dJ = dg + f(T)\,dT + \int_{t_0}^{T_0} (f_x \delta x + f_{\dot{x}} \delta \dot{x})\,dt, \tag{III.16}$$

where $\delta x = x_\epsilon d\epsilon$, and we have put $\epsilon = 0$, $d\epsilon = 1$. The right side of (III.16) must vanish if x_0 is the minimal arc. Holding T_0 fixed, but letting $x(t)$ vary subject to $x(t^0)$ and $x(T)$ being held fast, we conclude that the Euler equations (III.6) and (III.7) must hold. Hence, integrating by parts and observing that $dx^i = \dot{x}^i dt + \delta x^i$ gives

$$dg + [(f - \dot{x}^i f_{\dot{x}^i})\,dT + f_{\dot{x}^i} dX^i]^{t=T_0} = 0.$$

If g is absent from (III.15), then we have the transversality condition given above. If not, then the expression in square brackets must be equal to $-dg$.

Before we leave the classical theory we will discuss briefly the theory of multiple integrals. Consider

$$J = \int\int_A f(x, y, z, p, q)\,dx\,dy, \tag{III.17}$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. The first variation of the functional J is

$$\delta J = \int\int_A (f_z \delta z + f_p \delta z_x + f_q \delta z_y)\,dx\,dy, \tag{III.18}$$

where $\delta z = 0$ on the boundary C of A. If we define the inner product of the two functions u and v to be

$$(u, v) = \int\int_A (u_x v_x + u_y v_y)\,dx\,dy,$$

then δJ is expressible in the form

$$\delta J = (u, \delta z).$$

Take the function u to be the solution of the system

$$\Delta u = \frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q - f_z, \qquad u = 0 \text{ on } C.$$

The function $u(x, y)$ is the gradient of J along the surface $z(x, y)$. If z minimizes J on the class of surfaces having the same boundary

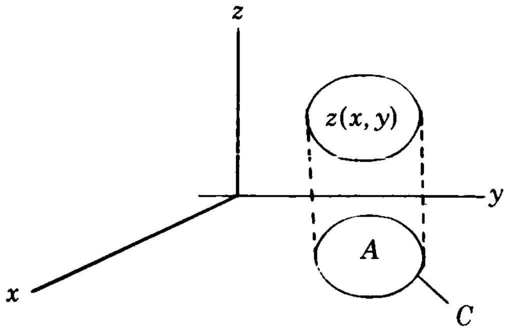

FIGURE 10. Minimum Problem for a Double Integral

values, then the gradient of J is zero and the condition for J to be a minimum is

$$\Delta u = \frac{\partial}{\partial x} f_p + \frac{\partial}{\partial y} f_q - f_z = 0. \tag{III.19}$$

Finally, let us discuss briefly functionals containing higher derivatives:

$$x = x(t) \quad \text{for} \quad t^0 \leqq t \leqq t^1,$$

$$J(x) = \int_{t^0}^{t^1} f(t, x, \dot{x}, \ddot{x})\, dt = \min,$$

and let $x(t^0)$, $\dot{x}(t^0)$, $x(t^1)$, $\dot{x}(t^1)$ be held fast. The Euler equation, which can again be derived by means of the directional derivative concept, is

$$f_x - \frac{d}{dt} f_{\dot{x}} + \frac{d^2}{dt^2} f_{\ddot{x}} = 0, \tag{III.20}$$

and the Weierstrass condition is as before with

$$E(t, x, \dot{x}, \ddot{x}, X) = f(t, x, \dot{x}, \ddot{X}) - f(t, x, \dot{x}, \ddot{x}) - (\ddot{X} - \ddot{x}) f_{\ddot{x}}(t, x, \dot{x}, \ddot{x}).$$

It is interesting to note how the above problem can be cast into the form of a control problem, as introduced earlier or discussed in more detail in §V. Write

$$x^1 = x, \quad x^2 = \dot{x}, \quad u = \ddot{x}.$$

Then the differential equations of the process are

$$\dot{x}^1 = x^2, \qquad \dot{x}^2 = u,$$

with $x^i(t^0)$, $x^i(t^1)$ given. Then we wish to find x, u for which

$$J = \int_{t^0}^{t^1} f(t, x^1, x^2, u)\, dt = \min,$$

which is a "control problem."

IV. **Theory of cones.** The theory of cones in n-dimensional geometry is useful for discussing advanced theories of the calculus of variations. The following is a brief introduction to the theory.

If we have a vector $k = (k_0, k_1, k_2, \cdots, k_m)$, we define the following:

DEFINITION. A *hyperplane* is the plane $L(k) = 0$ where

$$L(k) = \alpha_0 k_0 + \alpha_1 k_1 + \cdots + \alpha_m k_m$$

or

$$\begin{aligned} L(k) &= \lambda_0 k_0 - \lambda_1 k_1 - \cdots - \lambda_m k_m \\ &= \lambda_0 k_0 - \lambda_\gamma k_\gamma \ \ (\gamma = 1, 2, \cdots, m), \end{aligned}$$

where $\lambda_0, \lambda_1, \cdots, \lambda_m$ are not all zero. For example, in two dimensions, a hyperplane is any line through the origin. A hyperplane divides the $m + 1$ dimensional space into two half spaces $L(k) \geqq 0$ and $L(k) \leqq 0$.

DEFINITION. A *ray* is a vector $k \neq 0$ and all αk $(\alpha \geqq 0)$, i.e., all nonnegative multiples of a vector.

DEFINITION. A *cone* K is a collection of rays. If k is in K, so also is αk, $\alpha \geqq 0$.

DEFINITION. A *convex cone* K is a cone such that if k and k' are in K, so also is $k + k'$.

LEMMA I. *If K and $\bar{K}$ are convex cones such that no $\bar{k}$ in $\bar{K}$ is interior to K, there exists a hyperplane $L(k) = 0$ which separates them in the sense that*

$$L(k) \geqq 0 \ \textit{if} \ k \ \textit{is in} \ K,$$

$$L(\bar{k}) \leqq 0 \ \textit{if} \ \bar{k} \ \textit{is in} \ \bar{K}.$$

DEFINITION. If K, $\bar{K}$ are convex cones, the set $K - \bar{K}$ consists of all vectors of the form $k - \bar{k}$, where k is in K and $\bar{k}$ is in $\bar{K}$. The set $K - \bar{K}$ is a convex cone.

Lemma I is not sufficiently general to cover all cases that arise.

However, it is useful in nondegenerate cases. To take care of degenerate cases, Lemma I should be replaced by Lemma I*, given below. However, because the results are more intuitive when Lemma I is applicable, we shall limit ourselves to this case.

LEMMA I*. *If K, $\bar{K}$ are convex cones and $K^* = K - \bar{K}$ does not contain all vectors, there exists a hyperplane $L(k) = 0$ such that $L(k^*) \geqq 0$ for all k^* in K^*. The hyperplane $L(k) = 0$ separates K and $\bar{K}$ in the sense that*

$$L(k) \geqq 0 \text{ if } k \text{ is in } K,$$
$$L(\bar{k}) \leqq 0 \text{ if } \bar{k} \text{ is in } \bar{K}.$$

DEFINITION. By the tangent cone $\mathscr{C}$ of R at a point x_0 in R will be meant a collection of rays, each of which is the limit of a sequence $\{L_q\}$ of rays eminating from x_0, the ray L_q containing a point $x_q \neq x_0$ in R at a distance of at most $1/q$ from x_0. For example, for a smooth closed region R, the cone tangent to R at a point x_0 on the boundary is the half space containing R. If x_0 is interior to R, the tangent cone is the whole space. At the boundary point of a region R with corners, the tangent cone may be less or more than a half space, depending on whether the corner is re-entrant.

THEOREM. *Let X be a closed, well-behaved set in x-space, $x = (x^1, x^2, \cdots x^m)$, let x_0 be a boundary point of X, and let $\mathscr{C}$ be the cone tangent to X at x_0. Assume that $\mathscr{C}$ is convex and has an interior point. Let $f_0(x), f_1(x), \cdots, f_m(x)$ be functions on X having derivatives $f_0'(x_0,h), f_1'(x_0,h), \cdots, f_m'(x_0,h)$ at x_0, and let K be all vectors k defined by the formula*

$$k_\rho = f_\rho'(x_0, h), \text{ where } h \text{ is in } \mathscr{C}, \qquad (\rho = 0, 1, \cdots, m).$$

Then K is a closed convex cone.

A proof of this theorem will not be given here.

LEMMA II. *If $\bar{k}$ is interior to K, there is an $\bar{h}$ interior to $\mathscr{C}$ such that $\bar{k}_\rho = f_\rho'(x_0, \bar{h})$, and there exists a curve $x(t)$ in X such that*

$$f_\rho(x(t)) - f_\rho(x_0) = t\bar{k}_\rho \qquad (0 \leqq t \leqq \epsilon),$$

and

$$x(0) = x_0, \qquad \dot{x}(0) = \bar{h}.$$

Let us now apply some of these results to one of the problems we considered earlier. Suppose that x_0 minimizes $f_0(x)$ in X subject to the constraints

$$f_\alpha(x) = 0 \qquad (\alpha = 1, 2, \cdots, m'),$$

$$f_\beta(x) \geqq 0 \qquad (\beta = m' + 1, \cdots, m).$$

Suppose further that

$$f_\alpha(x_0) = 0,$$

$$f_{\beta'}(x_0) = 0 \qquad (\beta' = m' + 1, \cdots, m''),$$

$$f_{\beta''}(x_0) > 0 \qquad (\beta'' = m'' + 1, \cdots, m).$$

Let $\bar{K}$ be all vectors $\bar{k} = (\bar{k}_0, \cdots, \bar{k}_m)$, where $\bar{k}_0 < 0$, $\bar{k}_\alpha = 0$, $\bar{k}_{\beta'} \geqq 0$, $\bar{k}_{\beta''}$ arbitrary. Then no $\bar{k}$ in $\bar{K}$ is interior to K, where K is defined by the previous theorem. Intuitively K represents the directional derivatives that f_ρ can have at x_0 and $\bar{K}$ is a class of vectors that cannot be directional derivatives of f_ρ at x_0.

To see these results, we suppose the last statement is untrue and show a contradiction. If it is untrue, then by Lemma II, there is a curve $x(t)$ $(0 \leqq t \leqq \epsilon)$ in X such that $x(0) = x_0$ and

$$f_0(x(t)) = f_0(x_0) + t\bar{k}_0 \qquad 0 \leqq t \leqq \epsilon,$$

$$f_\alpha(x(t)) = f_\alpha(x_0) + t\bar{k}_\alpha,$$

$$f_{\beta'}(x(t)) = f_{\beta'}(x_0) + t\bar{k}_{\beta'},$$

and

$$f_{\beta''}(x(t)) = f_{\beta''}(x_0) + t\bar{k}_{\beta''}.$$

But the first equation leads to the conclusion that $f_0(x(t)) < f_0(x_0)$, because $t > 0$, $\bar{k}_0 < 0$, a clear contradiction.

THEOREM. *If x_0 is a solution of the above problem, there exist multipliers $\lambda_0 \geqq 0$, $\lambda_1, \cdots, \lambda_m$ such that*

(1) $\lambda_{\beta'} \geqq 0$,

(2) $\lambda_{\beta''} = 0$,

(3) *The function $F = \lambda_0 f_0 - \lambda_\gamma f_\gamma$ has the property that $F'(x_0, h) \geqq 0$ for all h in $\mathscr{C}$.*

PROOF OF (3).

$$\begin{aligned} F'(x_0,h) &= \lambda_0 f_0'(x_0,h) - \lambda_\gamma f_\gamma'(x_0,h) \\ &= \lambda_0 k_0 - \lambda_\gamma k_\gamma \\ &= L(k) \geqq 0 \text{ for } k \text{ in } K. \end{aligned}$$

PROOF OF (1) AND (2). Choose $\bar{k} = (-1,0,0,\cdots,0)$ in $\bar{K}$. Then $L(\bar{k}) = -\lambda_0 \leqq 0$ so that $\lambda_0 \geqq 0$.

Now choose $\bar{k} = (-1,0,\cdots,0,\bar{k}_\sigma,0,\cdots,0)$ such that there are at least $m'+1$ zeros before $\bar{k}_\sigma$. Then

$$L(\bar{k}) = -\lambda_0 - \lambda_\sigma \bar{k}_\sigma \leqq 0, \qquad m'+1 \leqq \sigma \leqq m'',$$

where if $\bar{k}_\sigma$ is any positive number, $\lambda_\sigma \geqq 0$, and if $\bar{k}_\sigma$ $(\sigma > m'')$ is any nonpositive number, $\lambda_\sigma = 0$.

V. **Control theory.** In control problems it is customary to think of the states of the systems being controlled as being represented by the vector

$$x(t) = (x^1(t), x^2(t), \cdots, x^q(t))$$

and the control by another vector

$$u(t) = (u^1(t), u^2(t), \cdots, u^n(t)).$$

The process, as it takes place in time, is governed by differential equations

$$\dot{x}^i = f^i(t,x,u),$$

and usually starts at some initial point

$$x^i(t^0) = b^i.$$

A given choice of $u(t)$ gives an initial value problem for the state

$$\begin{aligned} \dot{x}^i &= f^i(t,x(t),u(t)) = g^i(t,x), \\ x^i(t^0) &= b^i. \end{aligned}$$

The problem in control theory is to determine $u(t)$ so that we hit some target while minimizing something, say time, fuel consumption or cost.

An example of a simple control problem is to choose $u(t)$ so that at a fixed time T you reach $x^i(T) = c^i$ in such a way that

(V.1) $$J = \int_{t^0}^{T} g(t, x(t), u(t))\, dt = \min.$$

It can be seen that this problem is contained in the classical variational problem discussed in §I, when T is replaced by t^1 and $\dot{x}^i = f^i = u^i$.

The problem can be modified in several ways to make it more meaningful, but more complicated. We could add constraints of the form

$$|u^i(t)| \leqq C$$

or, say inequality constraints

$$\phi_\alpha(t, u(t)) \geqq 0,$$

$$\phi_\beta(t, u(t)) = 0,$$

or

$$\phi_\alpha(t, x(t), u(t)) \geqq 0, \qquad \text{etc.}$$

Let us translate the results for the classical fixed end point problems into the language and notation of optimal control theory. Let

(V.2) $$p_i(t) = f_{\dot{x}^i}(t, x_0(t), \dot{x}_0(t))$$

and let

$$u^i = \dot{x}^i, \qquad u_0^i(t) = \dot{x}_0^i(t).$$

We now define a new function

(V.3) $$H(t, x, u, p) = p_i u^i - f(t, x, u).$$

The minimizing arc x_0 determined by u_0 has the property that

(V.4) $$H(t, x_0(t), u, p(t)) \leqq H(t, x_0(t), u_0(t), p(t)),$$

i.e., $-H$ is minimized over all admissible elements $(t, x_0(t), u)$, which means that $+H$ is maximized. Hence we must have

$$H_{u^i} = p_i - f_{\dot{x}^i}(t, x, u) = 0.$$

By (V.2) this verifies the Euler condition

$$\frac{d}{dt} f_{\dot{x}} = f_x.$$

The classical Weierstrass condition comes directly from (V.4):

$$
\begin{aligned}
0 &\leqq H(t, x_0, u_0, p) - H(t, x_0, u, p) \\
&= p_i(t)\, u_0^i - f(t, x_0, u_0) - [p_i u^i - f(t, x_0, u)] \\
&= f(t, x_0, u) - f(t, x_0, u_0) - (u^i - u_0^i)\, f_{x^i}(t, x_0, u_0) \\
&- E(t, x_0, u_0, u).
\end{aligned}
$$

At this point one can make an analogy to the theory of Hamilton-Jacobi dynamics. If H were the Hamiltonian, then the Hamilton-Jacobi equations would be

$$
\text{(V.5a)} \qquad \begin{aligned} \dot{x}_i &= H_{p_i} = u^i, \\ \dot{p}_i &= -H_{x^i} = f_{x^i}. \end{aligned}
$$

The Hamiltonian would be defined by the definition of $H(t, x, u, p)$ and the equation

$$
\text{(V.5b)} \qquad H_{u^i} = 0.
$$

In the classical calculus of variations theory, (V.5) are the Euler equations.

Recall the modern brachistochrone problem discussed earlier:

$$
\dot{x}^1 = x^3, \qquad \dot{x}^2 = x^4, \qquad \dot{x}^3 = F \cos u, \qquad \dot{x}^4 = F \sin u - g,
$$

with $x^i(0)$ and $x^i(T)$ given, choose u such that $T = \min$. This problem also fits very easily into the general context of Leitman [2], who discusses problems of the form

$$
\ddot{x} = X(t, x, y) + \frac{c\beta}{m} \cos \psi,
$$

$$
\ddot{y} = Y(t, x, y) + \frac{c\beta}{m} \sin \psi,
$$

$$
\dot{m} = -\beta, \qquad 0 \leqq \beta \leqq \beta_{\max},
$$

$$
G(T, x(T), y(T), \dot{x}(T), \dot{y}(T), m(T)) = \min,
$$

with an initial point given. Such a problem is called a Problem of Mayer in classical texts.

Let us now state the necessary conditions for the solution to the following general control problem:

$$
\begin{aligned}
x &= \text{state variable } x^i(t) \qquad (i = 1, \cdots, q), \\
u &= \text{control variable } u^k(t) \qquad (k = 1, \cdots, n),
\end{aligned}
$$

where $t^0 \leqq t \leqq T$. The governing differential equations are

$$\dot{x}^i = f^i(t, x, u).$$

We are given $[t^0, x^i(t^0)]$ fixed and $x^i(T)$ fixed, and we wish to make

$$J(x) = g(T) + \int_{t^0}^{T} f(t, x, u)\, ds = \min. \tag{V.6}$$

Assume that $x_0^i(t)$, $u_0^k(t)$ for $t^0 \leqq t \leqq T_0$ is the solution, and define as before the function

$$H(t, x, u, p) = p_i f^i - \lambda_0 f. \tag{V.7}$$

Then there exist multipliers $\lambda_0 \geqq 0$ and $p_i(t)$, not all zero, such that

$$\dot{x}^i = H_{p_i} = f^i, \quad \dot{p}_i = -H_{x^i}, \tag{V.8}$$

and

$$H(t, x_0(t), u, p(t)) \leqq H(t, x_0(t), u_0(t), p(t)) \tag{V.9}$$

for all $(t, x_0(t), u)$ that are admissible. Admissibility may be defined by constraints of the following general form:

$$0 \leqq u^k \leqq C, \quad |u^k| \leqq C, \quad \phi_\alpha(u) \geqq 0.$$

Equations (V.8) and (V.9) combined constitute the Euler equations and the Weierstrass condition for this problem.

The transversality condition takes the form

$$\lambda_0 g'(T) - H(T, x_0(T), u_0(T), p(T)) = 0. \tag{V.10}$$

The analogous form of the transversality condition for the classical approach is given in §III. In nondegenerate problems the constant λ_0 is positive and can be chosen to be unity.

Let us solve the rocket problem (modern brachistochrone):

$$H = p_1 x^3 + p_2 x^4 + F(p_3 \cos u + p_4 \sin u) - p_4 g.$$

But $(d/dt)\, H = H_t = 0$; therefore $H = \text{constant} = \lambda_0 g'(T)$; $g(T) = T$; hence $H = \lambda_0 \geqq 0$ along the minimal curve. Now

$$\begin{aligned}
\dot{p}_1 &= -H_{x^1} = 0, && \therefore p_1 = \text{constant},\\
\dot{p}_2 &= -H_{x^2} = 0, && \therefore p_2 = \text{constant},\\
\dot{p}_3 &= -H_{x^3} = -p_1, && \therefore p_3 \text{ is linear in } t,\\
\dot{p}_4 &= -H_{x^4} = -p_2, && \therefore p_4 \text{ is linear in } t.
\end{aligned}$$

Let $\xi = p_3$, $\eta = p_4$, and we see that $\ddot{\xi} = 0$, and $\ddot{\eta} = 0$; hence the point (ξ, η) moves at a constant rate. Since we have no constraints of the form $\phi_\alpha(u) \geqq 0$, we must choose u such that $H = \max$ on the minimal curve, $H_u = 0$ on the curve,

$$0 = H_u = F(-p_3 \sin u + p_4 \cos u) = 0.$$

Hence, $\tan u = p_4/p_3 = \eta/\xi$.

The properties of the solution have been obtained without finding an explicit solution. The solution says that the thrust force F is always directed to a point that moves on a straight line at a constant rate as shown in Figure 11.

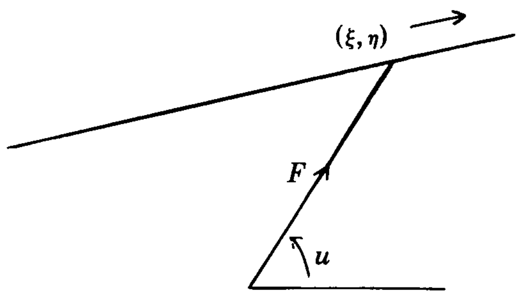

FIGURE 11. Minimum Time Path for a Rocket

Not all problems in control theory have solutions, i.e., not all systems are controllable. To illustrate the concept of controllability, let us suppose a problem is governed by a set of differential equations

$$\dot{x}^i = f^i(t, x, u).$$

We now ask whether there are functions u which can get us from P_0 to P_1. Moreover (see Figure 12), if we can get to P_1, can we

FIGURE 12. A Question of Controllability

get to P_2, a neighboring point, also? It might not be possible. To be explicit, consider the geodesic problem

$$y = y(x) \quad \text{for} \quad x_1 \leqq x \leqq x_2,$$

$$y(x_1) = 0, \qquad y(x_2) = b,$$

$$J(y) = \int_{x_1}^{x_2} (1 + (y')^2)^{1/2} dx = \min.$$

Let us introduce the function

$$z(x) = \int_{x_0}^{x} (1 + (y')^2)^{1/2} dx,$$

and put $y' = \tan u$, $z' = \sec u$, $x_1 = 0$, $x_2 = 1$. The problem is now to determine $u(x)$ such that subject to the constraints $y' = \tan u, Z' = \sec u, y(x_1) = 0, Z(x_1) = 0, y(x_2) = b$ we have

$$z(x_2) = \int_{0}^{x_2} (1 + (y')^2)^{1/2} dx = \min,$$

which is a Problem of Mayer. The solution $n = 6$ is known a priori. The properties of the solution can be most easily shown in a figure.

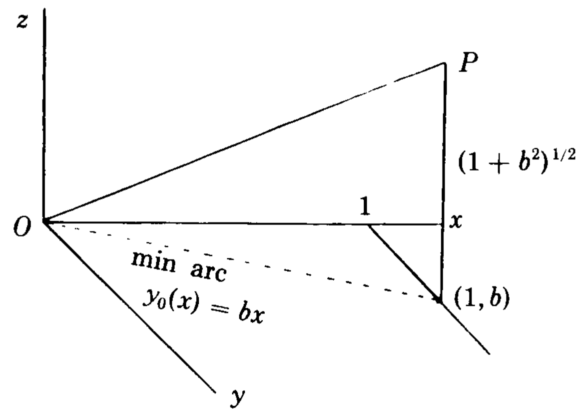

FIGURE 13. Locus for Solution Curve

The line OP represents the locus of points of z for the solution curve $y_0(x)$. For *any* variation from the true solution, the corresponding value of z must be larger than the value of z on OP for the same set of values of (x, y). We see that there is a hyperbolic cone of reachable points. The line OP is on the boundary of the cone. If we draw the intersection of the cone with the plane $x_2 = 1$, we obtain Figure 14. Even for the simple problem discussed here, there are points $z(x_2)$ that cannot be reached, regardless of the control available.

This lack of complete controllability is typical of Problems of Mayer.

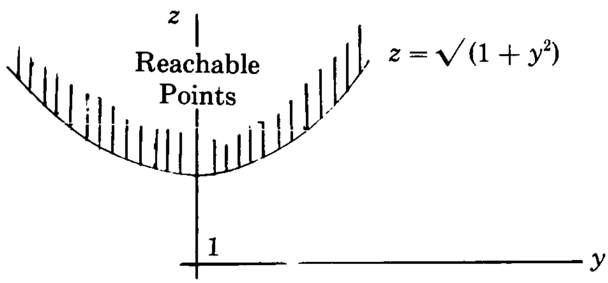

FIGURE 14. Reachable Points

Finally, let us discuss a particular case of the control problem where we have constraints of the form $|u| \leqq 1$.
Let the problem be to approach the origin in $\dot{x}$, x phase space in minimum time, subject to a control u and differential equations

$$\dot{x}^1 = x^2, \;\; \dot{x}^2 = u.$$

The function $H = p_1 x^2 + p_2 u$. We must choose u to maximize H subject to $|u| \leqq 1$. Carrying out the steps

$$\dot{p}_1 = 0, \qquad \text{hence } p_1 = c_1 = \text{constant},$$

$$\dot{p}_2 = -p_1, \qquad \text{hence } p_2 \text{ is linear in } t,$$

i.e., $p_2 = c_2 - c_1 t$. For fixed time, we will maximize H by selecting u. It is straightforward to show that if $p_2(t) > 0$, $u = 1$, and if $p_2(t) > 0$, $u = -1$. The solution can be written

$$u = \operatorname{sign}(c_2 - c_1 t)$$

in phase space. Starting at, say point P_1 in phase space, the trajectory follows the curve shown in Figure 15. First $u = -1$ up to point A; then $u = 1$ to the origin. Similar remarks hold for point P_2.

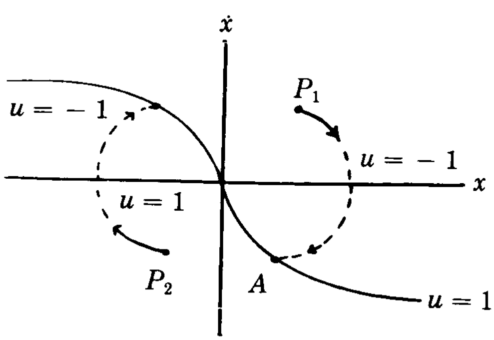

FIGURE 15. Control with Constraints

VI. **Problem transformations.** By means of simple transformations, we can show that all of the problems are, in principle, the same.

The general control problem is given in terms of $x = x^i(t)$ state variables, $t^0 \leqq t \leqq t^1$, $i = 1, \cdots, q$; $u = u^k(t)$ control variables, $t^0 \leqq t \leqq t^1$, $k = 1, \cdots, n$, subject to differential equations

$$\dot{x}^i = f^i(t, x, u),$$

and the formulation may depend on other parameters

$$w^\sigma \qquad \sigma = 1, \cdots, r,$$

and may be constrained by conditions such as

$$\phi_\alpha(u) = 0,$$

$$\phi_\beta(u) \geqq 0,$$

$$\phi_\alpha(t, u) = 0,$$

$$\phi_\alpha(t, x, u) \geqq 0, \text{ etc.}$$

As Case (i), consider the constraint

$$\phi_\alpha(t, x, u) = 0 \tag{VI.1}$$

with end points expressed parametrically as

$$\begin{aligned} t^0 &= T^0(w), & x^i(t^0) &= X^{i0}(w), \\ t^1 &= T^1(w), & x^i(t^1) &= X^{i1}(w). \end{aligned} \tag{VI.2}$$

We impose isoperimetric conditions

$$J(x) = g_\gamma(w) + \int_{t^0}^{t^1} f_\gamma(t, x(t), u(t))\, dt = 0, \tag{VI.3}$$

and we wish to make

$$J(x) = g(w) + \int_{t^0}^{t^1} f(t, x(t), u(t))\, dt = \min. \tag{VI.4}$$

If we have a problem with constraints of the form

$$\phi_{\alpha'}(t, x, u) = 0, \qquad \alpha' = 1, \cdots, m'$$

$$\phi_{\alpha''}(t, x, u) \geqq 0, \qquad \alpha'' = m' + 1, \cdots, m,$$

then to get the formulation, Case (i), we can introduce more functions u by writing

$$\phi_{m'+j}(t, x, u) = (u^{n+j})^2$$

or

$$\phi_{m'+j}(t, x, u) - (u^{n+j})^2 = 0, \qquad j = 1, \cdots, m - m',$$

which are just $m - m'$ more constraints of the desired form $\phi(t, x, u) = 0$. This method, however, does introduce singularities, so caution is in order. Isoperimetric inequalities can be similarly transformed. In principle, Case (i) contains all problems which include inequality constraints.

Let us discuss now the Isoperimetric Problem of Bolza:

$$x = x^i(t) \quad \text{for} \quad t^0 \leqq t \leqq t^1 \qquad i = 1, \cdots, q$$

with constraints $\phi_\alpha(t, x, \dot{x}) = 0$, $\alpha = 1, \cdots, m$

$$\text{(VI.5)} \qquad J_\gamma(x) = g_\gamma(t^0, x(t^0), t^1, x(t^1)) + \int_{t^0}^{t^1} f_\gamma(t, x(t), \dot{x}(t))\, dt = 0, \qquad \gamma = 1, \cdots, p.$$

$$\text{(VI.6)} \qquad J_0(x) = g_0(t^0, x(t^0), t^1, x(t^1)) + \int_{t^0}^{t^1} f_0(t, x(t), \dot{x}(t))\, dt = \min.$$

Remarks similar to those made above hold here for inequality constraints.

Consider, as special cases

Case (i)

$$\text{(VI.7)} \qquad f_\gamma \equiv 0, \quad \text{the Problem of Bolza,}$$

Case (ii)

$$\text{(VI.8)} \qquad f_\gamma \equiv 0, \qquad f_0 \equiv 0, \quad \text{the Problem of Mayer,}$$

Case (iii)

$$\text{(VI.9)} \qquad f_\gamma \equiv 0, \qquad g \equiv 0, \quad \text{the Problem of Lagrange.}$$

We will show that all three problems are basically the same, first showing that the functions f can be eliminated, i.e., we can write equivalent problems involving no integrals.

Let the problem be

$$x = x^i(t),$$

$$y = y^\rho(t)$$

where

$$y^{\rho}(t) = \int_{t^0}^{t} f_\rho \, dt, \qquad \rho = 0, 1, \cdots, p.$$

The differential equations are now

$$\phi_\alpha(t, x, \dot{x}) = 0,$$
$$\dot{y}^{\rho} - f_\rho(t, x, \dot{x}) = 0,$$

with side conditions

$$J_\gamma(x) = g_\gamma + y^{\gamma}(t^1) = 0,$$
$$y^{\rho}(t^0) = 0.$$

The problem reduces to a Problem of Mayer, for we now wish to make

$$J_0 = g_0 + y^0(t^1) = \min.$$

This transformation does not preserve the concept of strong neighborhoods.

Let us consider a more general Isoperimetric Problem of Bolza. Let the state variables be

$$x^i(t), w^\sigma \qquad i = 1, \cdots, q;\ \sigma = 1, \cdots, p;\ t^0 \leqq t \leqq t^1,$$

assuming that the state also depends on parameters w, and with the differential equations

$$P_\alpha(t, x, \dot{x}) = 0. \tag{VI.10}$$

We have end conditions

$$t^s = T^s(w), \qquad x^i(t^s) = X^{is}(w), \quad s = 0, 1 \tag{VI.11}$$

and constraints

$$J_\gamma(x) = g_\gamma(w) + \int_{t^0}^{t^1} f_\gamma(t, x, \dot{x}) \, dt = 0 \qquad \gamma = 1, \cdots, r; \tag{VI.12}$$

and we wish to make

$$J(x) = g(w) + \int_{t^0}^{t^1} f(t, x, \dot{x}) \, dt = \min. \tag{VI.13}$$

If w appears in the integrand of (VI.12), we merely add the new state variables $x^{q+\sigma}$ and the differential equations $\dot{x}^{q+\sigma} = 0$. Then the integrand in the constraints corresponding to (VI.12) contains no terms in w^σ.

To see that the problem consisting of (VI.5), (VI.6) and (VI.7) is a special case of this, let

$$t^0 = w^1, \qquad x^i(t^0) = w^{1+i}, \qquad t^1 = w^{q+2}, \qquad x^i(t^1) = w^{q+2+i}.$$

The problem (VI.10), (VI.11), (VI.12) and (VI.13) is, conversely, identical to (VI.5), (VI.6) and (VI.7). If we append to the set of differential equations associated with the latter problem the following, $\dot{w}^\sigma = 0$, i.e., $w^\sigma =$ constant, the end values, then we obtain the problem

$$x^i(t), \qquad w^\sigma(t).$$

Differential equations

$$\dot{w}^\sigma = 0,$$

$$P_\alpha = (t, x, \dot{x}) = 0,$$

with end conditions becoming the constraints with $f_\gamma \equiv 0$,

$$t^0 - T^0(w(t)) = 0, \qquad x^i(t^0) - X^{i0}(w(t^0)) = 0,$$

$$t^1 - T^1(w(t^1)) = 0, \qquad x^i(t^1) - X^{i1}(w(t^1)) = 0,$$

and we wish to make

$$J = g(w(t^0)) + \int_{t^0}^{t^1} f(t, x, \dot{x})\, dt = \min.$$

By similar arguments and transformations, it is possible to eliminate the constraint functions $J_\gamma(x)$ by transforming the variational problems to control problems. It is straightforward, conversely, to show that the control problem is a variational problem of one of the special types (VI.7), (VI.8) or (VI.9). Thus, all of the special types of problems we have formulated and discussed are basically the same. The type of formulation one chooses is a matter of taste, or convenience in application.

VII. **Methods of computation.** The method of steepest descent, or gradient method, can be most easily discussed in terms of functions of a finite number of variables. Let $f(x)$ be a function of n variables $x = (x^1, x^2, \cdots, x^n)$. The derivative of f in the direction h is

$$f'(x,h) = g \quad h = |g|\,|h| \cos\theta, \tag{VII.1}$$

where $g = \operatorname{grad} f = (\partial f/\partial x^1, \cdots, \partial f/\partial x^n)$, and θ is the angle between g and h. For fixed $|h|$, f' is greatest in the g direction.

Recall that at the minimum point x_0 we can expand

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0, x - x_0) + \cdots. \tag{VII.2}$$

If we truncate this expression at the second order term and set $f(x) =$ constant, we have an equation for an ellipsoid in x^n-space. Thus, starting with some approximate value of the solution x_0, we use the concept of the gradient, or direction of greatest change of f, to follow the "flow lines" from some ellipsoid $f(x_0) =$ constant to the minimum point of f, i.e., we must solve the set of equations

$$\frac{dx^i}{dt} = -f_{x^i} = -g^i. \tag{VII.3}$$

For numerical computations, (VII.3) is discretized to

$$\Delta x^i = -g^i \Delta t$$

or the iterative form

$$x^i_{n+1} = x^i_n - \alpha g^i_n. \tag{VII.4}$$

Equation (VII.4) is the Gradient Iteration Formula, and embodies in it the Method of Steepest Descent.

An advantage of gradient methods is that they pull the solution away from saddle points. However, they do encounter the difficulty in application that one may have to deal with narrow ellipsoids. To overcome this difficulty, one must apply special methods to choose α in (VII.4).

To discuss the gradient method for integrals, consider the problem

$$x = x(t) \qquad (t^0 \leqq t \leqq t^1)$$

$[t^0, x(t^0)]$, $[t^1, x(t^1)]$ held fast

$$J(x) = \int_{t^0}^{t^1} (f(t), x(t), \dot{x}(t))\, dt = \min.$$

We will admit corners in the minimizing arc

$$x_0 = x_0(t) \qquad (t^0 \leqq t \leqq t^1),$$

and we will call a variation h admissible if

$$h = h(t) \qquad (t^0 \leqq t \leqq t^1)$$

and $h(t^0) = 0$; $h(t^1) = 0$. Note the vectorial character of $h(t)$. If h is admissible, so is αh. If, in addition, g is admissible, so is

$$\alpha_1 h + \alpha_2 g.$$

We define the inner product of h and g as

(VII.5) $$g \cdot h = (g,h) = \int_{t^0}^{t^1} \dot{g}(t)\, \dot{h}(t)\, dt.$$

Let us define

(VII.6) $$g(t) = \int_{t^0}^{t} [f_{\dot{x}}(\tau, x_0(\tau), \dot{x}_0(\tau)) - \int_{t^0}^{\tau} f_x(x, x_0(s), \dot{x}_0(s))\, ds - c] d\tau,$$

where c is chosen so that

(VII.7) $$g(t^1) = 0.$$

Since $g(t^0) = 0$, g is an admissible variation. In fact, g is the gradient of J at x_0, hence

(VII.8) $$J'(x_0, h) = \int_{t^0}^{t^1} \dot{g}(t)\, \dot{h}(t)\, dt = g \cdot h,$$

where

$$\dot{g}(t) = f_{\dot{x}} - \int_{t^0}^{t} f_x ds - c.$$

If

$$|h|^2 = \int_{t^0}^{t^1} \dot{h}^2 dt$$

is held fast, then $J'(x_0, h)$ has a maximum value when $h = \alpha g$.

For numerical solution of this type of problem, we can use the concepts developed earlier for functions of n variables, i.e., (VII.4), but to use our definition of g, we would rewrite (VII.4) as

(VII.9) $$x_{n+1}(t) = x_n(t) - \alpha g_n(t).$$

For a general discussion of gradient methods, see [11].

The freedom of choice of the definition of gradient in these numerical methods is unconstrained. Suppose we define the dot or inner product of the functions $g(t)$, $h(t)$ as

$$g \cdot h = (g, h) = \int_{t^0}^{t^1} g(t)\, h(t)\, dt. \tag{VII.10}$$

For J, defined as before,

$$J'(x_0, h) = \int_{t^0}^{t^1} (f_x h + f_{\dot{x}} \dot{h})\, dt. \tag{VII.11}$$

Integrating by parts with h taken as admissible, we obtain

$$J'(x_0, h) = \int_{t^0}^{t^1} g h\, dt \tag{VII.12}$$

with

$$g = f_x - \frac{d}{dt}(f_{\dot{x}}). \tag{VII.13}$$

We could call g in (VII.13) the gradient. Analogous to (VII.3), we would have to solve

$$\frac{\partial x}{\partial s}(t, s) = \frac{d}{dt}(f_{\dot{x}}) - f_x = -g \tag{VII.14}$$

$x(t^0, s) = 0$, $x(t^1, s) = 0$, where $x(t, 0) = x_0$. For example, if

$$f = \frac{1}{2}\dot{x}^2 = \frac{1}{2}\left(\frac{\partial x}{\partial t}\right)^2,$$

$$f_{\dot{x}} = \dot{x},$$

then we have the system

$$\frac{\partial x}{\partial s} = \frac{\partial^2 x}{\partial t^2},$$

with $x(t^0, s) = 0$, $x(t^1, s) = 0$, and $x(t, 0) = x_0(t)$. Note that by using the gradient approach in this simple example, we obtain a heat equation which gives the set of flow lines of the energy integral.

Of course, the above problem of minimization could have been handled by what Courant and Hilbert [**3**] call Indirect Methods; that is, by solving the corresponding Euler equation

$$\frac{d}{dt}(f_{\dot{x}}) = f_x$$

subject to the two point conditions $x(t^0)$, $x(t^1)$ fixed. In the same book, Direct Methods are discussed. For example, if we define $\mu = \inf J(x)$ for all admissible x, the problem is to find μ by constructing a minimizing sequence x_q such that

$$\lim_{q \to \infty} J(x_q) = \mu.$$

On the other hand, we could approach the problem by (a) finding μ, (b) showing that $\overline{x_q} \to x_0$, and then (c) $\mu = \lim J(x_q) \geqq J(x_0)$. The latter approach is that of the Tonelli School in Italy, and stems from work by Weierstrass and Hilbert.

As the first direct method, consider the following basically Eulerian technique for obtaining a minimizing sequence. Suppose the interval of interest is $(t^0 \leqq t \leqq t^1)$. Divide the interval into q sub-intervals of length

$$h = \frac{t^1 - t^0}{q},$$

where q is some integer. Then the integral $\int_{t^0}^{t^1} f\,dt$ can be approximated by a function of q variables $\xi_1, \xi_2, \cdots, \xi_q$,

$$F(\xi_1, \xi_2, \cdots, \xi_q),$$

and the minimum of this function can be obtained by the usual methods for functions of n variables. One disadvantage of this method is that it usually involves too many variables.

A second direct method, which is very useful when the side conditions are linear and the integrand functions are quadratic, is the Rayleigh-Ritz. The details of this method are discussed in Courant-Hilbert [**3**].

For another approach, we observe that the admissible variation

$$h = x(t) - x_0(t)$$

has the properties

$$h(t^0) = h(t^1) = 0.$$

We can estimate h by choosing a complete set of functions $h_k(t)$, $k = 1, 2, \cdots$, which vanish at t^0 and t^1, for example, if $t^0 = 0$, $t^1 = 1$:

(VII.15) $$h_k(t) = \sin \frac{\omega_k t}{t^1} \qquad (\omega_k = k\pi).$$

We then write

(VII.16) $$h(t) = \sum_{k=1}^{\infty} \alpha_k h_k(t).$$

Hence

(VII.17) $$x(t) = x_0(t) + \alpha_1 h_1(t) + \cdots + \alpha_q h_q(t) + \cdots.$$

Thus

(VII.18) $$J(x) = F(\alpha_1, \cdots, \alpha_q)$$

if we terminate (VII.17) at the qth term. Thus we again have the problem of minimizing a function of q variables. The effectiveness of this method depends, as does the effectiveness of Raleigh-Ritz, on the choice of the functions $h_k(t)$. It is a type of Rayleigh-Ritz method.

Side conditions of the form

$$K(x) = \int_{t0}^{t1} g(x,t)\, dt = C$$

merely impose on the problem of minimizing $F(\alpha)$ conditions of the form

$$G(\alpha_1, \alpha_2, \cdots, \alpha_q) = \text{constant}.$$

Iterative methods play a dominant role in the problem of minimizing integrals and functions of a finite number of variables. In general, we are given the task of finding the minimum of $F(x,y,z)$. If we guess a set x_0, y_0, z_0 that is close to the answer, we will get convergence of an iterative scheme, which we can formulate as follows:

Given x_0, y_0, z_0:

(1) minimize $F(x, y_0, z_0)$, solution: x_1

(2) minimize $F(x_1, y, z_0)$, solution: y_1

(3) minimize $F(x_1, y_1, z)$, solution: z_1,

and so on; this is a Gauss-Seidel-like procedure.

Let us define formally an iterative procedure. Let x_0 be an initial guess (a vector). Then we write

(VII.19) $$x_{q+1} = x_q + \alpha_q h_q$$

as our iterative process; h_q is essentially a choice of direction along which we go from the qth estimate to the $q+1$th estimate; α_q determines how far we go in that direction. To use (VII.19), we must have a program for selecting α_q and h_q.

One form of Newton's method for finding the minimum of a function of n variables is basically written in the form (VII.19). If

(VII.20) $$F(x+\alpha h) = F(x) + \alpha F'(x,h) + \frac{\alpha^2}{2} F''(x,h),$$

truncating the Taylor Series at second order terms, then we minimize the right-hand side with respect to α and obtain

$$\alpha = \frac{-F'(x,h)}{F''(x,h)},$$

and hence we take

(VII.21) $$\alpha_q = \frac{-F'(x_q,h_q)}{F''(x_q,h_q)}.$$

The Method of Conjugate Gradients given by Hestenes and Steifel in [12] is a variant of (VII.19). For a discussion of iterative methods for linear systems, see [13]. The gradient method, as applied to the problem of minimizing an integral, is discussed in [14].

Let us discuss in some detail an iterative method for finding the minimum of a function of n variables $F(x) = F(x^1, x^2, \cdots, x^n)$. By iterating on x, we hope to improve a given approximation of the solution by choosing a δx such that

$$\mathfrak{x} = x + \delta x = x + \alpha h,$$

i.e., $x_{q+1} = x_q + \alpha_q h_q$. Program for α:

(a) $\alpha_q = \alpha =$ constant. Usually if α is chosen too large for convergence, choose $\alpha^* = \alpha/2$; if too small, choose $\alpha^* = 2\alpha$. One can also step α to find the value for quickest convergence.

(b) $\alpha_q = -\beta(F'(x_q,h_q)/F''(x_q,h_q))$, where we have added a scale factor, β $(1-\epsilon \leqq \beta \leqq 1+\epsilon)$. If $\beta < 1$, one is under-relaxing; if $\beta > 1$, over-relaxing.

Program for h_q:

(a) Choose n linearly independent vectors

$$u_1, u_2, \cdots, u_n.$$

Choose h_q successively from the sequence

$$u_1, u_2, \cdots, u_n, u_1, \cdots .$$

This is the usual Gauss-Seidel procedure. Any combination of the u_j can be made.

(b) $h_q = -\operatorname{grad} F$. Recall $F'(x_0, h) = F_{x^i} h^i = \operatorname{grad} F \cdot h$. Usually we define the dot product of the two vectors x, y as

$$x \cdot y = x_i y_i.$$

We could define

$$x \cdot y = \sum g_{ij} x_i y_j,$$

a positive definite form. Then, for quick convergence, we could write

$$(\operatorname{grad} F)_i = g^{ij} \frac{\partial F}{\partial x^j},$$

where $g_{ij}, g^{jk} = \delta_i^k$, and then choose g^{ij} so that $\operatorname{grad} F$ points toward the minimum point, not normal to $F =$ constant as is usually the case. This implies a particular choice of h_q.

Newton's method appears in all phases of numerical analysis. When solving the equation

$$G(x) = 0,$$

we write

$$G(x + \delta x) = G(x) + G'(x)\, \delta x,$$

set the right-hand side equal to zero, and pick

$$\delta x = -\frac{G(x)}{G'(x)}.$$

For a system of equations $G_i(x) = 0$, we put

$$0 = G_i(x + \delta x) = G_i(x) + \frac{\partial G_i}{\partial x^j} \delta x^j.$$

Then we put $g_{ij}(x) = (\partial G_i / \partial x^j)$, not necessarily a symmetric matrix, and then let

$$\delta x^j = -g^{jk} G_k(x).$$

If $G_i = \partial F / \partial x^i$, then $g_{ij} = (\partial F)/\partial x^i \partial x^j$ is symmetric. For quadratic functions F, we choose $(\operatorname{grad} F)_i = g^{ij}(\partial F/\partial x^j)$; but if F is of an indefinite form, we may get saddle points. Suppose we are solving a minimization problem with constraints

$$f(x) = \min,$$
$$g(x) = 0.$$

The first necessary condition is

$$f_{x^i} + \lambda g_{x^i} = 0.$$

By Newton's method,

$$f_{x^i} + \lambda g_{x^i} + (f_{x^i x^j} + \lambda g_{x^i x^j})\,\delta x^i + \delta\lambda g_{x^i} = 0,$$

and

$$g + g_{x^j}\delta x^j = 0.$$

To solve these equations for δx^j and $\delta\lambda$, we must have

$$\begin{vmatrix} f_{x^i x^j} + \lambda g_{x^i x^j} & g_{x^i} \\ g_{x^j} & 0 \end{vmatrix} \neq 0,$$

and then to iterate, we put

$$x^j_{q+1} = x^j_q + \delta x^j,$$
$$\lambda_{q+1} = \lambda_q + \delta\lambda_q.$$

Finally, let us consider Newton's method for finding the solution of a simple differential equation

$$T = 1 + y'^2 - yy'' = 0 \qquad \text{subject to } y(a) = A, y(b) = B,$$
$$\text{with } y > 0;$$
$$\text{hence } y'' > 0.$$

This is the catenary problem. To solve this, guess a function $y(x)$ to satisfy the boundary conditions, and set $T + \delta T = 0$, i.e.,

$$T + 2y'\delta y' - \delta y y'' - y\delta y'' = 0$$

with $\delta y(a) = 0$, $\delta y(b) = 0$. Improve on the guess by solving this linear equation for δy.

This method is also applicable to simple and multiple integrals.

References

1. L. S. Pontryagin, *The mathematical theory of optimal processes*, Interscience, New York, 1962.

2. G. Leitmann, ed. *Optimization techniques, with applications to aerospace systems*, Academic Press, New York, 1962.

3. R. Courant and D. Hilbert, *Methods of mathematical physics*. VI, Interscience, New York, 1953.

4. C. Carathéodory, *Variationsrechnung und Partialle Differentialgleichungen Erster Ordnung*, B. G. Teubner, Berlin, 1935.

5. G. A. Bliss, *Calculus of variations*, Open Court, LaSalle, Illinois, 1925.

6. ———, *Lectures on calculus of variations*, Univ. Chicago Press, Chicago, Illinois, 1946.

7. R. Weinstock, *Calculus of variations, with applications to physics and engineering*, McGraw-Hill, New York, 1952.

8. O. Bolza, *Lectures on the calculus of variations*, Stechert-Hafner, New York, 1946.

9. ———, *Vorlesungen über Variationsrechnung*, B. G. Teubner, Berlin, 1909; reprinted by K. F. Koehlers, Leipzig, 1933.

10. N. I. Akheizer, *Calculus of variations*, Blaisdell, New York, 1962.

11. Marvin L. Stein, *On methods for obtaining solutions of fixed end-point problems in the calculus of variations*, J. Res. Nat. Bur. Standards, RP 2418, **50** (1953), 277-297.

12. M. R. Hestenes and E. S. Steifel, J. Res. Nat. Bur. Standards, RP 2379, **49**(1952), 409-436.

13. Robert M. Hayes, *Iterative methods of solving linear problems on Hilbert space*, Contributions to the Solution of Systems of Linear Equations, Nat. Bur. Standards Appl. Math. Ser. No. 39, U.S. Gov. Printing Office, Washington, D.C., 1954, pp. 71-103.

14. M. R. Hestenes, *Numerical methods for obtaining solutions of fixed end-point problems in the calculus of variations*, Rand Report No. 102, 1949.

15. ———, *A general problem in the calculus of variations with applications to paths of least time*, RM 100, The RAND Corp., 1950; see also ASTIA Document No. 112382.

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