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Periodic Motions in Banach Space and Applications to Functional-Differential Equations

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1. Introduction

In establishing the existence of periodic solutions for nonautonomous differential equations of the form

$$\dot{x} = g(x, t),$$

where g is periodic in t of period ω for fixed x, it is often convenient to consider the translation operator $T(x(t)) = x(t + \omega)$. If corresponding to each initial vector chosen in an appropriate region there corresponds a unique solution of our equation, then periodicity may be established by proving the existence of a fixed point under T. This same technique is also useful for more general functional equations and can be extended in a number of interesting ways. In this paper we shall consider a variable type of translation operator which is useful in investigating periodicity for autonomous differential and functional equations where the period involved is less obvious.

Let us denote by X a Banach space, by R^+ the nonnegative real line, and by $\mathscr{F}$ the set of continuous functions F: $X \times R^+ \to X$ such that $F(x, 0) = x$ for arbitrary x in X, and

$$F(F(x, t_1), t_2) = F(x, t_1 + t_2) \tag{1.1}$$

for arbitrary x in X and t_1, t_2 in R^+. We shall say that F in $\mathscr{F}$ has a periodic motion of period $p > 0$ if there exist y in X such that

$$F(y, t + p) = F(y, t)$$

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for all t in R^+. Our interest is in determining conditions under which maps in $\mathscr{F}$ are nontrivially periodic and in determining, where possible, information of a more quantitative nature concerning this motion. To this end let us call F in $\mathscr{F}$ λ-continuous relative to $Y \subset X$ if there exist numbers a and b, where $b \geqq a > 0$, and a continuous functional λ: $Y \to [a, b]$ such that the mapping f: $Y \to X$ defined by the formula

$$f(x) = F(x, \lambda(x))$$

is completely continuous. For subsets A and B of X we adopt the notation $\bar{A}$, A^0, $\partial(A)$, and $A \setminus B$ for the closure of A, the interior of A, the boundary of A, and the set of all elements in A not contained in B respectively. For x a point in X, $\|x\|$ denotes the norm of x, and $N(x, \epsilon) = \{y\colon \|y - x\| < \epsilon\}$, $\epsilon > 0$. For x and y in X, $\widehat{xy}$, $\overline{xy}$, and $r(xy)$ shall denote the sets $\{u\colon u = x + \lambda(y - x),\ \lambda \text{ in } (0, 1)\}$, $\{u\colon u = x + \lambda(y - x),\ \lambda \text{ in } [0, 1]\}$, and $\{u\colon u = x + \lambda(y - x),\ \lambda \geqq 0\}$ respectively. We shall prove the following two closely related theorems.

THEOREM 1. *Let F in $\mathscr{F}$ be λ-continuous relative to $\bar{Y}$, where $Y \subset X$ is a bounded open convex set such that* 0 *is not contained in Y. Let $f(\bar{Y}) \subset \bar{Y}$ and suppose that there exists a constant $k > 0$ such that f maps $\bar{Y} \setminus N(0, \epsilon)$ into $\bar{Y} \setminus N(0, k\epsilon)$ for all ϵ sufficiently small. Suppose further that for any constant $M > 0$ there exists an integer n such that f^n maps $\bar{Y} \setminus N(0, \epsilon)$ into $\bar{Y} \setminus N(0, M\epsilon)$ for all ϵ sufficiently small. Then F has a nontrivial periodic motion of period p such that $a \leqq p \leqq b$.*

THEOREM 2. *Let F in $\mathscr{F}$ be λ-continuous relative to $\bar{Y}$, where $Y \subset X$ is a bounded open convex set such that* 0 *is not contained in Y. Let $f(\bar{Y}) \subset \bar{Y}$ and suppose there exists a constant $k > 0$ such that $\|f(x)\| \geqq k\,\|x\|$ for $\|x\|$ sufficiently small. Suppose further that for any constant $M > 0$ there exists an integer n such that $\|f^n(x)\| \geqq M\,\|x\|$ for $\|x\|$ sufficiently small. Then F has a nontrivial periodic motion of period p such that $a \geqq p \geqq b$.*

We observe that if there is a point x^* in $\bar{Y}$ such that $x^* \neq 0$ and $f(x^*) = x^*$, then (1.1) implies

$$F(x^*, t) = F(F(x^*, \lambda(x^*)), t) = F(x^*, t + \lambda(x^*)).$$

Hence Theorem 1 and Theorem 2 are proved once it is established that f has a nontrivial fixed point in $\bar{Y}$ under the specified hypotheses. In section 2 we shall develop and prove fixed point theorems for this purpose.

In Section 3 we shall apply Theorem 2 in establishing the existence of periodic solutions of functional-differential equations of the form

$$\dot{x}(t) = L(H_t x) + G(H_t x), \tag{1.2}$$

where for some specified $h > 0$, $H_t x(\theta) = x(t + \theta)$ for θ in $[-h, 0]$, L is a continuous linear operator, and G is a continuous operator of higher order. Finally in Section 4 we shall discuss some specific examples of equations of the form (1.2) which have surprisingly simple and very interesting periodic behavior.

2. Asymptotic Fixed Point Theorems

A theorem asserting the existence of a fixed point under a mapping f by utilizing information known about f^n for n sufficiently large is suggestively called an asymptotic fixed point theorem. In this section we prove the following two essentially equivalent theorems of this type.

THEOREM 3. *Let Y be a bounded open convex subset of a Banach space X and suppose the point x_0 is not contained in Y. Let f be a completely continuous mapping of $\bar{Y}$ into itself and suppose there exists a constant $k > 0$ such that f maps $\bar{Y} \setminus N(x_0, \epsilon)$ into $\bar{Y} \setminus N(x_0, k\epsilon)$ for all ϵ sufficiently small. Suppose further that for any constant $M > 0$ there exists an integer n such that f^n maps $\bar{Y} \setminus N(x_0, \epsilon)$ into $\bar{Y} \setminus N(x_0, M\epsilon)$ for all ϵ sufficiently small. Then f has a fixed point $x^* \neq x_0$* in $\bar{Y}$.

THEOREM 4. *Let Y be a bounded open convex subset of a Banach space X and let x_0 be a boundary point of Y. Let f be a completely continuous mapping of $\bar{Y}$ into itself. Suppose there exists a constant $k > 0$ such that $\|f(x) - x_0\| \geqq k\| x - x_0\|$* for *$\|x - x_0\|$ sufficiently small. Suppose further that for any constant $M > 0$ there exists an integer n such that $\|f^n(x) - x_0\| \geqq M \|x - x_0\|$ for $\|x - x_0\|$ sufficiently small. Then f has a fixed point $x^* \neq x_0$ in $\bar{Y}$.*

Let us define a mapping f to be compact if for any subset $S \subset X, f(S)$ is contained in a compact subset of X. It is worth noting that if we require f in Theorem 3 and Theorem 4 to be compact in place of completely continuous, then we can drop the condition that Y be bounded. This follows from the fact that in this situation $f(Y)$ is contained in a bounded open convex set W and Y may be replaced by $Y \cap W$.

Referring to the remarks immediately following Theorem 2, it is clear that Theorem 1 is an immediate consequence of Theorem 3 and Theorem 2 is an immediate consequence of Theorem 4.

Now letting A be an arbitrary subset of X and ξ an arbitrary point in $X \setminus A$ we introduce the notation $C(\xi, A)$ for the cone with a vertex at ξ and generated by A. That is,

$$C(\xi, A) = [x : x = (1 - \lambda)\xi + \lambda y, y \text{ in } A, \lambda > 0].$$

We recall that a set $S \subset X$ and containing a point x_1 such that x in S implies $\overline{x_1 x} \subset S$ is referred to as a star set. A set $S \subset A$ is called a star body if there exists a point x_1 in S^0 with respect to which S is a star set and if for each y in X the ray $r(x_0 y)$ intersects $\partial(S)$ in at most one point.

We shall now present a series of lemmas concerning convex and topologically convex subsets of X which will be useful in establishing Theorem 3.

LEMMA 1. *Let S be a bounded star body of X with respect to an interior point ξ. Then there exists a unique positive continuous functional λ on $X \setminus \{\xi\}$ and a unique continuous mapping $\omega: X \setminus \{\xi\} \to \partial(S)$ such that for each x in $X \setminus \{\xi\}$,*

$$x = (1 - \lambda(x))\xi + \lambda(x)\omega(x).$$

LEMMA 2. *Let A be a convex subset of X with a boundary point x_0 and an interior point x_1. For each $\mu > 0$ let*

$$x_\mu = x_1 + [\mu/(\|x_0 - x_1\|)](x_0 - x_1).$$

Then there exist positive constants v and η such that

$$C(x_\mu, N(x_0, \eta\mu)) \cap A \subset N(x_0, \mu) \cap A$$

for all μ in $(0, v)$.

Since Lemma 1 and Lemma 2 are special cases of Lemmas 1 and 3 of [11], they will not be proved in this paper.

LEMMA 3. *Let Y be a convex subset of X with a boundary point x_0 and an interior point ξ. Let $N(x_0, \epsilon)$ be such that $\|\xi - N(x_0, \epsilon)\| > 0$ and let $x_1 = (1 - \lambda_1)\xi + \lambda_1 x_0$, where $0 < \lambda_1 < 1$ and $\|x_1 - N(x_0, \epsilon)\| > 0$. Let $C = C(x_1, N(x_0, \epsilon) \cap Y)$ and let $x_2 = (1 - \lambda_2)x_1 + \lambda_2\xi$ for some $\lambda_2 > 0$. Then any ray $r(x_2 y)$ for y in $\overline{C \cap Y}$ must intersect $\partial(C)$ and $\partial(Y)$, once and only once each. Furthermore, if $u_1 = (1 - \rho_1)x_2 + \rho_1 y$, $\rho_1 > 0$, is contained in $\partial(C)$ and $u_2 = (1 - \rho_2)x_2 + \rho_2 y$, $\rho_2 > 0$, is in $\partial(Y)$, then $\rho_2 \geqq 1 \geqq \rho_1$.*

Proof. For the convenience of the reader the construction under consideration is suggested by Figure 1.

Let us observe that the interior of C is precisely the set of all points x which may be expressed by the formula

$$x = x_1 + \lambda(x_0 - x_1) + \lambda y_1, \tag{2.1}$$

where $\lambda > 0$ and $\|y_1\| < \epsilon$. Now for arbitrary y in $\overline{C \cap Y}$,

$$\begin{aligned} r(x_2 y)(\rho) &= x_2 + \rho(y - x_2) \\ &= x_2 + \rho(x_1 + \eta(x_0 - x_1) + \eta y_2 - x_2), \end{aligned}$$

where $\rho > 0$, $1 \geqq \eta \geqq 0$, and $\|y_2\| \leqq \epsilon$. Since for some constant $\eta_1 > 0$, we have $x_2 = x_1 - \eta_1(x_0 - x_1)$ it follows that

$$\begin{aligned} r(x_2y)(\rho) &= x_1 - \eta_1(x_0 - x_1) + \rho(\eta(x_0 - x_1) + \eta y_2 + \eta_1(x_0 - x_1)) \\ &= x_1 + [\rho(\eta + \eta_1) - \eta_1](x_0 - x_1) + \rho\eta y_2. \end{aligned}$$

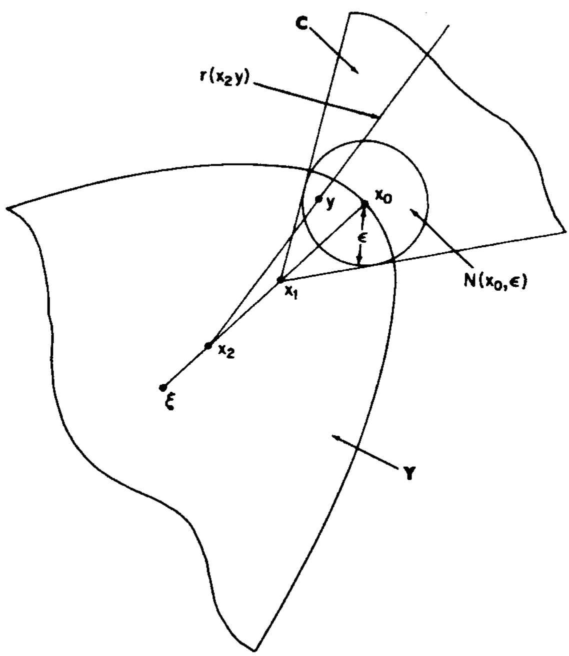

Figure 1

Since $[\rho(\eta + \eta_1) - \eta_1] > \rho\eta$ for all $\rho > 1$, we may conclude from 2.1 that $r(x_2y)(\rho)$ is an interior point of C for all $\rho > 1$. Hence the convexity of C implies $r(x_2y)$ intersects $\partial(C)$ at a unique point $u_1 = (1 - \rho_1)x_2 + \rho_1 y$ contained in $\bar{Y}$ and it is clear that $\rho_1 \leqq 1$. By the convexity of Y, $r(x_2y)$ can intersect $\partial(Y)$ in at most one point and by the boundedness of $C \cap Y$ it must do so in at least one point $u_2 = (1 - \rho_2)x_2 + \rho_2 y$ where $\rho_2 \geqq 1$. Therefore, we have established our lemma.

LEMMA 4. *Let Y be a closed and bounded convex subset of X with an interior point x_1 and an exterior point x_2. Let $x_0 = \widehat{x_1x_2} \cap \partial(Y)$ and let $v_1 = (1 - \lambda_1)x_1 + \lambda_1 x_0$ and $v_2 = (1 - \lambda_2)x_1 + \lambda_2 x_0$, where $1 > \lambda_1 > \lambda_2 > 0$. If $\|v_1 - x_0\| > r_2 > r_1 > 0$ and $C(x_2, \overline{N(x_0, r_2)}) \subset C(x_2, Y^0)$, then there* *exists a homeomorphism φ defined on Y such that $\varphi(Y \setminus C(v_2, N(x_0, r_2)))$ and $\varphi(Y \setminus C(v_1, N(x_0, r_1)))$ are convex.*

Proof. Let $\xi = (1 - \lambda_3)x_1 + \lambda_3 x_0$ where $\lambda_2 > \lambda_3 > 0$ and let $C_1 = C(v_1, N(x_0, r_1))$ and $C_2 = C(v_2, N(x_0, r_2))$. Clearly ξ is an interior point of the convex sets Y and $Y \cap C(x_1, C_2 \cap Y)$ and an exterior point of C_1

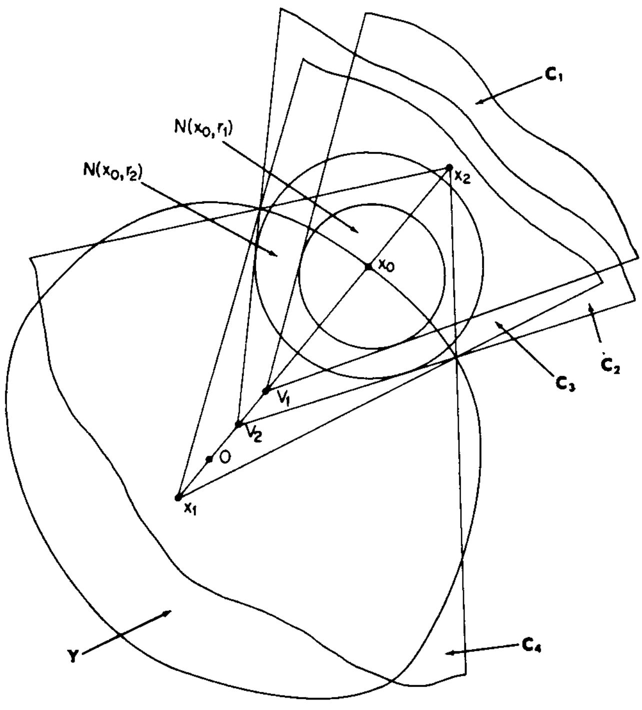

Figure 2

and C_2. Letting $C_3 = C(x_1, C_2 \cap Y)$ and $C_4 = C(x_2, C_2 \cap Y)$, we have that ξ is also an interior point of the convex set $C_3 \cap C_4$. We assume without loss of generality that $\xi = 0$ and depict our construction thus far in Figure 2.

Now by Lemma 1 we have the existence of a unique positive continuous functional α on $X \setminus \{0\}$ and a unique continuous mapping ω: $X \setminus \{0\} \to \partial(Y)$ such that for each x in $X \setminus \{0\}$

$$x = \alpha(x)\omega(x).$$

For each x in $Y \setminus C(0, C_2)$, we define

$$\varphi_1(x) = \beta(\omega(x))x,$$

where $\beta(\omega(x))\omega(x)$ is the unique point of intersection of the segment

$$\overline{0\omega(x)} = \{y : y = \lambda\omega(x), \lambda \text{ in } [0, 1]\}$$

and $\partial(C_3)$. Again utilizing Lemma 1 we have that β is a continuous functional on $\partial(C_3) \cap Y$ and we observe that it is both bounded and bounded away from zero on $\partial(C_3) \cap Y$. Hence clearly φ_1 and $\varphi_1{}^{-1}$ are well defined and continuous. We observe that $\varphi_1(x) = x$ for $\omega(x)$ in $\partial(C_2)$. For x in $\overline{C(0, C_2)} \setminus C_2$, we define φ_2 by the formula

$$\varphi_2(x) = \frac{1}{\gamma(\omega(x))}\, x,$$

where $\gamma(\omega(x))\omega(x)$ is the unique point of intersection between $\overline{0\omega(x)}$ and $\partial(C_2)$. Lemma 3 clearly implies $Y \setminus C_2$ is a star body, so Lemma 1 implies γ is continuous. Since we may easily observe that γ is both bounded and bounded away from zero it follows that φ_2 and $\varphi_2{}^{-1}$ are well defined and continuous. Since $\varphi_2(x) = x$ for $\omega(x)$ in $\partial(C_2)$, it is clear that defining $\varphi = \varphi_1$ on $Y \setminus C(0, C_2)$ and $\varphi = \varphi_2$ on $\overline{C(0, C_2)} \setminus C_2$ we have that φ is a homeomorphism on $\overline{Y \setminus C_2}$ and

$$\varphi(\overline{Y \setminus C_2}) = C_3 \cap Y.$$

Now for x in $C_2 \cap Y$, let $\sigma(\omega(x))\omega(x)$, $\sigma(\omega(x)) \geqq 1$, be the unique point of this form in $\partial(C_4)$. Let $\theta(\omega(x))\omega(x)$ denote the unique point of $\overline{0\omega(x)} \cap \partial(C_1)$ if $\overline{0\omega(x)}$ intersects C_1 in $\overline{C_3 \cap C_4}$ and let $\theta(\omega(x)) = 1$ otherwise. We define φ_3 on $C_2 \cap Y$ by the formula

$$\varphi_3(x) = \omega(x) + \frac{(\sigma(\omega(x)) - 1)}{(\theta(\omega(x)) - \gamma(\omega(x)))}(x - \gamma(\omega(x))\omega(x)).$$

Now it is clear that $\theta(\omega(x)) - \gamma(\omega(x)) > 0$ on $C_2 \cap Y$, and by Lemma 1 we have that ω, σ, θ, and γ are continuous. Hence φ_3 is continuous. Our hypothesis that C_2 be contained in $C(x_2, Y^0)$ implies that $\sigma(\omega(x)) > 1$, so it follows that $\varphi_3{}^{-1}$ is well defined and continuous. Extending our definition of φ to $C_2 \cap Y$ be the formula $\varphi(x) = \varphi_3(x)$ we observe that

$$\varphi(Y \setminus C(v_1, N(x_0, r_1))) = \overline{C_3 \cap C_4}$$

which is, of course, convex. Hence to complete our proof we have only to show that φ and φ^{-1} are continuous on $\partial(C_2) \cap Y$ and $\partial(Y) \cap \bar{C}_2$ respectively.

Let x be contained in $\partial(C_2) \cap Y^0$. Then $x = \gamma(\omega(x))\omega(x)$, where $\gamma(\omega(x)) < 1$. Since $\theta(\omega(x)) < 1$ implies $\theta(\omega(x)) > \gamma(\omega(x))$, we have that $\theta(\omega(x)) - \gamma(\omega(x)) = a > 0$. Now for y in C_2 we have

$$\varphi(y) - \varphi(x) = \omega(y) - \omega(x) + \frac{(\sigma(\omega)(y)) - 1)}{(\theta(\omega(y)) - \gamma(\omega(y)))}(y - \gamma(\omega(y))\omega(y)).$$

Clearly we can choose a neighborhood $N(x, \epsilon)$ such that y in $N(x, \epsilon) \cap C_2$ implies $\theta(\omega(y)) - \gamma(\omega(y)) > a/2$, and

$$\|\varphi(y) - \varphi(x)\| \leqq \|\omega(y) - \omega(x)\| + \frac{2(\sigma(\omega(y)) - 1)}{a}(\|y - x\| + \|\gamma(\omega(x))\omega(x) - \gamma(\omega(y))\omega(y)\|).$$

Therefore the continuity of φ for x in $\partial(C_2) \cap Y^0$ follows from the continuity of ω and γ.

Next let us suppose x to be a point in $\partial(C_2) \cap \partial(Y)$. Then $\theta(\omega(x)) = \gamma(\omega(x)) = \sigma(\omega(x)) = 1$, and in fact, we may choose a neighborhood $N(x, \epsilon_1)$ such that y in $N(x, \epsilon_1)$ implies $\theta(\omega(y)) = 1$. Let us also observe that

$$\|y - \gamma(\omega(y))\omega(y)\| \leqq (1 - \gamma(\omega(y)))\,\|\omega(y)\|,$$

for y in C_2. Hence for y in $N(x, \epsilon_1) \cap C_2$ we have

$$\varphi(y) - \varphi(x) = \omega(y) - \omega(x) + \frac{(\sigma(\omega(y)) - 1)}{(1 - \gamma(\omega(y)))}(y - \gamma(\omega(y))\omega(y))$$

$$\|\varphi(y) - \varphi(x)\| \leqq \|\omega(y) - \omega(x)\| + (\sigma(\omega(y)) - 1)\,\|\omega(y)\|.$$

Therefore, the continuity of φ for x in $\partial(C_2) \cap \partial(Y)$ follows from the continuity of ω and σ.

We now observe that φ^{-1} on $\overline{C_4} \setminus Y$ takes the form

$$\varphi^{-1}(x) = \gamma(\omega(x))\omega(x) + \frac{(\theta(\omega(x)) - \gamma(\omega(x)))}{(\sigma(\omega(x)) - 1)}(x - \omega(x)).$$

For x in $\partial(Y) \cap C_2$ clearly $x = \omega(x)$, $\sigma(\omega(x)) > 1$, and for y in $\overline{C_4} \setminus Y$

$$\varphi^{-1}(y) - \varphi^{-1}(x) = \gamma(\omega(y))\omega(y) - \gamma(\omega(x))\omega(x) + \frac{\theta(\omega(y)) - \gamma(\omega(y))}{(\sigma(\omega(y)) - 1)}(y - \omega(y)).$$

We may choose a neighborhood $N(x, \epsilon_2)$ such that y in $N(x, \epsilon_2)$ implies $(\sigma(\omega(y)) - 1) > a_1$, so we have for y in $N(x, \epsilon_2) \cap (\overline{C_4} \setminus Y)$ that

$$\varphi^{-1}(y) - \varphi^{-1}(x) = \gamma(\omega(y))\omega(y) - \gamma(\omega(x))\omega(y) + \frac{\theta(\omega(y)) - \gamma(\omega(y))}{a_1}[(y - x) + (\omega(x) - \omega(y))].$$

Hence it is clear that the continuity of φ^{-1} for x in $\overline{C_4} \setminus Y$ follows from the continuity of γ and ω.

For x in $\partial(Y) \cap \partial(C_2)$ and y in $\overline{C_4} \setminus Y$ we have $x = \omega(x)$, $\theta(\omega(x)) = \gamma(\omega(x)) = 1$, and $\|y - \omega(y)\| \leqq (\sigma(\omega(y)) - 1)\,\|\omega(y)\|$. Thus

$$\begin{aligned}\|\varphi^{-1}(y) - \varphi^{-1}(x)\| \leqq\ & \|\gamma(\omega(y))\omega(y) - \gamma(\omega(x))\omega(x)\| \\ & + (\theta(\omega(y)) - \gamma(\omega(y)))\,\|\omega(y)\|\end{aligned}$$

which, of course, implies that the continuity of φ^{-1} on $\partial(Y) \cap \partial(C_2)$ follows from the continuity of γ, θ, and ω. This completes the proof of our lemma.

In Browder [3] the following generalization of the Schauder Fixed Point Theorem is proved.

BROWDER'S THEOREM. *Let S and S_1 be open convex subsets of a Banach space X, S_0 a closed convex subset of X, $S_0 \subset S_1 \subset S$, f a compact mapping of S into X. Suppose that for a positive integer m, f^m is well-defined in S_1, $\bigcup_{j=0}^{m-1} f^j(S_0) \subset S_1$, while $f^m(S_1) \subset S_0$. Then f has a fixed point in S_0.*

We shall make essential use of Browder's Theorem in proving our next theorem which in turn will be used to complete our proof of Theorem 3.

THEOREM 5. *Let Y be a closed and bounded convex subset of a Banach space X with an interior point v_2 and an exterior point x_2, and let f be a completely continuous mapping of Y into X. Let $x_0 = \widehat{v_2x_2} \cap \partial(Y)$, $\|v_2 - x_0\| > r_2 > r_1 > 0$, $C(x_2, N(x_0, r_2)) \subset C(x_2, Y^0)$ and let v_1 be a point in $\widehat{v_2x_0} \setminus \overline{N(x_0, r_2)}$. If for some positive integer m, f^m is well-defined on $Y \setminus C(v_1, N(x_0, r_1))$,*

$$\bigcup_{j=0}^{m-1} f^j(Y \setminus C(v_2, N(x_0, r_2))) \subset Y \setminus \overline{C(v_1, N(x_0, r_1))}$$

and

$$f^m(Y \setminus C(v_1, N(x_0, r_1))) \subset Y \setminus C(v_2, N(x_0, r_2)).$$

Then f has a fixed point in $Y \setminus C(v_2, N(x_0, r_2))$.

Proof. Let us define ψ to be the support function of $Y - v_2$ defined relative to the origin. We extend the function f to the domain

$$W = \{x : \psi(x - v_2) < \tfrac{3}{2}\}$$

by defining

$$f(v_2 + (1 - \lambda)(y - v_2)) = f(v_2 + \tfrac{1}{3}(1 + 3\lambda)(y - v_2))$$

for all y in $\partial(W)$ and $0 < \lambda < \frac{1}{3}$ (see Fig. 3).

Defining $W_0 = Y \setminus C(v_2, N(x_0, r_2))$, $W_1 = \overline{W \setminus C(v_1, N(x_0, r_1))}$, and $W_2 = W \setminus C(v_2, N(x_0, r_2))$, we easily observe that our hypotheses imply that

$$\bigcup_{j=0}^{m-1} f^j(W_2) \subset W_1,$$

and

$$f^m(W_1) \subset W_0.$$

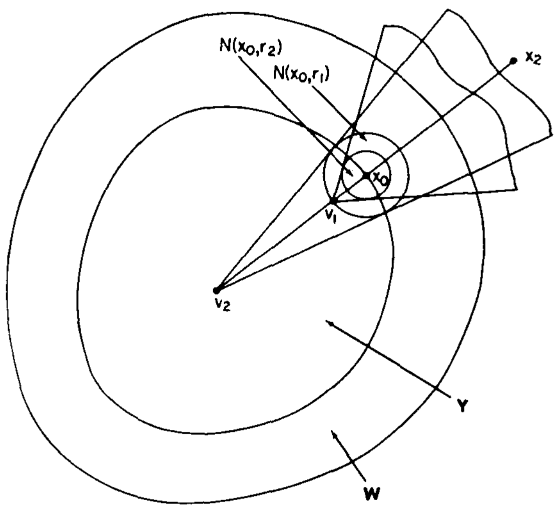

Figure 3

Lemma 4 implies the existence of a homeomorphism φ defined on $\overline{W}$ such that $\varphi(\overline{W}_1) = D_1$ and $\varphi(\overline{W}_2) = D_2$ are convex. Hence letting D_0 denote the convex hull of $\varphi(W_0)$ we have $D_0 \subset D_2$,

$$\varphi f^i \varphi^{-1}(D_0) = (\varphi f \varphi^{-1})^i(D_0) \subset (\varphi f \varphi^{-1})^i(D_2) \subset D_1$$

for $i = 1, 2, \ldots, m-1$, and

$$(\varphi f \varphi^{-1})^m(D_1) \subset D_0.$$

Invoking Browder's Theorem it follows that there is an element x^* in D_0 such that $(\varphi f \varphi^{-1})(x^*) = \varphi^{-1}(x^*)$ which, of course, implies $f(\varphi^{-1}(x^*)) = \varphi^{-1}(x^*)$. But $\varphi^{-1}(x^*)$ is contained in W_0, so our proof is complete.

Proof of Theorem 3. For ϵ sufficiently small the hypotheses of Theorem 3 imply that we may choose an integer n_1 such that $f^{n_1}(\bar{Y} \setminus N(x_0, \epsilon))$ is contained in $\bar{Y} \setminus N(x_0, \epsilon)$ and k such that $0 < k < 1$ and

$$f^i(\bar{Y} \setminus N(X_0, \epsilon)) \subset \bar{Y} \setminus N(x_0, k^i\epsilon)$$

for $i = 0, 1, \ldots, n_1 - 1$. Clearly it follows that $f^{mn_1+i}(\bar{Y} \setminus N(x_0, \epsilon))$ must be contained in $\bar{Y} \setminus N(x_0, k^i\epsilon)$ for $m = 1, 2, \ldots$. Also for an arbitrarily chosen constant c, we may select ϵ sufficient small and n_2 such that $f^{n_2}(\bar{Y} \setminus N(x_0, k^{n_1}\epsilon))$ is contained in $Y \setminus N(x_0, c\epsilon)$. Hence we have for ϵ sufficiently small that

$$\bigcup_{i=0}^{\infty} f^i(\bar{Y} \setminus N(x_0, \epsilon)) \subset \bar{Y} \setminus N(x_0, k^{n_1}\epsilon) \tag{2.2}$$

and

$$f^{n_2}(\bar{Y} \setminus N(x_0, k^{n_1}\epsilon)) \subset \bar{Y} \setminus N(x_0, c\epsilon). \tag{2.3}$$

Now let x_1 be a point in Y, and for $\mu > 0$ let

$$x_\mu = x_1 + [\mu/(\|x_0 - x_1\|)](x_0 - x_1).$$

By Lemma 2 there exist positive constants η and ν such that

$$C(x_\mu, N(x_0, \eta\mu)) \cap Y \subset N(x_0, \mu) \cap Y$$

for all μ in $(0, \nu]$. Choosing $\epsilon < \eta^2\nu$ and $\eta \leqq k^{n_1}$ we have

$$\overline{C(x_{\eta\epsilon}, N(x_0, \eta^2\epsilon))} \cap \bar{Y} \subset N(x_0, \eta\epsilon) \cap \bar{Y}$$

and

$$\overline{C\left(x_{\epsilon/\eta^2}, N\left(x_0, \frac{\epsilon}{\eta}\right)\right)} \cap \bar{Y} \subset N\left(x_0, \frac{\epsilon}{\eta^2}\right) \cap \bar{Y}.$$

Figure 4 is suggestive of the construction under consideration. Referring to (2.2) we may observe that for ϵ sufficiently small we have

$$\bigcup_{i=0}^{\infty} f^i(\bar{Y} \setminus C(x_{\epsilon/\mu}, N(x_0, \epsilon))) \subset \overline{Y \setminus C(x_{\mu\epsilon}, N(x_0, \mu^2\epsilon))}.$$

Also setting $c = k^{n_1}\mu^{-3}$ and referring to (2.3) we have that

$$f^{n_2}(\bar{Y} \setminus C(x_{\mu\epsilon}, N(x_0, \mu^2\epsilon))) \subset \bar{Y} \setminus C(x_{\epsilon/\mu}, N(x_0, \epsilon)),$$

for ϵ sufficiently small. At this point we can invoke Theorem 5 to conclude the existence of a fixed point under f in $\bar{Y} \setminus C(x_{\epsilon/\mu}, N(x_0, \epsilon))$, and thereby the proof of Theorem 3 is complete.

LEMMA 5. *Let A be a closed convex set in X having interior and let x_0 be a boundary point of A. Let f be a continuous mapping of A into itself, let ϵ_0 and ϵ_1 be positive constants with $\epsilon_1 > \epsilon_0$, and suppose*

$$f(\partial(N(x_0, \epsilon_1)) \cap A) \subset A \setminus N(x_0, \epsilon_0).$$

If g is a function defined on A by the formula $g(x) = f(x)$, for x in $N(x_0, \epsilon_1)$, $g(x) = f(x)$, for x in $A \setminus N(x_0, \epsilon_1)$ and $f(x)$ in $A \setminus N(x_0, \epsilon_0)$, $g(x) = \widehat{xf(x)} \cap \partial(N(x_0, \epsilon_0))$, for x in $A \setminus N(x_0, \epsilon_1)$ and $f(x)$ in $N(x_0, \epsilon_0)$, then g is continuous and has the same fixed points in A as f.

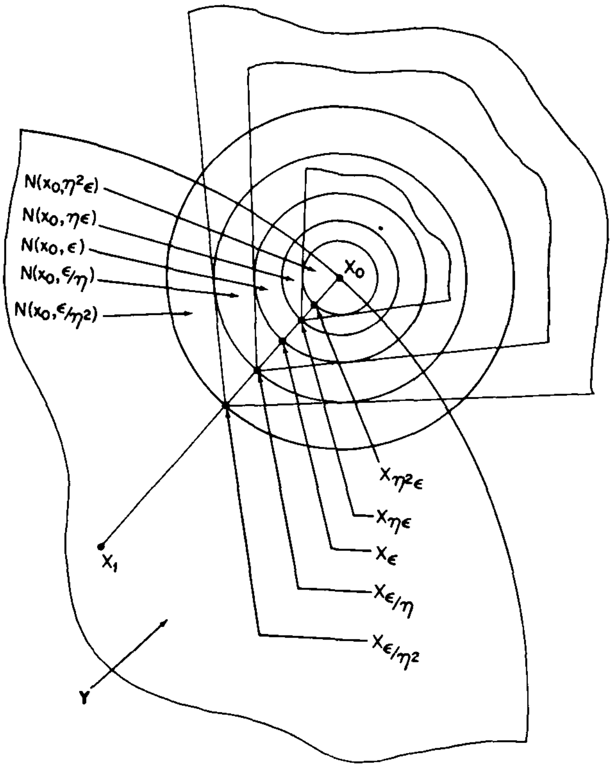

Figure 4

Proof. It is clear from the convexity of $N(x_0, \epsilon_0)$ that g is well defined. Furthermore, it is immediate from its definition that g has the same fixed points in A as f. Hence our only problem is to show that g is continuous.

g is, of course, continuous on $N(x_0, \epsilon_1)$, so we shall suppose for the remainder of our argument that x is a point in $A \setminus N(x_0, \epsilon_1)$. If $f(x)$ is contained in $A \setminus \overline{N(x_0, \epsilon_0)}$ it is again immediately obvious from its definition that g is continuous at x. Thus we also suppose $f(x)$ is contained in $\overline{N(x_0, \epsilon_0)}$. Now let ξ be some point in $(r(xf(x)) \setminus \overline{xf(x)}) \cap N(x_0, \epsilon_0)$ and let

$\eta > 0$ by an arbitrary positive constant such that $N(g(x), \eta)$ does not contain ξ. By Lemma 2 there exists $v > 0$ such that

$$C(\xi, N(g(x), v)) \cap \partial(A) \subset N(g(x), \eta).$$

Furthermore using the continuity of f at x we have the existence of positive constants δ_1 and δ_2 such that the convex hull H of $N(x, \delta_2) \cup N(f(x), \delta_1)$ is contained in $C(\xi, N(g(x), v))$ and $f(N(x, \delta_2)) \subset N(f(x), \delta_1)$. Thus we have

$$\begin{aligned} g(N(x, \delta_2)) \subset H \cap \partial(N(x_0, \epsilon_0)) &\subset C(\xi, N(g(x), v)) \\ &\cap \partial(N(x_0, \epsilon_0)) \subset N(g(x), \eta), \end{aligned}$$

so we may conclude that g is continuous at x and the proof of our lemma is complete.

Proof of Theorem 4. Under the hypotheses of Theorem 4 let us suppose that x_0 is not a fixed point under f. Then since f maps $\bar{Y}$ into $\bar{Y}$ it follows from the Schauder fixed point theorem [16] that f has a fixed point x^* in $\bar{Y}$ and $x^* \neq x_0$. Therefore, we may assume for the remainder of this argument that $f(x_0) = x_0$.

From our hypotheses it is clear that there exists $\epsilon_1 > 0$ such that for all x in $N(x_0, \epsilon_1) \cap \bar{Y}$ and some positive constant $k < 1$ we have

$$\|f(x) - x_0\| \geqq k\, \|x - x_0\|. \tag{2.4}$$

Furthermore, ϵ_1 may be chosen so that for some positive integer n_1 we have

$$\|f^{n_1}(x) - x_0\| \geqq k^{-1}\, \|x - x_0\|, \tag{2.5}$$

for x in $N(x_0, \epsilon_1) \cap \bar{Y}$. For arbitrarily chosen constants $a > 1$, $b \leqq k$ and $m > n_1$ we observe that the continuity of f implies the existence of a constant δ_1 in $(0, b\epsilon_1)$ such that for x in $N(x_0, a\delta_1) \cap \bar{Y}$ and $i = 1, 2, \ldots, m$

$$\|f^i(x) - x_0\| < b\epsilon_1. \tag{2.6}$$

Hence for all x in $(N(x_0, \delta_1) \setminus N(x_0, k\delta_1)) \cap \bar{Y}$ we have

$$\|f^i(x) - x_0\| \geqq k^{i+1}\delta_1,\ i = 1, 2, \ldots, n_1 - 1. \tag{2.7}$$

Now for an arbitrarily specified constant $c > 1$ let us specify $a = k^{-n_1}c$ and let $b = k$. Clearly our hypotheses also allow us to choose δ_1 sufficiently small so that we have an integer $n_2 \geqq n_1$ such that

$$\|f^{n_2}(x) - x_0\| \geqq ck^{-n_1}\, \|x - x_0\|$$

for all x in $N(x_0, ck^{-n_1}\delta_1) \cap \bar{Y}$. Hence we have that

(2.8) $$f^{n_2}((N(x_0, ck^{-n_1}\delta_1) \setminus N(x_0, k^{n_1}\delta_1)) \cap \bar{Y}) \subset \bar{Y} \setminus N(x_0, c\delta_1).$$

Now let us specify g to be a function defined on $\bar{Y}$ by the following formula:

$$g(x) = f(x), \qquad \text{for } x \text{ in } N(x_0, \epsilon_1)$$

$g(x) = f(x)$, for x in $\bar{Y} \setminus N(x_0, \epsilon_1)$ and $f(x)$ in $\bar{Y} \setminus N(x_0, k\epsilon_1)$, $g(x) = \widehat{xf(x)} \cap \partial(N(x_0, k\epsilon_1))$, for x in $\bar{Y} \setminus N(x_0, \epsilon_1)$ and $f(x)$ in $N(x_0, k\epsilon_1)$. We have by Lemma 5 that g is continuous and has the same fixed points in $\bar{Y}$ as f. From its definition and (2.4) and (2.5), clearly

$$g^i(\bar{Y} \setminus N(x_0, \delta_1)) \subset \bar{Y} \setminus N(x_0, k^{n_1}\delta_1)$$

for all positive integers i, so

$$\bigcup_{i=0}^{\infty} g^i(\bar{Y} \setminus N(x_0, \delta_1)) \subset \bar{Y} \setminus N(x_0, k^{n_1}\delta_1).$$

Furthermore, we observe that

$$g^i(\bar{Y} \setminus N(x_0, ck^{-n_1}\delta_1)) \subset \bar{Y} \setminus N(x_0, c\delta_1)$$

for all i and this fact together with (2.8) imply that

(2.10) $$g^{n_2}(\bar{Y} \setminus N(x_0, k^{n_1}\delta_1)) \subset \bar{Y} \setminus N(x_0, c\delta_1)$$

Replacing f by g and using (2.9) and (3.0) in the same fashion as (2.2) and (2.3) are used in the proof of Theorem 3, we obtain the result, using the same argument, that there is a point $x^* \neq x_0$ in $\bar{Y}$ which is a fixed point under g. But, of course, $g(x^*) = x^*$ implies $f(x^*) = x^*$, so the proof of Theorem 4 is complete.

3. Periodic Solutions for Functional-Differential Equations

Many problems in the theory of control, biological behavior, econometrics, and other active areas of our technology are associated with the periodic behavior of systems which are governed at least in part by the after effects of their previous states. For this reason the theory of periodic solutions for differential-difference equations and more general functional-differential equations, which describe such systems, is emerging as an important area of mathematical research from a practical as well as a theoretical point of view. For discussions of some of the interesting and often intriguing interpretations of these generalized differential equations the reader is referred to references [1], [2], [5], [14], [17], [18], [19]. A guide to the extensive Russian literature on this subject is contained in [20].

As an application of the results of the previous section we consider a class of autonomous functional-differential equations. We begin by denoting the space of n-dimensional real vectors by R^n and defining $|x|$ to be max $\{|x_i|: \ i = 1, 2, \ldots, n\}$ for all vectors $x = (x_1, \ldots, x_n)$ in R^n. $C([-h, 0])$ is defined to be the space of continuous functions mapping a specified closed interval $[-h, 0]$, $h > 0$, into R^n where n is arbitrary but specified. A topology on $C([-h, 0])$ is determined by the norm functional

$$\|\varphi\| = \max \{|\varphi(\theta)|: \ \theta \text{ in } [-h, 0]\},$$

defined for all $\varphi = (\varphi_1, \ldots, \varphi_n)$ in $C([-h, 0])$. The component norm is specified by the formula

$$\|\varphi_i\| = \max \{|\varphi_i(\theta)|: \ \theta \text{ in } [-h, 0]\}, \qquad i = 1, 2, \ldots, n.$$

For each $b > 0$ let $C([-h, 0], b) \subset C([-h, 0])$ be such that φ in

$$C([-h, 0], b)$$

implies $\|\varphi\| < b$. Let $D([-h, 0]) \subset C([-h, 0])$ denote the set of all continuously differentiable elements and let φ in $D([-h, 0], c)$ imply that $\|\dot{\varphi}\| < c$. We now define F to be a continuous mapping of $C([-h, 0])$ into R^n. We shall relate the results of Section 1 and 2 to the question of existence of periodic solutions for functional-differential equations of the form

$$\dot{x}(t) = F(H_t x), \tag{3.1}$$

where $H_t x$ denotes the translation to $[-h, 0]$ of the restriction of the function x to $[t - h, t]$. That is, $H_t x(\theta) = x(t + \theta)$ for θ in $[-h, 0]$. Specifying a function φ in $C([-h, 0])$, we shall call a function x defined on $[-h, t^*]$, $t^* > 0$, a solution of eq. (3.1) corresponding to the initial function φ, if $H_0 x = \varphi$ and if x satisfies eq. (3.1) for t in $[0, t^*]$. We assume conditions sufficient to assure the existence and uniqueness of a solution of (3.1) corresponding to each initial function in $C([-h, 0])$ and require that these solutions be jointly continuous under translation and variation in initial data. That is, for each pair (t_1, φ_1) and every $\epsilon > 0$ there exist $\delta > 0$ such that $|t - t_1| < \delta$ and $\|\varphi - \varphi_1\| < \delta$ imply

$$\|H_{t_1}(\varphi_1) - H_t(\varphi)\| < \epsilon.$$

Now let $C_0([-h, 0]) \subset C([-h, 0])$ be such that φ in $C_0([-h, 0])$ implies $\varphi_1(0) = 0$ and $\varphi_1(t) > 0$ on $(-h, 0)$. Let $C_0([-h, 0], b)$ denote

$$C_0([-h, 0]) \cap C([-h, 0], b).$$

We impose the following condition on the oscillatory behavior of our system:

(a) For specified constants $a_1 \geqq h$, $b_2 > b_1$, and $c_2 > 0$, and for each φ in $C_0([-h, 0], b_1)$ the first component x_1 of the corresponding solution x of eq. (3.1) has a smallest zero $z(\varphi)$ with $h \leqq z(\varphi) \leqq a_1$ and such that $H_{z(\varphi)}x$ is contained in $C_0([-h, 0], b_2) \cap D([-h, 0], c_2)$. Furthermore, the first component F_1 of F is such that $F_1(H_{z(\varphi)}x) < 0$ for all φ in

$$C_0([-h, 0], b_2).$$

Now let us denote by $H_t x(\varphi)$ the solution segment of eq. (3.1) at time t corresponding to φ in $C([-h, 0])$. We define $T: C([-h, 0]) \times R^+ \to C([-h, 0])$ by assigning to each pair (φ, t) in $C([-h, 0]) \times R^+$ the element $H_t x(\varphi)$ in $C([-h, 0])$. From our assumption that each initial function determines a solution uniquely it follows that T is well defined and we easily observe that $T(\varphi, 0) = \varphi$ and $T(T(\varphi, t_1), t_2) = T(\varphi, t_1 + t_2)$. Furthermore, since solutions of eq. (3.1) depend continuously on their initial data it follows that T is continuous on $C_0([-h, 0]) \times R^+$. Defining f by the formula

$$f(\varphi) = T(\varphi, z(\varphi)) \tag{3.2}$$

we observe that the boundedness of the functional z, the fact that

$$F_1(H_{z(\varphi)}x) < 0$$

for all φ in $C_0([-h, 0], b_1)$, and the continuity of solutions under translation imply that f is continuous on $C_0([-h, 0], b_1)$.

Since it is now clear that we have a formulation of the type presented in Section 1, it seems reasonable to expect that the techniques we have developed an applicable. This is indeed the case and as our first result we present the following theorem:

THEOREM 6. *Let there exist a constant* $k > 0$ *such that* $\|f(\varphi)\| \geqq k\|\varphi\|$ *for* $\|\varphi\|$ *sufficiently small. Furthermore suppose that for any constant* $M > 0$ *there exists an integer* n *such that* $\|f^n(\varphi)\| \geqq M\,\|\varphi\|$ *for* $\|\varphi\|$ *sufficiently small. Then eq.* (3.1) *has a manifold of non-constant periodic solutions of period* p *where* $h \leqq p \leqq a_1$.

Proof. Turning our attention to formulations of the type discussed in Section 1 immediately preceding Theorem 1, we have with z playing the role of λ that T is z-continuous relative to $\overline{C_0([-h, 0], b_1)}$ and

$$C_0([-h, 0], b_1)$$

is a bounded open convex set in the Banach space $B = \{\varphi\colon\ \varphi \text{ in } C([-h, 0]),$

$\varphi(0) = 0\}$. Hence we see that this theorem is a consequence of Theorem 2.

Now let L and G be continuous functions mapping $C([-h, 0])$ into R^n where, in addition, L is linear. Letting $F = L + G$ we shall investigate equations of the form

$$\dot{x}(t) = L(H_t x) + G(H_t x), \tag{3.3}$$

and consider a somewhat more constructive approach to verifying the existence of periodic motions. That is, we shall formulate a situation which utilizes the structure of L and a "smallness" condition on G near the origin to yield the hypotheses of Theorem 6.

A very natural way to approach eq. (3.3) is, of course, to begin by considering the linear homogeneous equation

$$\dot{u}(t) = L(H_t u). \tag{3.4}$$

We have by virtue of a well known theorem of Riesz [15] that L may be written as a Stieltjes integral with a matrix integrator η of bounded variation. That is,

$$\dot{u}(t) = \int_{-h}^{0} u(t + \theta) d\eta(\theta). \tag{3.5}$$

The characteristic eq. of (3.4) takes the form

$$\det\left(sI - \int_{-h}^{0} e^{\theta s} d\eta(\theta)\right) = 0 \tag{3.6}$$

where s is a complex variable and I denotes the $n \times n$ identity matrix. We assume (3.6) has a unique pair of conjugate complex roots $\sigma \pm i\nu$ with largest positive real parts. For every φ in $C_0([-h, 0])$ we assume the existence of a maximal infinite sequence $\{\eta_n(\varphi)\}$, $n = 1, 2, \ldots$ such that $u_1(\varphi)(\eta_n(\varphi)) = 0$ and $H_{\eta_n(\varphi)}u(\varphi)$ is contained in $C_0([-h, 0])$ for all n. Furthermore, we assume the existence of a positive constant c_1 such that for φ in $C_0([-h, 0])$ that

$$\limsup_{\|\varphi\| \to 0} \left(\frac{\dot{u}_1(\eta_1(\varphi))}{\|\varphi\|}\right) < -c_1. \tag{3.7}$$

Finally, we assume a type of uniform instability of the trivial solution with respect to initial functions in $C_0([-h, 0])$. That is, we require the existence of a positive constant c_3 such that

$$\|H_{\eta_n(\varphi)}u(\varphi)\| \geqq c_3 \|\varphi\| e^{\sigma \eta_n(\varphi)} \tag{3.8}$$

for all φ in $C_0([-h, 0])$.

It is worth remarking that the reasonableness of a condition such as (3.8) depends very strongly on having a restriction on the oscillation of initial functions such as that imposed by virtue of being an element in $C_0([-h, 0])$. When we have true hereditary dependence in our equation condition (3.8) can not be satisfied for all solutions corresponding to initial functions restricted only to lie in a bounded subset of $C([-h, 0])$, since in general there is always a subspace of $C([-h, 0])$ with respect to which the trivial solution is asymptotically stable. This follows from the fact, essentially established by Langer [13], that (3.6) in general has a sequence of zeros with real parts tending to minus infinity. We also remark that the existence of the roots $\sigma \pm iv$ as specified is in itself enough to establish the existence of a cone in $C([-h, 0])$ on which the type of instability suggested by (3.8) is present. Hence the only essentially independent condition one needs to impose is that $C_0([-h, 0])$ is contained in this cone of instability. The reader is referred to [6] for the analysis needed in verifying these comments.

Let us denote by U^* the solution of the $n \times n$ matrix equation

$$\dot{U}(t) = \int_{-h}^{0} U(t + \theta) d\eta(\theta) \tag{3.9}$$

corresponding to the initial matrix function $\Phi(t) = 0$ for t in $[-h, 0)$ and $\Phi(0) = I$.

LEMMA 6. *Consider the vector equation*

$$\dot{y}(t) = L(H_t y) + g(t, \varphi), \tag{3.10}$$

where φ in $C([-h, 0])$ is the initial function for y and g is continuous on $R^+ \times C([-h, 0])$. Suppose there exist positive constants δ_1 and δ such that for all $\|\varphi\|$ sufficiently small

$$\|g(t, \varphi)\| = 0(e^{(\sigma-\delta_1)t})\|\varphi\|^{1+\delta}. \tag{3.11}$$

Then for arbitrary $M > 0$, there exists $\epsilon_1 > 0$ sufficiently small and $t_1 > 0$ such that

$$\|H_{\eta_n(\varphi)} y\| > M\|\varphi\| \tag{3.12}$$

for all φ in $C_0([-h, 0], \epsilon_1)$ and all $\eta_n(\varphi) > t_1$. Furthermore, for arbitrary $t_2 > 0$ and $\delta_2 > 0$ we may choose ϵ_1 such that for each φ in $C_0([-h, 0], \epsilon_1)$ the first component of each solution $y(\varphi)$ has zeros $\{z_n(\varphi)\}$ such that $|z_n(\varphi) - \eta_n(\varphi)| < \delta_2$ for all $\eta_n(\varphi) \leqq t_2$.

Proof. It is easily verified that

$$y(t) = u(t) + \int_0^t g(\tau, \varphi) U^*(t - \tau) d\tau, \tag{3.13}$$

where y and u are solutions of eqs. (3.9) and (3.4) respectively and both correspond to the initial function φ. Now by our assumption concerning the eigenvalues of (3.4) it follows that $U^*(t)$ may be written

$$U^*(t) = e^{\sigma t}K(t)$$

where $K(t)$ is a bounded matrix. Hence for $\|\varphi\|$ sufficiently small,

$$\begin{aligned} \left\|\int_0^t g(\tau, \varphi)U^*(t-\tau)d\tau\right\| &\leqq e^{\sigma t}\left\|\int_0^t e^{-\sigma\tau}g(\tau, \varphi)K(t-\tau)\,d\tau\right\| \\ &\leqq e^{\sigma t}\int_0^t e^{-\sigma\tau}\|g(\tau\,\varphi)\|\;\|K(t-\tau)\|\,d\tau \\ &\leqq c_4 e^{\sigma t}\|\varphi\|^{1+\delta}, \end{aligned} \tag{3.14}$$

where c_4 is a positive constant independent of φ. Referring to (3.8) and (3.11) we have

$$\begin{aligned} \|H_{\eta_n(\varphi)}y\| &\geqq c_3\|\varphi\|e^{\sigma\eta_n(\varphi)} - c_4\|\varphi\|^{1+\delta}e^{\sigma\eta_n(\varphi)} \\ &= \|\varphi\|e^{\sigma\eta_n(\varphi)}(c_3 - c_4\|\varphi\|^{\delta}) \\ &\geqq \|\varphi\|e^{n\sigma a}(c_3 - c_4\|\varphi\|^{\delta}). \end{aligned} \tag{3.15}$$

Clearly (3.15) implies we may choose M arbitrarily and satisfy (3.12).

Now using (3.7) and (3.8) we may verify without difficulty that there must exist a constant $\gamma > 0$ such that

$$\limsup_{\|\varphi\|\to 0}\left(\frac{\dot u_1(\eta_n(\varphi))}{\|\varphi\|}\right) < -\gamma \tag{3.16}$$

for all $\eta_n(\varphi) \leqq t_2$. (3.16) in turn implies the existence of a positive constant $\rho < \delta_2$ such that

$$\dot u_1(t) < -\frac{\gamma}{2}\|\varphi\| \tag{3.17}$$

for all t in $Z = \cup\{t\colon\ |t - \eta_n(\varphi)| < \rho,\ \eta_n(\varphi) \leqq t_2\}$ when $\|\varphi\|$ is sufficiently small. Referring to (3.10) and (3.11) we have the existence of a constant c_5 such that

$$|\dot y_1(t, \varphi) - \dot u_1(t, \varphi)| \leqq c_5\|\varphi\|^{1+\delta}$$

for $\|\varphi\|$ sufficiently small and $t \leqq t_2$. Hence for $\|\varphi\|$ small enough and $t \leqq t_2$,

$$\dot y_1(t, \varphi) < -\frac{\gamma}{4}\|\varphi\|. \tag{3.18}$$

Referring to (3.13) and (3.14) we observe that for $\|\varphi\|$ small and $\eta_n(\varphi) \leqq t_2$,

$$|y_1(\eta_n(\varphi))| < c_6\|\varphi\|^{1+\delta}$$

where c_6 is a constant. If $y_1(\eta_n(\varphi))$ is positive, then

$$y_1(\eta_n(\varphi) + \rho) < c_6\|\varphi\|^{1+\delta} - \rho\frac{\gamma}{4}\|\varphi\|. \tag{3.19}$$

If $y_1(\eta_n(\varphi))$ is negative, then

$$y_1(\eta_n(\varphi) - \rho) > - c_6\|\varphi\|^{1+\delta} + \rho\frac{\gamma}{4}\|\varphi\|. \tag{3.20}$$

Clearly (3.19) and (3.20) imply that for $\|\varphi\|$ is sufficiently small the function y_1 must change signs in each of the intervals $(\eta_n(\varphi) - \delta_2, \eta_n(\varphi) + \delta_2)$, $\eta_n(\varphi) \leqq t_2$, so the proof of our lemma is complete.

THEOREM 7. *Suppose the conditions specified on eqs.* (3.3) *and* (3.4) *are satisfied and that for* φ *in* $C([-h, 0])$ *we have* $|G(\varphi)| = 0(\|\varphi\|^{1+\delta})$, $\delta > 0$, *as* $\|\varphi\| \to 0$. *Furthermore, suppose that for every element* φ *in* $C([-h, 0], b_1)$ *we have that the solution* $x(\varphi)$ *of eq.* (3.3) *is such that for specified positive constants* k_1 *and* K_1,

$$k_1\|H_{t-h}x(\varphi)\| \leqq \|H_t x(\varphi)\| \leqq \exp\{K_1\|H_{t-h}x\|\} - 1. \tag{3.21}$$

Then eq. (3.3) *has a manifold of periodic solutions of period* p *where*

$$h \leqq p \leqq a_1.$$

Proof. Let φ be contained in $C_0([-h, 0], 1)$ and let $t(\varphi)$ be the first point where $\|H_t x(\varphi)\|$ equals $\|\varphi\|^{1-(\delta/2)}$. Let $g(t, \varphi)$ be defined by the formula

$$\begin{aligned} g(t, \varphi) &= G(H_t x(\varphi)), && \text{for } t \text{ in } [0, t(\varphi)) \\ &= G(H_{t(\varphi)}x(\varphi)), && \text{for } t \geqq t(\varphi). \end{aligned} \tag{3.22}$$

Since $\|G(\varphi)\| = 0(\|\varphi\|^{1+\delta})$, $\delta > 0$ as $\|\varphi\| \to 0$, it is clear that there exists $\epsilon_1 > 0$ so that $G(H_t x(\varphi)) < \|\varphi\|^{1+(\delta/3)}$ for all φ sufficiently small. Hence by Lemma 6 we have that for an arbitrarily specified positive constant M we may choose $\epsilon_1 > 0$ and $t_1 > 0$ such that

$$\|H_{\eta_n(\varphi)}y(\varphi)\| > M\|\varphi\|$$

for all φ in $C_0([-h, 0], \epsilon_1)$ and all $\eta_n(\varphi) > t_1$. Also, for arbitrary $t_2 > 0$ and $\delta > 0$ we may choose ϵ_1 such that for each φ in $C_0([-h, 0], \epsilon_1)$ the first component of each solution $y(\varphi)$ has zeros $\{z_n(\varphi)\}$, $n = 1, 2, \ldots,$ such that $|z_n(\varphi) - \eta_n(\varphi)| < \delta$ for all $\eta_n(\varphi) \leqq t_2$. Since we are free to do so let us choose $t_2 > t_1$ and such that the intersection of $\{z_n(\varphi)\}$, $n = 1, 2, \ldots,$ and the interval (t_1, t_2) is non-void for all φ in $C_0([-h, 0], \epsilon_1)$.

Since $\|H_t y(\varphi)\|$ is a continuous function of t and consequently uniformly continuous on $[0, t_2]$, it follows that we may choose ϵ_1 such that

$$\|H_{z_n(\varphi)} y(\varphi)\| > \frac{M}{2} \|\varphi\| \tag{3.23}$$

for all φ in $C_0([-h, 0], \epsilon_1)$ and all $z_n(\varphi)$ in $[t_1, t_2]$.

It is clear from definition that

$$x(\varphi)(t) = y(\varphi)(t) \tag{3.24}$$

for all $t \leqq t(\varphi)$ and (3.12) implies $t(\varphi)$ tends to infinity as $\|\varphi\| \to 0$. Hence, in particular, we may choose ϵ_1 such that (3.24) is satisfied for all $t \leqq t_2$, and we have that

$$\|H_{z_n(\varphi)} x(\varphi)\| > \frac{M}{2} \|\varphi\| \tag{3.25}$$

for all φ in $C_0([-h, 0], \epsilon_1)$ and all $z_n(\varphi)$ in $[t_1, t_2]$. Defining f as in (3.2), it is clear that

$$f^n(\varphi) = H_{z_n(\varphi)} x(\varphi)$$

for all φ in $C([-h, 0], b_1)$ and $z_n(\varphi) \leqq t_2$, and (3.25) implies $\|f^n(\varphi)\| \geqq M \|\varphi\|$ for $\|\varphi\|$ sufficiently small. As a direct consequence of (3.21) we may conclude the existence of a positive constant k such that $\|f(\varphi)\| \geqq k \|\varphi\|$ for $\|\varphi\|$ sufficiently small. Hence the hypotheses of Theorem 6 are satisfied and its application completes the proof of this theorem.

We remark that with some added effort the results of this section can be extended to perturbed systems of the form

$$\dot{x}(t) = L(H_t x) + G(H_t x) + \epsilon J(\epsilon, H_t x), \tag{3.26}$$

where J is jointly continuous in its arguments. Furthermore, our condition that $H_{z(\varphi)} x$ be contained in $C_0([-h, 0], b_2) \cap D([-h, 0], c_2)$ can be weakened to require only that f^n be completely continuous for n sufficiently large and that solutions of (3.1) be uniformly ultimately bounded. It is also possible to drop the requirement that the positive functional z on $C_0([-h, 0], b)$ be bounded below by h and require only that it be bounded below by some constant $a > 0$. For techniques which are useful in such extensions the reader is referred to [12]. [12] also contains results useful in investigating the continuity of periodic solutions with respect to the parameter ϵ.

4. Examples

As an example in the class of functional-differential equations discussed in the previous section let us first consider the scalar equation

$$\dot{x}(t) = -\alpha \int_{-h}^{0} x(t+\theta)d\eta(\theta)\{1 + x(t)\}, \tag{4.1}$$

where η is a non-constant non-decreasing function defined on $[-h, 0]$. Let $Y(\tau)$ denote the class of continuous real valued functions y defined for $t \geqq \tau - h$ and such that $-1 < y(t) < \exp\{\alpha(\eta(0) - \eta(-h))h\}$.

We consider the mapping T: $Y(\tau) \to Y(\tau)$ defined by the formula

$$\begin{aligned} T(y(t)) &= y(t), \qquad \text{for } t \text{ in } [\tau - h, \tau] \\ &= \{1 + y(\tau)\} \exp\left\{-\alpha \int_{\tau}^{t}\int_{-h}^{0} y(v+\theta)d\eta(\theta)\,dv\right\} - 1 \end{aligned} \tag{4.2}$$

for $t > \tau$. We easily verify that for arbitrary y_1 and y_2 in $Y(\tau)$ such that $y_1(t) = y_2(t)$ on $[\tau - h, \tau]$ we have that

$$\|T(y_1) - T(y_2)\| \leqq (t - \tau)K\,\|y_1 - y_2\|.$$

where $\|\ \|$ denotes the maximum value norm taken with respect to the interval $[\tau - h, t]$ and K is a parameter depending on the fixed value of y_1 and y_2 at τ. It follows that there exists a number $t_1 > \tau$ such that T is a contraction on $[\tau - h, t_1]$ and consequently has a unique fixed point y^* in $Y(\tau)$ when restricted to $[\tau - h, t_1]$. Since the properties of $Y(\tau)$ are preserved under T, it follows that y^* can be extended indefinitely. Hence corresponding to each continuous function φ defined on an interval $[t_0 - h, t_0]$ and such that $-1 < \varphi(t) < \exp\{\alpha(\eta(0) - \eta(-h))h\}$ on this interval there exists a unique function y in $Y(t_0)$ such that

$$y(t) = \varphi(t) \qquad \text{for } t \in [t_0 - h, t_0]$$

and

$$y(t) = \{1 + y(t_0)\} \exp\left\{-\alpha \int_{t_0}^{t}\int_{-h}^{0} y(v+\theta)\,d\eta(\theta)\,dv\right\} - 1. \tag{4.3}$$

for $t \geqq t_0$. Differentiation will verify that y is a solution of (4.1). Asol since every solution corresponding to an initial function φ with $\varphi(t_0) > -1$ must satisfy (4.3) it follows that y is the unique solution of (4.1) corresponding to φ. If $t_2 > t_1 > t_0$, we have from formula (4.3) that

$$y(t_2) = \{1 + y(t_1)\} \exp\left\{-\alpha\int_{t_1}^{t_2}\int_{-h}^{0} y(v+\theta)\,d\eta(\theta)\,dv\right\} - 1, \tag{4.4}$$

which in turn implies that $y(t_1) = y(t_2)$ if and only if

$$\int_{t_1}^{t_2}\int_{-h}^{0} y(v+\theta)\,d\eta(\theta)\,dv = 0. \tag{4.5}$$

Interchanging the order of integration it becomes clear that any possible periodic solution y^* of period p for eq. (4.1) must have the property that

$$\int_{t}^{t+p} y^*(v)\,dv = 0 \tag{4.6}$$

for all t.

Now since we have assumed η is non-decreasing it follows from the First Main Value Theorem for Stieltjes integrals that eq. (4.1) may be written in the form

$$\dot{x}(t) = -\alpha[\eta(0) - \eta(-h)]x(t - \tau(t))\{1 + x(t)\} \tag{4.7}$$

where $0 \leqq \tau(t) \leqq h$. From (4.7) it is easily observed that x must either tend to zero as $t \to \infty$ or take on the value zero on an unbounded set of points. If η is such that $\tau(t)$ can be uniformly bounded away from zero for large α then all conditions specified for the existence of periodic solutions in Section 3 can be shown to be satisfied for an appropriately chosen class of initial functions and α sufficiently large. In addition to the existence of periodic solutions, one may establish bounds on their periods as functions of α. The arguments involved are essentially the same as those used in [5] in discussing the equation

$$\dot{x}(t) = -\alpha x(t-1)\{1 + x(t)\}. \tag{4.8}$$

Computed periodic solutions over one period of the equation

$$\dot{x}(t) = -\{\alpha x(t-1) + \beta x(t-2)\}\{1 + x(t)\} \tag{4.9}$$

are exhibited in Figure 5 where $(\alpha, \beta) = (1, 1), (0.8, 0.8), (0.5, 1)$ and $(2, 0)$. We remark that experimental evidence obtained in carrying out the calculation of these curves strongly support a conjecture of asymptotic stability for the manifolds of periodic solutions represented by these curves. The more detailed discussion and numerical study contained in [6] clarifies the nature of this behavior as related to the special case of eq. (4.8).

Another example in our class of functional-differential equations which has interesting periodic behavior is the equation

$$\dot{x}(t) = -\alpha\int_{-h}^{0} x(t+\theta)\,d\eta(\theta)\{1 - x(t_\eta^{\,2})\} \tag{4.10}$$

where as before η is assumed to be a non-constant and non-decreasing function defined on $[-h, 0]$. We shall see that this equation has a connection with the theory of elliptic functions.

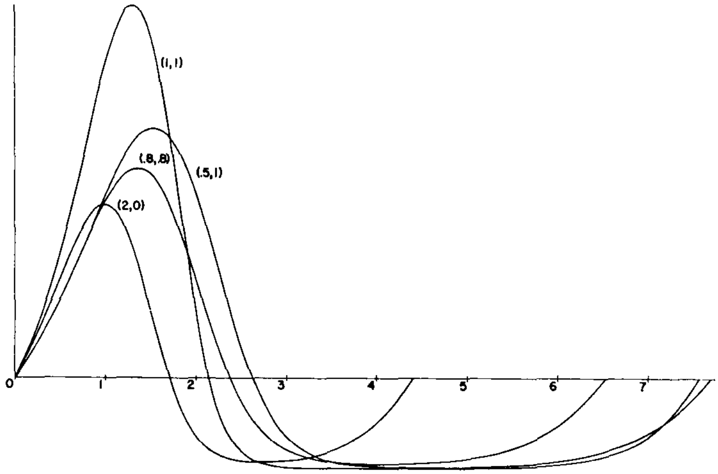

Figure 5

Let $W(\tau)$ denote the class of continuous real valued functions y defined for $t \geqq \tau - h$ and such that $|y(t)| < 1$ for all t. Let T^*: $W(\tau) \rightarrow W(\tau)$ be defined by the formula

$$T^*(y(t)) = y(t), \qquad \text{for } t \in [\tau - h, \tau]$$

$$= \frac{\dfrac{1 + y(\tau)}{1 - y(\tau)} \exp \left\{ -2\alpha \displaystyle\int_\tau^t \int_{-h}^0 y(v + \theta)\, d\eta(\theta)\, dv \right\} - 1}{\dfrac{1 + y(\tau)}{1 - y(\tau)} \exp \left\{ -2\alpha \displaystyle\int_\tau^t \int_{-h}^0 y(v + \theta)\, d\eta(\theta)\, dv \right\} + 1}$$

for $t > \tau$. It is easily verified that for arbitrary y_1 and y_2 in $W(\tau)$ such that $y_1(t) = y_2(t)$ on $[\tau - h, \tau]$ we have

$$\|T^*(y_1) - T^*(y_2)\| \leqq (t - \tau)K^* \|y_1 - y_2]$$

where K^* is a parameter which tends to infinity only as $y_i(\tau) \rightarrow \pm 1$. It follows that there exist $t_2 > \tau$ such that on $[\tau - h, t_2]$, T^* is a contraction and consequently has a unique fixed point y^* when restricted to $[\tau - h, t_2]$. Since the properties of $W(\tau)$ are preserved under T^* it follows that y^* can be extended indefinitely. Proceeding now as in our discussion of eq. (4.1) we may conclude that eq. (4.10) has a unique solution y corresponding to each continuous function φ defined and bounded by one on an interval $[t_0 - h, t_0]$, and $y(t)$ may be expressed as follows:

$$y(t) = \varphi(t), \qquad \text{for } t \text{ in } [t_0 - h, t_0],$$

$$y(t) = \frac{\dfrac{1 + y(t_0)}{1 - y(t_0)} \exp\left\{-2\alpha \displaystyle\int_{t_0}^{t} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv\right\} - 1}{\dfrac{1 + y(t_0)}{1 - y(t_0)} \exp\left\{-2\alpha \displaystyle\int_{t_0}^{t} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv\right\} + 1}, \qquad \text{for } t \geq t_0.$$

If $y(t_0) = 0$, then

$$\text{(4.12)} \qquad y(t) = \tanh\left\{-\alpha \int_{t_0}^{t} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv\right\}, \quad \text{for } t \geqq t_0.$$

We have that for $t_2 > t_1 > t_0$,

$$\text{(4.13)} \quad y(t_2) = \frac{\dfrac{1 + y(t_1)}{1 - y(t_1)} \exp\left\{-2\alpha \displaystyle\int_{t_1}^{t_2} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv\right\} - 1}{\dfrac{1 + y(t_1)}{1 - y(t_1)} \exp\left\{-2\alpha \displaystyle\int_{t_1}^{t_2} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv\right\} + 1},$$

which implies that $y(t_2) = y(t_1)$ if and only if

$$\text{(4.14)} \qquad \int_{t_1}^{t_2} \int_{-h}^{0} y(v + \theta)\, d\eta(\theta)\, dv = 0.$$

Hence we have that any possible periodic solution y^* of period p for eq. (4.10) must have the property that

$$\text{(4.15)} \qquad \int_{t}^{t+p} y^*(v)\, dv = 0$$

for all t.

We may write eq. (4.10) in the form

$$\dot{x}(t) = -\alpha\, [\eta(0) - \eta(-h)]\, x(t - \tau(t))\{1 - x(t)^2\},$$

where $0 \leqq \tau(t) \leqq h$, and it follows that either $x(t) \to 0$ as $t \to \infty$ or $x(t)$ is zero on an unbounded set of points. If $\tau(t)$ can be uniformly bounded away from zero for large α, then we may use the procedure of Section 3 to establish the existence of periodic solutions and bounds for their periods.

Computed periodic solutions over one period for the equation

$$\text{(4.16)} \qquad \dot{x}(t) = -\{\alpha x(t - 1) + \beta x(t - 2)\}\{1 - x(t)^2\}$$

are presented in Figure 6 where $(\alpha, \beta) = (3, 0), (6, 0), (1, 1), (3, 3), (1, 2)$, and $(2, 4)$. As was the case for eq. (4.9) there is strong experimental evidence to suppose the conjecture that the manifold of periodic solutions represented by these curves is asymptotically stable. Furthermore it can be observed that the periods of these solutions is invariant under multiplication of the parameters α and β by a common factor. That is, if α and β are such that the corresponding eq. (4.16) has a periodic solution of period p, then replacing α and β by $\alpha_1 = k\alpha$ and $\beta_1 = k\beta$ respectively,

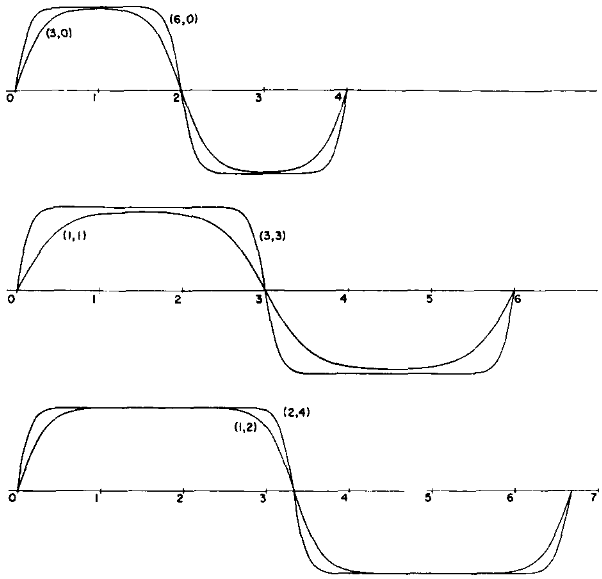

Figure 6

$k > 1$, we again have that eq. (4.16) has a periodic solution of period p. Finally we observe a very distinct quasi-sinusoidal symmetry in these curves and that larger values of the parameters α and β produce squarer wave forms.

Let us now indicate how one may prove the quasi-sinusoidal nature of periodic solutions of (4.16) when $\alpha = \beta$. That is, let us outline a procedure by which it can be shown that corresponding to each $\alpha \geqq 1$ the equation

(4.17) $$\dot{x}(t) = -\alpha\{x(t-1) + x(t-2)\}\{1 - x(t)^2\}$$

has a periodic solution of period p such that the following conditions are satisfied.

(1) $y(t) = 0$ if and only if $t = np/2$ when $n = 0, \pm 1, \pm 2, \ldots$, and $y(t) = -y(t + p/2)$ for all t.

(2) If $y(z) = 0$, then $y(z + p/4 - t) = y(z + p/4 + t)$, for all t.

(3) $y(t) > 0$ implies $\ddot{y}(t) < 0$ and $y(t) < 0$ implies $\ddot{y}(t) > 0$.

First of all we reason that if y is a periodic solution with the above mentioned symmetries, then we have that $\dot{y}(p/4) = 0$ which implies by our differential equation that

$$x(p/4 - 1) + x(p/4 - 2) = 0. \tag{4.18}$$

Furthermore properties (1) and (2) dictate that $x(-\frac{1}{2}) + x(\frac{1}{2}) = 0$, so by (4.18) we conclude that $p/4 = \frac{3}{2}$ and $p = 6$. We observe that this value of p checks with the indications of Figure 6.

Let Ω denote the set of all real valued periodic functions of period 6 defined on the real line which have continuous first derivatives, vanish at the origin, and satisfy symmetry conditions (1) and (2). For each y in Ω we define the norm $\| \ \|$ by the formula

$$\|y\| = \max \{\max \{|y(t)|: \ t \in R\}, \qquad \max \{|\dot{y}(t)|: \ t \in R\}/2\alpha\}.$$

$\Omega_1 \subset \Omega$ is the subset of all elements such that $\|y\| < 1$ and which are positive at $t = \frac{3}{2}$. $\Omega_2 \subset \Omega_1$ is defined to be the set of all elements y which have continuous second derivatives bounded by $12\alpha^2$ and satisfy symmetry condition (3).

Now for eq. (4.17), formula (4.12) reduces to

$$y(t) = \tanh \left\{-\alpha \int_{t_0}^{t} [y(v - 1) + y(v - 2)] \, dv\right\}. \tag{4.19}$$

Being guided by this formula we define an operator T on Ω by the formula

$$T(y(t)) = \tanh \left\{-\alpha \int_{0}^{t} [y(v - 1) + y(v - 2)] \, dv\right\} \tag{4.20}$$

for all y in Ω. It is easily verified that T is completely continuous and that $T(\Omega_1) \subset \Omega_2$. Utilizing the quasi-linear nature of T for $\|y\|$ small, we may also show that for $\epsilon > 0$ sufficiently small

$$T(\Omega_1 - N(0, \epsilon)) \subset \Omega_1 - N(0, k\epsilon), \qquad k > 1.$$

Hence we may conclude using Theorem 3 that T has a fixed point $y^* \neq 0$ in $\bar{\Omega}_2$. That is, there exists a non-trivial periodic function of period 6 which has symmetry properties (1), (2), and (3) and is such that

$$y^*(t) = \tanh \left\{-\alpha \int_{0}^{t} [y^*(v - 1) + y^*(v - 2)] \, dv\right\},$$

y^* is clearly a solution of (4.17) and we have our desired result.

Similarly it may be shown that for α sufficiently large the equation

$$\dot{x}(t) = -\alpha \int_{-h}^{0} x(t + \theta) \, d\theta \{1 - x(t)^2\} \tag{4.21}$$

has periodic solutions with symmetry properties (1), (2), and (3). In this case our symmetry conditions imply

$$\int_{-h}^{0} x(p/4 + \theta)d\theta = 0$$

and we easily conclude that p must be $2h$. Considering an operator T^* defined by the formula

$$T^*(y(t)) = \tanh\left\{-\alpha \int_0^t \int_{-h}^0 y(\tau + \theta)\, d\theta\, d\tau\right\}$$

for y in a space similar to Ω, we are able to conclude that there is a non-trivial periodic function of period $2h$ which has symmetry properties (1), (2), and (3) and is a fixed point under T^* and consequently a solution of (4.21).

As our next example we note that for each $\alpha > (\pi/2)$ the equation

$$\dot{x}(t) = -\alpha x(t - 1)\{1 - x(t)^2\}, \tag{4.22}$$

has periodic solutions of period 4 which are quasi-sinusoidal in the sense of symmetry conditions (1), (2), and (3). Furthermore, it may be shown that for each $\alpha \geqq 3$ the corresponding solution with the indicated properties is unique to within translations. The existence of such solutions may be established in the manner previously discussed for eqs. (4.17) and (4.21) using an operator T_1 defined by the formula

$$T_1(y(t)) = \tanh\left\{-\alpha \int_0^t y(\tau - 1)\, d\tau\right\}, \tag{4.23}$$

or an operator T_2 defined by the formula

$$T_2(y(t)) = \tanh\left\{\alpha \int_{1-t}^1 y(\tau)\, d\tau\right\}, \tag{4.24}$$

where in each case y is in an appropriate space. Our uniqueness result is simply a matter of observing that there is a neighborhood of the trivial element in the function space on which T_2 is defined which contains no non-trivial fixed points and that outside this neighborhood $T_2{}^2$ is a contracting mapping.

At this point it seems appropriate to consider a few function theoretical properties of the periodic solutions of the functional equations discussed in this section. First of all it may be shown that the periodic solutions of eq. (4.9) and eq. (4.16) are analytic in a strip of the complex plane containing the real line and it seems reasonable to suppose that the same is true for the periodic solutions of much more general equations in the class represented by eqs. (4.1) and (4.10).

Based on the following observations of Professor A. J. Macintyre, it seems clear that there is a very interesting connection between equations of type (4.10) and elliptic function theory. For let us consider the Jacoby elliptic functions $sn(u)$, $cn(u)$, and $dn(u)$ defined by the formulas

$$sn(u) = \sin \eta$$

$$cn(u) = \cos \eta$$

and

$$dn^2(u) = (1 - k^2 \sin^2 \eta)$$

where $0 < k^2 < 1$ and

$$u = \int_0^\eta \frac{dv}{\sqrt{1 - k^2 \sin^2 v}}.$$

Letting

$$\alpha = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

we have $(\pi/2) < \alpha < \infty$,

$$\begin{aligned}\frac{d}{du} sn(u) &= cn(u) \cdot dn(u) \\ &= sn(u + \alpha)\{1 - k^2 sn^2 u\},\end{aligned}$$

and

$$sn(u - \alpha) = -sn(u + \alpha).$$

Making the change of variables $y(u) = ksn(u)$ and $t = u/\alpha$ we get that

$$\frac{dy(t)}{dt} = -\alpha y(t - 1)\{1 - y(t)^2\}.$$

Hence $k\, sn(\alpha t)$ is a solution of (4.22) and it is well known that this function is periodic of period 4 for all $\alpha > (\pi/2)$ and satisfies symmetry condition (1), (2), and (3). By our previously mentioned uniqueness result concerning periodic solutions of eq. (4.22) it follows that to within translations this Jacoby elliptic function is the unique periodic solution of (4.22) of period 4. It may be shown, however, that equation (4.22) has other elliptic solutions of the same form but of smaller period if α is chosen sufficiently large.

As has been pointed out in [10] essentially the same type of periodic behavior as exhibited by eqs. (4.8) and (4.22) has been shown to exist for more general scalar differential-difference equations of the form

$$\dot{x}(t) = F(\alpha, x(t), x(t - 1)), \tag{4.25}$$

where α is a real parameter and $F(\alpha, u, v)$ is jointly continuous in its arguments and locally Lipschitzian in u. For these results it is assumed that $F(\alpha, u, v)$ can be written in the form $F(\alpha, u, v) = f(\alpha, v)g(u)$ when $f(\alpha, v) = \alpha v + 0(v^2)$ as $v \to 0, f(\alpha, v) = 0$ if and only if $v = 0$, and there exists a largest interval (a, b) containing the origin on which g is nonzero. One of the numbers a and b may be ∞ or $-\infty$ providing the other in a simple zero of g. It can be shown that for α properly chosen (4.25) has a manifold of periodic solutions of period ω with the following properties:

(*a*) If ψ is contained in M and $\psi(t_0) = 0$, then ψ has a single simple zero z_0 in $(t_0, t_0 + \omega)$.

(*b*)
$$\int_t^{t+\omega} \psi(\tau)\, d\tau = 0,$$
for all real t.

(*c*) For t in $(t_0, t_0 + \omega)$, $\dot{\psi}(t) = 0$ if and only if $t = t_0 + 1$ or $t = z_0 + 1$.

These results may be verified without difficulty using the techniques developed in [7] coupled with a straightforward perturbation procedure. Results of similar nature are also considered in [4].

As an example of a non-scalar system for which interesting results are obtainable using the techniques of this paper, we mention the following n-dimensional system:

$$(4.26) \quad \dot{x}_i(t) = \left(a_i - \sum_{j=1}^{n} \int_{-h}^{0} x_j(t + \theta) d\eta_{ij}(\theta)\right) x_i(t), \qquad i = 1, 2, \ldots, n.$$

(4.26) is a generalization of the famous Volterra-Lotka model used in mathematical biology to describe fluctuations in populations in a society of n species of organisms competing for food and mutually inhibiting or complimenting the growth of each other in various ways. The system also has other interpretations in biology and in the theory of mutually inhibiting and complimenting chemical reactions. Numerous special cases of (4.26) are considered from a heuristic point of view in biological literature of which references [17] and [18] are representative. We shall not present a detailed description of the behavior of solutions of this equation, since to do so properly would in itself require a paper of considerable length. However, a few superficial remarks seem appropriate at this point.

Let us define $C^+([-h, 0)] \subset C([-h, 0])$ to be such that $\varphi = (\varphi_1, \ldots, \varphi_n)$ in $C^+([-h, 0])$ implies $\varphi_i(\theta) > 0$ for $i = 1, 2, \ldots, n$ and all θ in $[-h, 0]$. One important observation to make in the analysis of (4.26) is that in the physically meaningful applications discussed above the parameters of this equation are such as to imply that T as defined in the previous section is a positive operator on $C^+([-h, 0])$ for all t. That is,

$t \geqq 0$ and φ contained in $C^+([-h, 0])$ imply $T(\varphi, t)$ is contained in $C^+([-h, 0])$. In addition, if φ is in $C([-h, 0])$ and $\varphi(0) = 0$, then $T(\varphi, t) = 0$ for $t \geqq h$. Furthermore, one may show in the most interesting cases that T is a uniformly bounded operator on bounded subsets of $C^+([-h, 0])$. More detailed analysis of (4.26) is centered around an investigation of the linear algebraic equation

$$Ax = a \tag{4.27}$$

when A is the $n \times n$ matrix $\{\eta_{ij}(0) - \eta_{ij}(-h)\}$ and a is the vector $(a_1, \ldots, a_n)$. Solutions of (4.27) obviously determine critical points of (4.26) and in many interesting cases these critical points can be shown to be asymptotically stable. In other cases one can establish the existence of periodic oscillations by the techniques discussed in the previous section of this paper.

As a closing remark we mention an observation made by K. L. Cooke concerning the existence of periodic solutions of higher frequency. In particular, let us consider functional equations of the form

$$\dot{x}(t) = \alpha f(x(t), x(t-1)), \tag{4.28}$$

and assume we have established the existence of at least one periodic solution of (4.28) for each value of $\alpha > \alpha_0$ where α_0 is some specified positive constant. Choosing a particular value of $\alpha_1 > \alpha_0$, let x_α denote a corresponding periodic solution of period ω_α for eq. (4.28). We make the change of variables $t = \theta\tau$, where θ is a parameter to be specified, and define $y(\tau) = x_\alpha(\theta\tau)$. Clearly we have

$$\begin{aligned} \dot{y}(\tau) &= \theta\alpha f(y(\tau), x_\alpha(\theta\tau - 1)) \\ &= \theta\alpha f(y(\tau), x_\alpha(\theta\tau - 1 - n\omega_\alpha)), \end{aligned}$$

where n is an arbitrarily chosen positive integer. If we choose $\theta = 1 + n\omega(\alpha)$, we have that

$$\dot{y}(\tau) = \theta\alpha f(y(\tau), y(\tau - 1)),$$

so y is a periodic solution of (4.26) when $\alpha = \theta\alpha_1$. Since the period of y is clearly equal to $\omega_\alpha/(1 + n\omega_\alpha)$ we see that we may select periodic solution of arbitrarily small period by choosing α sufficiently large. Obviously then if ω_α can be shown to be bounded away from zero as α tends to infinity (as can be done in the cases of eqs. (4.9) and (4.16)), there must exist more than one periodic solution of (4.26) corresponding to each α sufficient large.

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