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1. Dynamical Equations of an N-Port.

Consider an $N \times N$ matrix function $Z(\cdot)$ of the complex variable s. Assume that it is the impedance matrix of a finite, time-invariant, passive RLCTT (resistor, inductor, capacitor, ideal transformer, gyrator) N-port. Then, as is well known, $Z(\cdot)$ has the following properties

(i) Every element of $Z(\cdot)$ is a ratio of two relatively prime polynomials with real coefficients.

(ii) For every s with $\text{Re } s \geq 0$, the hermetian matrix

$$Z(s) + Z'(\bar{s}) \quad (' = \text{transpose}, \ \bar{} = \text{complex conjugate}) \tag{1.1}$$

is nonnegative definite.

If gyrators are not allowed, i.e., if the N-port is reciprocal, then we must have also

(iii) $Z(\cdot)$ is symmetrical, i.e., $z_{ij}(\cdot) = z_{ji}(\cdot)$.

We shall also assume:

(iv) $Z(\infty) = 0$, i.e., the degree of every numerator polynomial in $Z(\cdot)$ is less than the degree of the corresponding denominator.

Requirement (iv) is not really restrictive but it will simplify considerably the formulas which are to follow.

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The matrix function $Z(\cdot)$ represents the input-output relations of the N-port. It is natural to examine the relation between this "external" description of the N-port and the "internal" description in terms of dynamical or state variables. That is, how does one associate a state with the N-port described in terms of its impedance matrix?

This question of representation has been settled recently by the writer (see particularly [1]). It turns out that every "transfer function" matrix $Z(\cdot)$ which has properties (i) and (iv), but not necessarily (ii) and (iii), one may associate a system of vector differential equations of the form

$$dx/dt = Fx + Gu(t), \tag{1.2}$$

$$y(t) = Hx(t). \tag{1.3}$$

Here x, the <u>state</u>, is an n-vector; $u(\cdot)$, the <u>input</u> (current) is an N-vector, and $y(\cdot)$, the <u>output</u> (voltage), is also an N-vector. F, G, H are constant linear transformations. We call (1.2 -3) a <u>finite dimensional, constant linear dynamical system</u> [1].

If equations (1.2-3) are known, the matrix $Z(s)$ can be written down by inspection by taking the formal laplace transform of (1.2). The result is expressed by the formula

$$Z(s) = H(sI - F)^{-1}G \quad (I = \text{unit matrix}). \tag{1.4}$$

Given $Z(\cdot)$, the determination of F, G, and H in (1.2-3) is much less trivial. Some set $\{F, G, H\}$ satisfying (1.4) always exists. Moreover, there is a smallest integer n_o such that relations (1.2-4) hold simultaneously. Generally speaking, this smallest dimension n_o is a complicated function of the matrix $Z(\cdot)$. If n in (1.2-3) is larger than n_o, then the dynamical system (1.2-3) is said to be <u>reducible</u>. The number n_o is identical with the so-called degree of $Z(\cdot)$ as defined by McMillan [2-4]. A numerical method for machine computing n_o was given in [1]. Alternately, McMillan's definition yield n_o via the so-called Smith canonical form of polynomial matrices [5].

From the mathematical point of view, equations (1.2-3) may be viewed as representing an abstract dynamical system defined with respect to an abstract vector space X. To give these equations concrete meaning, we must choose a specific coordinate system or basis in X and express the abstract vector x and the abstract linear transformations F, G, and H in numerical form. Once this has been done, x becomes an n-tuple of real numbers and F, G, H become $n \times n$, $n \times N$, and $N \times n$ arrays (matrices) of real numbers.

Any system (1.2-3) given in numerical form, as just described, is called a realization of $Z(\cdot)$ (see [1]). Mathematically, the term "realization" means that we pass from the abstract to the concrete (numerical) description. Physically, the term "realization" is motivated by the fact [5] that any numerically given system of equations (1.2-3) may be interpreted as the program for an analog computer which simulates the given N-port.

Each realization corresponds to a specific choice of a coordinate system for the state vector. Our ultimate objective is to obtain that subclass of realizations which can be identified with a passive network, not merely with an analog computer program.

The next problem concerns the study of the relationships between various realizations of $Z(\cdot)$. This is indeed the main idea motivating the research discussed here. The problem is clearly of a group theoretical nature. We ask: What is the group of transformations which leave the properties of a given realization invariant?

Suppose we pick two bases for representing the abstract vector x. In the first, the vector is x described by the numerical n-tuple $\xi = (\xi_1, \ldots, \xi_n)$ and in the second it is represented by the n-tuple $\hat{\xi} = (\hat{\xi}_1, \ldots, \hat{\xi}_n)$. It is well known [7, p. 82] that $\xi_1 \hat{\xi}$ are related by a nonsingular linear transformation, so that

$$\hat{\xi} = T\xi \quad \text{or} \quad \hat{\xi}_i = \sum_{i=1}^{n} t_{ij}\xi_j, \quad i = 1, \ldots, n \tag{1.5}$$

where $T = [t_{ij}]$ is a constant, real, nonsingular matrix. The matrices T form the general linear group.

It is convenient to abuse notation and employ the symbols x, $\hat{x}$; F, $\hat{F}$, ... also for the n-tuples representing the vector x and matrices representing the linear transformation F with respect to certain specified bases. Now if the set $\{F, G, H\}$ specifies the dynamical system in the first basis, then the set $\{\hat{F}, \hat{G}, \hat{H}\}$ which specifies the same system with respect to the second basis is related to the first set by the relations

$$\begin{aligned} \hat{F} &= TFT^{-1} \\ \hat{G} &= TG \\ \hat{H} &= HT^{-1} \end{aligned} \tag{1.6}$$

which are easily derived using (1.2-3) and (1.5). (See [1].). Different choices of bases correspond to different realizations of the same $Z(\cdot)$. Therefore one would certainly expect that $Z(\cdot)$ is invariant with respect to a change in basis. This is verified with the aid of (1.6):

$$\begin{aligned} Z(s) &= H(sI - F)G = HT^{-1}T(sI - F)^{-1}T^{-1}TG \\ &= \hat{H}(sI - TFT^{-1})^{-1}\hat{G} = \hat{H}(sI - \hat{F})^{-1}\hat{G}. \end{aligned}$$

Conversely, one may ask: In what way do any two realizations of $Z(\cdot)$ differ from one another? The answer is [1] that if they are irreducible ($n = n_0$), then they differ only by a choice of basis. The criterion for irreducibility is that the triple $\{F, G, H\}$ be completely controllable and completely observable [1]. Thus if two completely controllable and completely observable triples $\{F, G, H\}$ and $\{\hat{F}, \hat{G}, \hat{H}\}$ yield the same transfer function matrix $Z(\cdot)$, then they are necessarily connected by the relations (1.6). Note that this is an abstract result; in practical cases is may be quite difficult to find the transformation T.

We now rephrase this important fact in such a way as to emphasize its group theoretical character:

THEOREM. Any two irreducible realizations (1.2-3) of a transfer function matrix $Z(\cdot)$ having properties (i) and (iv) are equivalent under the general linear group.

Now if $Z(\cdot)$ is the impedance matrix of a passive N-port, it will have certain other properties (namely (ii-iii) above) in addition to those needed to establish this theorem. One would therefore expect to find more restricted types of realizations which are invariant with respect to certain subgroups of the general linear group. The determination of these "network subgroups" is identical with the problem of studying all possible network realizations of a given $Z(\cdot)$, which is also called the problem of network equivalence. An important advantage of the group theoretic approach we wish to explore here is that it provides a unified way of studying the problem of synthesis by different classes of elements. For instance, the RLCT and RLCTT synthesis problems can be studied simultaneously. (See [8].)

It is important to bear in mind the conceptual distinction between the "impedance" transfer function and the "state variable" points of view in network theory.

Transfer and impedance functions are coordinate-free notions. They are most useful in studying properties of networks regardless of their internal structure. This is the deeper reason why existence criteria (such as positive realness) are stated more conveniently in terms of $Z(\cdot)$ then in terms of the triple $\{F, G, H\}$. This observation is not confined to network theory [9 - 10].

On the other hand, dynamical equations (1.2-3) always involve coordinates. These equations are most useful in the detailed study of the internal structure of a network. Such considerations have been generally absent from classical network theory, which may explain in part the difficulties encountered in resolving network equivalence problems.

The group-theoretical approach suggested here is completely analogous to the famous Erlanger Programm of Felix Klein. There have never been any systematic effort to apply Klein's ideas to network theory, as far as the writer is aware. It should be pointed out, however, that the work on network equivalence of Cauer [9, see particularly pages xviii, 49, and Chapter 10] was certainly a conscious step in the same direction. Much more can be done along these lines.

2. Restrictions Due the Passivity and Reciprocity.

The fact that $Z(\cdot)$ represents a passive N-port imposes certain restrictions on the matrices F, G, and H. These restrictions are the counterpart of properties (ii-iii) of the impedance matrix.

First of all it is necessary to identify the components x_i of the state n-tuple x with physical variables in the network. We adopt the following convention, which is both standard and convenient [12].

Let us consider an N-port which contains n_L inductors and $n_C = n - n_L$ capacitors. Then we define

$$x_i = \text{current through i-th inductor, when } i = 1, \ldots, n_L;$$

(2.1)

$$x_j = \text{voltage across i-th capacitor, when } j = n_L + 1, \ldots, n.$$

Assuming for a moment that none of the inductors are coupled with each other, and the same for the capacitors, it follows from linearity that the dynamical equations of the network may be written in the form

$$(2.2)\quad \begin{cases} L_i dx_i/dt = \sum_{k=1}^{n} a_{ik}x_k + \sum_{\ell=1}^{N} b_{i\ell}J_\ell(t) & i = 1, \ldots, n_L \\ C_j dx_j/dt = \sum_{k=1}^{n} a_{jk}x_k + \sum_{\ell=1}^{N} b_{j\ell}J_\ell(t) & j = n_L+1, \ldots, n \\ U_\ell(t) = \sum_{k=1}^{n} h_{\ell k}x_k(t), & \ell = 1, \ldots, N. \end{cases}$$

The $J_\ell(t)$ are the currents entering the ports and the $U_\ell(t)$ are the voltages across the ports.

To derive the numbers a_{ik} and $b_{i\ell}$ we may replace, for an instant, an inductor by a current source and a capacitor by a voltage source. We then obtain the following interpretation.

Let i_1, i_2 be integers belonging to $[1, n_L]$, j_1, j_2 integers belonging to $[n_L, n]$, and ℓ an integer belonging to $[1, N]$.

$a_{i_1 i_2}$ = voltage across i_1-th inductor when all capacitors are short circuited, all ports and all inductors save the i_2-th are open circuited, and the i_2-th inductor is replaçed by a unit current source.

$a_{i_1 j_2}$ = voltage across the i_1-th inductor when all ports and all inductors are open circuited, all capacitors save the j_2-th are short circuited, and the j_2-th capacitor is replaced by a unit voltage source.

$b_{i_1 \ell}$ = voltage across the i_1-th inductor when all capacitors are short circuited, all inductors are open circuited, all ports save the ℓ-th are open circuited, and a unit current source is connected across with ℓ-th port.

The other quantities are defined analogously.

It is clear that the matrices A and B depend only on that part of the network which contains the resistors and ideal transformers. Partioning these matrices according to the numbering scheme introduced above, we can easily see what restrictions are imposed by passivity and/or reciprocity. If we write

$$A = \left[\begin{array}{c|c} A_1 & C_2 \\ \hline C_1 & A_2 \end{array}\right] \tag{2.3}$$

then A_1 has the dimension of resistance, A_2 has the dimension of conductance, while C_1 and C_2 are dimensionless.

Passivity implies that A_1 and A_2 are nonpositive (but not necessarily symmetric) matrices i.e., their quadratic forms are nonpositive. Moreover, the quadratic form

$$x' \begin{bmatrix} 0 & C_1 \\ C_2 & 0 \end{bmatrix} x$$

must be identically 0, i.e., $C_2 = - C_1'$.

Reciprocity implies that A_1 and A_2 are symmetric (but not necessarily nonpositive).

As far as B and H are concerned,

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

If $B_1 \neq 0$ then $Z(\infty) \neq 0$ and assumption (iv) of the previous section is violated. $H_1 = 0$ for the same reason. Passivity requires that H_2B_2 be nonnegative definite. Reciprocity requires $B_2 = H_2'$.

After removing all inductors and capacitors the N-port becomes an (N + n)-port. This (N + n)-port does not necessarily possess either an impedance or admittance matrix because there may be open circuits or short circuits at certain ports. In such cases, one adopts a suitable mixed impedance-admittance description of the last n ports. We do not wish to dwell upon the complications — all trivial — which result in such cases.

Suppose we write the energy stored in the N-port as

$$E = \frac{1}{2} x'Px = \frac{1}{2} \sum_{i=1}^{n_L} L_i x_i^2 + \frac{1}{2} \sum_{j=n_L+1}^{n} C_j x_j^2 .$$

Then equations (2.2) take the simple form

(2.2') $$P dx/dt = Ax + BJ(t), \quad U(t) = Hx(t).$$

These equations are valid even if there is coupling between the inductors and capacitors. Thus in general P will have the form

(2.5) $$P = \left[\begin{array}{c|c} L & 0 \\ \hline 0 & C \end{array}\right].$$

The off-diagonal terms of P are 0 because in conventional networks there is no coupling between electric and magnetic fields. Since the stored energy must be a positive definite quadratic form, we assume that L and C are positive definite matrices. Reciprocity requires that they be symmetrical matrices. In the conventional RLCT cases L is symmetric, positive definite, while C is a matrix with positive entries on the diagonal and zeros elsewhere.

The case when L is merely nonnegative definite indicates the presence of ideal transformers. In this case the number of state variables is too large. We shall not discuss the resulting complications.

If the dynamical equations (2.2') of the N-port are reducible, i.e., may be replaced by a smaller set of equations having the same impedance matrix Z, then the network contains dynamical modes which are not specified by the impedance matrix Z but arise solely as a result of the synthesis procedure. For instance, the Darlington and the Bott-Duffin procedures introduce such extraneous modes. Although the presence of such additional modes may be necessary to carry out certain types of synthesis procedures, in this paper we shall be concerned only with the irreducible case. In other words, it will always be assumed that the N-port always contains a minimal number n_o of reactive elements.

3. Characterization of Passivity.

Now we shall state a relation between the impedance Z and the state variable description of an N-port. This relation was discovered in the course of studying the so-called Lur'e problem of construction Lyapunov functions for dynamical systems which are linear save for a single nonlinear element [9].

The result to be stated below is more general than the Main Lemma in [9], in that we admit $N \times N$ rather than 1×1 impedance matrices and we drop the assumption that all eigenvalues of F have negative real parts. On the other hand, we will assume here, as a matter of convenience, that $Z(\infty) = 0$. This is an unessential restriction which was not needed in the Main Lemma of [9]. For a full treatment of the general problem, including proofs, see [13, 14].

CHARACTERIZATION OF PASSIVITY THEOREM. *Let* $Z(\cdot)$ *be an* $N \times N$ *matrix of rational functions of the complex variable* s, *with* $Z(\infty) = 0$. *Let* $\{F, G, H\}$ *be a triple such that* (1.2-3) *is an irreducible realization of* $Z(\cdot)$. *Let* $\Psi(y)$ *be a continuous p-vector function of the p-vector* y *such that* $\Psi(0) = 0$ *and* $y'\Psi(y) \geq 0$ *for all* y.

Then the following statements are equivalent:

(I) $Z(\cdot)$ is nonnegative real, i.e., $\mathrm{Re}\, s \geq 0$ implies $Z(s) + Z'(s) =$ nonnegative definite hermetian matrix.

(II) There exists a symmetric, positive definite matrix P and a symmetric, nonnegative definite matrix Q such that

$$PF + F'P = -2Q, \tag{3.1}$$

$$PG = H. \tag{3.2}$$

(A matrix F satisfying (3.1-2) cannot have an eigenvalue with positive real part; in its Jordan form all imaginary eigenvalues are contained in 1×1 blocks; the null space of the matrix Q is necessarily contained in the eigenspace of F spanned by the eigenvectors corresponding to imaginary eigenvalues.)

(III) $V = x'Px$ is a Lyapunov function for the autonomous dynamical system $dx/dt = Fx - G\Psi(Hx)$ such that $\dot{V}(x) \leq 0$.

Let us give an indication of the proof. (II) implies (I) by direct substitution. Given (I), the nonnegative real character of $Z(i\omega)$ allows it to be factored as $Z(i\omega) + Z'(-i\omega) = W'(i\omega)W(i\omega)$. Every such factorization gives rise to a Q. The irreducibility of the realization is then utilized to verify that (3.1-2) hold. (II) implies (III) because

$$\begin{aligned}\dot{V}(x) &= 2x'PFx - 2x'G'P\Psi(Hx)\\ &= -2[x'Qx + y'\Psi(y)] \leq 0\end{aligned}$$

by (3.1-2). Finally, (III) implies (I) by direct computation.

Remarks. (1) In a 1958 paper, Desoer [15] pointed out that at that time the stability of passive nonreciprocal networks has not yet been proved and gave an argument showing that all such networks are, in fact, (Lyapunov) stable. This result is now confirmed in the strongest possible way. Since the nonnegative realness of $Z(\cdot)$ (as stated above, i.e., not requiring $Z(\cdot) = Z'(\cdot)$) is a necessary condition for realizing a passive RLCTT N-port, it is clear from the parenthetical remark in (II) that F cannot have eigenvalues with positive real parts, nor can it have solutions of the type $t^k \cos(\omega t + \alpha)$, $k > 0$.

(2) For networks, $V = x'Px$ can always be identified with the stored energy in the inductors and capacitors. Hence (III) shows that in unterminated passive networks ($\Psi = 0$) the energy is a nonincreasing function of time. For arbitrary passive resistive terminations, which may be either linear or nonlinear, the same conclusion holds. In the nonlinear case this result represents a considerable improvement over what was known before. (To my knowledge, the previous best general result here is that of Duffin [16].)

The characterization theorem expresses facts which are usually taken for granted from an intuitive physical point of view. As a result, the average engineer may be tempted to jump to the conclusion that nothing new has been done. But to do this would be a gross misunderstanding of the processes of scientific research. The theorem is a more precise and more general characterization of passivity than was heretofore available. The precision and the generality of this result in turn leads to deep insight into problems of network synthesis [8, 14], which can now be studied with a simplicity and explicitness that was impossible previously.

This theorem is a very convincing additional piece of evidence that passivity and network synthesis are intimately related. In [8] I shall present some preliminary results which show that the structural properties of the class of all networks which realize a given impedance function $Z(\cdot)$ are in 1-1 correspondence with certain algebraic invariants of $Z(\cdot)$.

4. Applications.

We shall illustrate the usefulness of the characterization theorem and the group theoretic ideas related to it by two classical examples: (i) LC 1-port synthesis according to Foster, and (ii) synthesis of lossless 2-ports.

EXAMPLE 1. We wish to prove that

A necessary and sufficient condition in order that the scalar impedance function $z(\cdot)$ be realizable as the 1-port shown in Fig. 1 is that

(i) $z(\cdot)$ is nonnegative real and

(ii) all poles of $z(\cdot)$ are imaginary.

Necessity: If the state variables for the network are assigned as shown in Fig. 1, it follows by inspection that the A, B, H, and P matrices are

(4.1)
$$A = \left[\begin{array}{c|cc|c|cc} 0 & & & & & \\ \hline & 0 & 1 & & & \\ & 1 & 0 & & & \\ \hline & & & \ddots & & \\ \hline & & & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right]$$

There are r 2×2 blocks on the diagonal

(4.2)
$$B = \left[\begin{array}{c} 1 \\ \hline 0 \\ 1 \\ \hline \vdots \\ \hline 0 \\ 1 \end{array}\right]$$

$$H = [1 \;\vdots\; 0 \;\; 1 \;\vdots\; \ldots \;\vdots\; 0 \;\; 1] \tag{4.3}$$

$$P = \left[\begin{array}{c:cc:c:cc} C_o & & & & & \\ \hdashline & L_1 & 0 & & & \\ & 0 & C_1 & & & \\ \hdashline & & & \ddots & & \\ \hdashline & & & & L_r & 0 \\ & & & & 0 & C_r \end{array}\right]. \tag{4.4}$$

It is clear that the quintuple

$$F = P^{-1}A, \quad G = P^{-1}B, \quad H = H, \quad P = P, \quad Q = 0$$

satisfies (3.1-2). Hence the impedance function of the 1-port, which is given by

$$z(s) = \frac{1}{C_o s} + \sum_{k=1}^{r} \frac{s}{C_k(s^2 + \omega_k{}^2)}, \qquad \omega_k^2 = 1/L_k C_k \tag{4.5}$$

is a nonnegative real function.

Sufficiency: Suppose that $z(\cdot)$ is nonnegative real. Then relations (3.1-2) are satisfied. If, in addition, all poles of $z(\cdot)$ are imaginary, then Q in (3.1) is necessarily zero.

We shall now carry out a series of changes of basis which will eventually exhibit the matrices $A = PF$, $B = PG$, $H = H$, and $P = P$ in the form (4.1-4), after which the existence of the network shown in Fig. 1 is obvious by inspection.

Step 1. We pick a basis that $P_{(1)} = I$. Then by (3.1) $F_{(1)} = -F'_{(1)}$. This is always possible because under a change of basis (1.5) we have

$$P = T'\hat{P}T \tag{1.6'}$$

Since P is positive definite, it can always be written as $P = T'IT$, T nonsingular, so that $\hat{P} = P_{(1)} = I$.

Step 2. We pick a second basis which leaves P invariant $(P_{(2)} = I)$ but transforms $F_{(1)}$ into the canonical form

$$F_{(2)} = \left[\begin{array}{c|cc|c|cc} 0 & & & & & \\ \hline & 0 & \omega_1 & & & \\ & -\omega_1 & 0 & & & \\ \hline & & & \ddots & & \\ \hline & & & & 0 & \omega_r \\ & & & & -\omega_r & 0 \end{array}\right].$$

This is always possible because [17] every skew symmetric matrix can be transformed into the canonical form (4.6) by means of an orthogonal transformation. Note that the element 0 in the upper left-hand corner of $F_{(2)}$ corresponds to the eigenvalue 0 (if $F_{(1)}$ happens to have that eigenvalue), while the 2×2 blocks along the diagonal correspond to the eigenvalues $\pm i\omega_k$, $k = 1, \ldots, r$. (Since $z(\cdot)$ has only imaginary poles, F has only imaginary eigenvalues; by nonnegative realness all eigenvalues must be simple. Thus $F_{(2)}$ is as it should be.)

In the basis (2), the column vectors B, H' have the form

$$G_{(2)} = B_{(2)} = H'_{(2)} = \begin{bmatrix} \beta_o \\ \hline \alpha_1 \\ \beta_1 \\ \hline \vdots \\ \hline \alpha_r \\ \beta_r \end{bmatrix}.$$

Since the triple $\{F, G, H'\}$ must be irreducible and therefore completely controllable and completely observable if (II) of the characterization theorem is to hold, it is easily verified that $\alpha_o > 0$ and $\alpha_k^2 + \beta_k^2 > 0$, $k = 1, \ldots, r$.

<u>Step 3.</u> Any proper (det = 1) orthogonal transformation (rotation) applied to the 2×2 matrix

$$\begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}$$

leaves it invariant. We apply r such rotations corresponding to the r blocks in F, in such a way that the vectors

$$\begin{bmatrix} \alpha_k \\ \\ \beta_k \end{bmatrix}, \qquad k = 1, \ldots, r$$

in B and H' are rotated into the vectors

$$\begin{bmatrix} 0 \\ \\ \gamma_k \end{bmatrix}, \qquad k = 1, \ldots, r.$$

By the comment made at the end of the preceding paragraph we know that $\gamma_k = \alpha_k^2 + \beta_k^2 \neq 0$. Thus $F_{(3)} = F_{(2)}$, $P_{(3)} = P_{(2)} = I$ and

$$G_{(3)} = B_{(3)} = \begin{bmatrix} \beta_0 \\ \hline 0 \\ \gamma_1 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline 0 \\ \gamma_r \end{bmatrix}.$$

Step 4. Now we obtain change to a basis where A, B and H have only 0, 1, -1 as elements. Fortunately this can be accomplished with the aid of a diagonal matrix. If $x_{(3)} = \Lambda x_{(4)}$, where Λ is diagonal, then

$$P_{(4)} = \Lambda P_{(3)} \Lambda = \Lambda^2,$$

$$B_{(4)} = P_{(4)}G_{(4)} = \Lambda P_{(3)}\Lambda\Lambda^{-1}G_{(3)} = \Lambda B_{(3)}$$

and

$$A_{(4)} = P_{(4)}F_{(4)} = \Lambda P_{(3)}\Lambda\Lambda^{-1}F_{(3)}\Lambda = \Lambda A_{(3)}\Lambda.$$

We choose

$$\Lambda = \left[\begin{array}{c|cc|c|cc}
1/\beta_o & & & & & \\
\hline
 & \gamma_1/\omega_1 & 0 & & & \\
 & 0 & 1/\gamma_1 & & & \\
\hline
 & & & \ddots & & \\
\hline
 & & & & \gamma_r/\omega_r & 0 \\
 & & & & 0 & 1/\gamma_r
\end{array}\right].$$

It is easily checked that $A_{(4)}$, $B_{(4)}$, $H_{(4)}$ assume the form (4.1 - 4.3). If we let

$$C_o = 1/\beta_o,$$
$$C_k = 1/\gamma_k, \qquad k = 1, \ldots, r;$$
$$L_k = \gamma_k/\omega_k, \qquad k = 1, \ldots, r;$$

then the same can be said also about $P_{(4)}$.

The proof is completed.

It may appear to the reader well-versed in classical synthesis theory that the usual Foster solution, obtained by a partial fraction expansion of (4.5), is a much simpler road to obtaining the network. Actually, the total number of arguments is not really smaller than in the matrix case. Our present method, however, has the very important additional feature that it gives insight into the group structure of passive 1-ports. Only a few more arguments are needed to completely describe in this way all passive 1-ports which contain a minimal number of reactive elements. The partial fraction manipulations used in the Foster theory can be interpreted as calculations based on the group representation provided by the laplace transform.

EXAMPLE 2. Let us consider the impedance matrix

$$Z(\cdot) = \begin{bmatrix} \frac{k_{11}s}{s^2+1} & \frac{k_{12}s}{s^2+1} \\ \frac{k_{12}s}{s^2+1} & \frac{k_{22}s}{s^2+1} \end{bmatrix}.$$

We wish to realize this matrix with a (lossless) reciprocal 2-port containing a minimum number of reactive elements.

Reciprocity requires that $k_{21} = k_{12}$. Passivity requires (as may be easily checked using the characterization theorem)

$$\Delta = k_{11}k_{22} - k_{12}^2 \geq 0,$$

which is popularly known as the "residue condition". In fact, the matrix of residues of $Z(\cdot)$ at either pole $s = \pm i$ is given by

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

and $\Delta = \det K$.

Two possibilities may arise: Either $\Delta > 0$ or $\Delta = 0$.

The minimum number of reactive elements to be used is equal to the minimum number n_o of state variables which are necessary and sufficient to realize $Z(\cdot)$ as a dynamical system (1.2-3). The number n_o can be computed with the aid of a theorem of E. G. Gilbert [Theorem 11], which states (in the special case when $Z(\cdot)$ has distinct poles)

$$n_o = \Sigma \text{ ranks of residue matrices of } Z(\cdot).$$

Hence $n_o = 4$ when $\Delta > 0$ and $n_o = 2$ when $\Delta = 0$. It should be noted that n_o is also equal to the McMillan degree of $Z(\cdot)$ [4].

Let us consider the case $\Delta > 0$. In [1, Sect. 8] two methods were given for constructing realizations of a given transfer function matrix. Using Method (B), it is found that the following matrices provide a realization of $Z(\cdot)$:

$$F_{(1)} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad G_{(1)} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad H_{(1)} = [0 \quad K].$$

If we let

$$P_{(1)} = \begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}, \quad Q_{(1)} = 0,$$

then the quintuple $\{F_{(1)}, G_{(1)}, H_{(1)}, P_{(1)}, Q_{(1)}\}$ satisfies (3.1-2). However, these matrices do not correspond to a network realization of $Z(\cdot)$. In order to obtain such a realization, we introduce a new basis defined by $x_{(1)} = Ux_{(2)}$, where

$$U = \begin{bmatrix} I & 0 \\ 0 & K^{-1} \end{bmatrix}.$$

Then

$$A_{(2)} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad B_{(2)} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad P_{(2)} = \begin{bmatrix} K & 0 \\ 0 & K^{-1} \end{bmatrix}.$$

Referring to the discussion of Sect. 1, it is easily seen that these matrices represent two resonant LC circuits in which there is coupling (represented by the matrices K and K^{-1} respectively) between both the inductors and capacitors.

Since coupled capacitors can always be realized by ordinary capacitors and ideal transformers, we have proved:

When $\Delta > 0$, $Z(\cdot)$ can be realized as an LCT network with a minimal number of reactive elements.

By further easy manipulations of the matrices involved, it can be shown -- as is well known [19, p. 447-453] -- that a single ideal transformer will always suffice.

The question then arises whether or not the 2-port can be realized using only coupled inductors (but not coupled capacitors) and without ideal transformers; further, whether or not the 2-port can be realized without any inductive or capacitive coupling. The answer to both questions is in general no. We shall not give the proof (straightforward) but merely state the result:

When $\Delta > 0$, $Z(\cdot)$ can be realized as an LC network with a minimal number of reactive elements, without ideal transformers, and without any coupling between inductors or capacitors if and only if either (i) $k_{12} = 0$, or (ii) $k_{12} = k_{11}$, or (iii) $k_{12} = k_{22}$. These conditions cannot be weakened if either coupled inductors or coupled capacitors (but not both) are admitted.

In the first case the 2-port consists of two unconnected 1-ports. In the second and third case the 2-port can be realized as an L-section.

I don't know how well known this easy result is. It is quite useful theoretically because it shows that transformerless synthesis implies very strong constraints on $Z(\cdot)$ in addition to nonnegative realness. This fact has a direct bearing on the transformerless synthesis of RLC 1-ports because such problems can always be reduced to the synthesis of lossless N-ports. (See [8].)

The case $\Delta = 0$ may be treated similarly.

5. Conclusions.

(1) Methods based on the impedance concept are coordinate-free. They do not display directly the structural properties of the realization.

(2) State-variable methods on the other hand, are closely related to the structural properties of networks.

(3) Every transfer function matrix admits realizations. Some of these realizations may correspond to networks, while others may require active elements (analog computers).

(4) Nonnegative real impedance matrices always admit passive realizations, i.e. realizations impose restrictions on F, G, H, P, Q in addition to those implied by passivity. In all cases that these restrictions are known they are expressible in terms of the impedance matrix.

This is where the story stands at the moment. I would like to suggest the following program for the future:

To classify all types of realizations of a given abstract dynamical system and to express the realizability conditions as coordinate-free properties (such as passivity, reciprocity, etc.) of abstract dynamical systems.

This problem suggests a partnership between mathematics and network theory which will be intellectually exciting and practically profitable.

Dynamical systems are the building blocks of modern technology. The resolution of the problem posed will tell us what technology is capable of doing at present. It will also suggest developing new components to realize that which is known to be possible for mathematical reasons.

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