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AN ENGINEERING APPROACH TO THE VARIABLE FLUID PROPERTY PROBLEM IN FREE CONVECTION*

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ABSTRACT

An analysis is made for the variable fluid property problem for laminar free convection on an isothermal vertical flat plate. For a number of specific cases, solutions of the boundary layer equations appropriate to the variable property situation were carried out for gases and for liquid mercury. Utilizing these findings, a simple and accurate shorthand procedure is presented for calculating free convection heat transfer under variable property conditions. This calculation method is well established in the heat transfer field. It involves the use of results which have been derived for constant property fluids, and of a set of rules (called reference temperatures) for extending these constant property results to variable property situations. For gases, the constant property heat transfer results are generalized to the variable property situation by replacing β (expansion coefficient) by $1/T_\infty$ and evaluating the other properties at $T_r = T_w - 0.38(T_w - T_\infty)$. For liquid mercury, the generalization may be accomplished by evaluating all the properties (including β) at this same T_r. It is worthwhile noting

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that for these fluids, the film temperature (with $\beta = 1/T_\infty$ for gases) appears to serve as an adequate reference temperature for most applications. Results are also presented for boundary layer thickness and velocity parameters.

NOMENCLATURE

a_n	coefficients in the polynomial representations of fluid properties of liquid mercury
A_1, A_2	dimensional constants in Sutherland's formulas of table 1, $A_1 = 362^O$ F abs, $A_2 = 198.7^O$ F abs
b	plate width
c	dimensional constant defined by eq. (6a)
c_p	specific heat at constant pressure
c_f	coefficient of friction defined by eq. (29)
F	dimensionless dependent variable defined by eq. (6b)
g	acceleration due to gravity
Gr_x	Grashof number based on x, dimensionless, see eqs. (13), (13a), (13b)
Gr_L	Grashof number based on L
h	local heat transfer coefficient, $q/(T_w - T_\infty)$
$\bar{h}$	average heat transfer coefficient, $Q/Lb(T_w - T_\infty)$
k	thermal conductivity
L	plate height
Nu_x	local Nusselt number, hx/k, dimensionless
$\overline{Nu_L}$	average Nusselt number, $\bar{h}L/k$, dimensionless
p	static pressure
Pr	Prandtl number, $c_p\mu/k$, dimensionless
q	local heat transfer rate per unit area of plate

Q	over-all heat transfer rate, $b \int_0^L q \, dx$
T	absolute temperature
t	Fahrenheit temperature
T_f	film temperature, $(T_w + T_\infty)/2$; or $t_f = (t_w + t_\infty)/2$
u	velocity component in x direction
u_{max}	maximum value of u across the boundary layer
v	velocity component in y direction
W	rate of fluid flow generated by free convection, $b \int_0^\infty \rho u \, dy$
x	coordinate measuring distance along plate from leading edge
y	coordinate measuring distance normal to plate
y_u	distance from plate at which u_{max} occurs
β	coefficient of thermal expansional, $-\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_p$
δ_i	boundary layer thickness defined as distance from plate at which $(T - T_\infty)/(T_w - T_\infty) = i$
η	dimensionless similarity variable defined by eq. (6a)
θ	dimensionless temperature variable, $(T - T_\infty)/(T_w - T_\infty)$
μ	absolute viscosity
ν	kinematic viscosity, μ/ρ
ρ	density
τ_w	shear stress at plate surface
ψ	stream function

Subscripts

w	denotes wall conditions
∞	denotes ambient conditions
r	denotes conditions at reference temperature

INTRODUCTION

The presence of a buoyancy force is a requirement for the existence of a free convection flow. Ordinarily, the buoyancy arises from density differences which are a consequence of temperature gradients within the fluid. Any analytical treatment of free convection must include density variations at least to the extent that a buoyancy force enters the problem.

A characteristic common to previous analytical studies of free convection has been the neglect of all fluid property variations, except for the essential density differences noted above*. Such a simplified treatment does not appear unreasonable when the temperature differences involved are small. This intuitive feeling has been corroborated in a formal manner by Ostrach (2). For situations where there are large temperature differences, the adequacy of the results derived from the constant property analysis has been in doubt.

An analytical treatment of the variable property problem, including numerical solutions, first appears to have been given in the thesis (1) from which this paper is taken. Some time after the appearance of Ref. 1, a much less extensive study was described in a Russian article by Tanaev (3). There is no significant overlap between Ref. 3 and the presentation here.

The analysis is made for an isothermal vertical plate, and the flow is taken to be laminar. For a large number of specific cases, numerical

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*The fluid properties entering the problem are the thermal conductivity, viscosity, specific heat, and density.

solutions of the boundary layer equations appropriate to the variable property situation were carried out for gases. It was initially planned to also study the variable property problem for several liquid metals. However, the nature of the property variations and the exceedingly time consuming numerical computations forced the current study to be limited to one liquid metal, namely, mercury.

Prime attention is focused on the heat transfer. Utilizing the heat transfer results corresponding to special cases for which numerical solutions were obtained, a simple and accurate shorthand procedure is presented for computing heat transfer under variable property conditions in gases and liquid mercury. This method is well known in the heat transfer field. It involves the use of results derived for constant property fluids, and of a set of rules for extending the constant property results to variable property situations. These rules are commonly termed reference temperatures. Not only will reference temperatures be derived for the heat transfer, but also for boundary layer thickness and velocity parameters. Those who are interested primarily in results are invited to pass over the section on analysis.

ANALYSIS

Physical model and coordinates. - The physical model and the coordinate system are portrayed in an elevation view in Fig. 1. Two physical situations are shown which come within the scope of the analysis. The left hand sketch depicts the case where the wall temperature, T_w, exceeds the ambient temperature T_∞. Under these circumstances the free-convection motion is upward as shown. The right hand sketch shows the situation where T_w is lower than the ambient temperature T_∞. In this case, the fluid flow is downward along the plate.

If the coordinate systems are taken as indicated, the mathematical distinction between the two situations vanishes when the conservation equations, as written later, are made dimensionless. So, separate analyses need not be made. Since it seems easier to visualize occurrences associated with the hot wall case, i.e., $T_w > T_\infty$, the analysis will be directed toward that situation. However, the results will be presented in a manner applicable to both $T_w > T_\infty$ and $T_w < T_\infty$.

Conservation laws. - The equations expressing conservation of mass, momentum, and energy for steady flow in a boundary layer on a vertical plate are

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} - \rho g + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial p}{\partial y} = 0 \tag{3}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \tag{4}$$

Viscous dissipation and work against the gravity field have been neglected.

The boundary conditions appropriate to the problem are

$$\left.\begin{array}{l} v = 0 \\ u = 0 \\ T = T_w \end{array}\right\} y = 0 \qquad \left.\begin{array}{l} u = 0 \\ T = T_\infty \end{array}\right\} y = \infty \tag{5}$$

where T_w and T_∞ are prescribed constants.

From Eq. (3), it follows that the pressure p is a function of x alone (i.e., a function only of height along the plate). So, $\partial p/\partial x$ is

constant across the boundary layer and may be evaluated as

$$\partial p / \partial x = - \rho_\infty g$$

With this substitution, Eq. (2) becomes

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = g(\rho_\infty - \rho) + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \tag{2a}$$

There appears in this equation a density difference, $\rho_\infty - \rho$, which provides the buoyancy force for the free convection motion.

The customary approach followed in the constant property analysis is first to replace the density difference by a temperature difference, and then to assume that ρ, μ, c_p, and k are constant.

The present analysis continues without placing any restrictions on the nature of the property variations. The solution of Eq. (1) may, as usual, be written in terms of a stream function ψ defined by the relations

$$\frac{\rho}{\rho_w} u = \frac{\partial \psi}{\partial y}, \qquad \frac{\rho}{\rho_w} v = - \frac{\partial \psi}{\partial x} \tag{1a}$$

where ρ_w, the fluid density at the wall, is regarded as a constant. Then, the velocity components u and v which appear in Eqs. (2a) and (4) are replaced in favor of the stream function ψ. The result of the substitution is a rather complicated looking pair of simultaneous partial differential equations for ψ and T as functions of x and y. Rather than deal with these two formidable partial differential equations directly, experience suggests a method of transforming them to ordinary differential equations, which are easier to solve. In the usual terminology of boundary layer theory, such a transformation is called a similarity transformation.

Reduction to ordinary differential equations. - A new independent
variable η, called a similarity variable, is defined by

$$\eta = cx^{-1/4} \int_0^y \frac{\rho}{\rho_w} \, dy$$

where

$$c = \left[\frac{g(\rho_\bullet - \rho_w)/\rho_w}{4\nu_w^2} \right]^{1/4}$$

(6a)

New dependent variables F and θ are given by

$$F(\eta) = \frac{\psi}{x^{3/4}} \cdot \frac{1}{4\nu_w c}, \qquad \theta(\eta) = \frac{T - T_\bullet}{T_w - T_\bullet}$$

(6b)

The function θ is a dimensionless temperature and F is related to the
velocities in the following way

$$u = 4\nu_w c^2 x^{1/2} F', \qquad v = \frac{\rho_w}{\rho} \cdot \frac{\nu_w c}{x^{1/4}} \left[\eta F' - 3F \right]$$

(7)

The primes represent differentiation with respect to η.

Under the transformation Eqs. (6a) and (b), the partial differential
equations for ψ and T become

$$\frac{d}{d\eta} \left[\frac{\rho\mu}{\rho_w\mu_w} F'' \right] + 3FF'' - 2(F')^2 + \frac{(\rho_\bullet/\rho) - 1}{(\rho_\bullet/\rho_w) - 1} = 0$$

$$\frac{d}{d\eta} \left[\frac{\rho k}{\rho_w k_w} \theta' \right] + 3Pr_w \left(\frac{c_p}{c_{p_w}} \right) F\theta' = 0$$

(8)

The Prandtl number is represented by Pr and the subscript w denotes
conditions at the wall ($y = 0$). The boundary conditions, Eq. (5), trans-
form to

$$\left. \begin{array}{l} F = 0 \\ F' = 0 \\ \theta = 1 \end{array} \right\} \eta = 0 \qquad \left. \begin{array}{l} F' = 0 \\ \theta = 0 \end{array} \right\} \eta = \bullet$$

(9)

It may be noted that the transformation relations, as well as the resulting ordinary differential equations, can be reduced directly to the well-known equations for the constant property fluid. Further, all solutions of Eqs. (8) subject to the boundary conditions (9) have the characteristic that $u = 0$ and $T = T_\infty$ along the line $x = 0$.

Simplifications associated with $p = \rho RT$. - For the special case of fluids which obey the perfect gas law, $p = \rho RT$, there are certain simplifications which can be introduced into the analysis. Consider first the buoyancy force, $g(\rho_\infty - \rho)$, which appears in Eq. (2a). For a perfect gas, this density difference can be transformed into a temperature difference without making any approximations. It is only necessary to notice that both ρ_∞ and ρ are to be evaluated at the same value of x, and consequently, at the same static pressure p. With this in mind, it follows directly that

$$\rho_\infty - \rho = \frac{\rho}{T_\infty}(T - T_\infty) \tag{10}$$

In the constant property analysis, it has been customary to write

$$\rho_\infty - \rho = \beta \rho (T - T_\infty)$$

where β is the coefficient of expansion. So, it is seen that the buoyancy force used in the constant property analysis is precisely correct for a perfect gas, provided that β is replace by $1/T_\infty$.

Further, in the differential Eqs. (8), the dimensionless buoyancy force (the last term) can be simplified to

$$\frac{(\rho_\infty/\rho) - 1}{(\rho_\infty/\rho_w) - 1} = \frac{T - T_\infty}{T_w - T_\infty} = \theta \tag{10a}$$

So, Eqs. (8) become

$$\frac{d}{d\eta}\left[\frac{\rho\mu}{\rho_w \mu_w} F''\right] + 3FF'' - 2(F')^2 + \theta = 0$$

$$\frac{d}{d\eta}\left[\frac{\rho k}{\rho_w k_w} \theta'\right] + 3Pr_w\left(\frac{c_p}{c_{p_w}}\right)F'\theta = 0 \qquad\Bigg\}\qquad (8a)$$

It is worthwhile noting that for a very special sort of variable property gas, Eqs. (8a) reduce to the equations for a constant property fluid. Consider a gas having the property variations: $p = \rho RT$, $\rho\mu =$ constant, $\rho k =$ constant, $c_p =$ constant. Inspection of Eqs (8a) show the tremendous simplification afforded by the assumption that $\rho\mu =$ constant and $\rho k =$ constant. It follows that

$$\rho\mu/\rho_w \mu_w = 1, \qquad \rho k/\rho_w k_w = 1$$

and also that $k/\mu =$ constant. Further, it is seen that the Prandtl number ($= c_p\mu/k$) is also a constant. Then Eqs. (8a) may be rewritten as

$$F''' + 3FF'' - 2(F')^2 + \theta = 0$$

$$\theta'' + 3Pr\, F\theta' = 0 \qquad (8b)$$

But, these equations are precisely those for the constant property problem. Also identical are the boundary conditions. So, from the mathematical point of view, the constant property problem is identical to that for the special variable property fluid: $p = \rho RT$, $\rho k =$ constant, $\rho\mu =$ constant, $c_p =$ constant. All solutions which have been obtained for the constant property differential equations become available for this special variable property fluid.

It is interesting to note that a similar finding applies in forced convection. In fact, a common procedure for accounting for variable

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property effects for forced convection over a flat plate is to postulate that the real gas is sufficiently well approximated by the special gas just considered.

HEAT TRANSFER PARAMETERS AND GRASHOF NUMBERS

<u>Local heat transfer</u>. - The local heat flux from the surface to the fluid may be calculated using Fourier's Law

$$q = - \left[k \frac{\partial T}{\partial y} \right]_{y=0}$$

Introducing the dimensionless variables from Eqs. (6a) and (b), the expression for q becomes

$$q = - k_w (T_w - T_\infty) cx^{-1/4} \left[\frac{d\theta}{d\eta} \right]_{\eta=0} \tag{11}$$

The derivative $\left[d\theta/d\eta \right]_{\eta=0}$, normally abbreviated $\theta'(0)$, is found from the solutions of Eqs. (8). A dimensionless representation of the results is achieved by use of the local heat transfer coefficient and local Nusselt number, which are written in the usual way as

$$h \equiv \frac{q}{T_w - T_\infty}, \qquad Nu_x \equiv \frac{hx}{k} \tag{12}$$

Further, a generalized Grashof number which is applicable to both constant and variable property fluids is defined by

$$Gr_x \equiv \frac{gx^3 \left\{ |\rho_\infty - \rho_w|/\rho_w \right\}}{\nu^2} \tag{13}$$

The absolute magnitude sign removes the necessity for separate consideration of $T_w > T_\infty$ and $T_w < T_\infty$.

Using these definitions of h, Nu_x, and Gr_x the local heat flux given by Eq. (11) becomes

$$Nu_{x,w} = \frac{\left[-\theta'(0) \right]}{\sqrt{2}} Gr_{x,w}^{1/4} \tag{14}$$

where the second subscript on the Nusselt and Grashof numbers indicates the location at which k and ν are evaluated.

Overall heat transfer. - The overall heat transfer Q is found by integrating Eq. (11). So,

$$Q = b \int_0^L q \, dx \tag{15}$$

where b is the plate width. Introducing the following dimensionless groups

$$\bar{h} \equiv \frac{Q}{bL(T_w - T_\infty)}, \quad \overline{Nu}_L \equiv \frac{\bar{h}L}{k}, \quad Gr_L \equiv \frac{gL^3 \left\{ |\rho_\infty - \rho_w| / \rho_w \right\}}{\nu^2} \tag{16}$$

leads to the following dimensionless result for the overall heat transfer

$$Nu_{L,w} = \frac{4}{3} \frac{\left[-\theta'(0) \right]}{\sqrt{2}} Gr_{L,w}^{1/4} \tag{17}$$

The Grashof number. - The generalized Grashof number defined by Eq. (13) arises naturally from the analysis of the general variable property fluid. The form taken by this definition for certain special cases is of interest.

For instance, for the constant property analysis, Eq. (13) simplifies to

$$Gr_x = \frac{g\beta |T_w - T_\infty| x^3}{\nu^2} \tag{13a}$$

This is identical to usual Grashof number definition.

For a perfect gas, the Grashof number becomes

$$Gr_x = \frac{g |T_w - T_\infty| x^3}{T_\infty \nu^2} \tag{13b}$$

DESCRIPTION OF THE GASES STUDIED

In previous variable properties analyses for gases, it has been common to use idealized forms of the property variations. There have been a few instances where real gas properties were used. In the present study, both idealized and real gases have been included, the real gas being a close approximation to air. Table I describes the five gases to be considered here in the order (reading to left to right) in which they will be discussed. All are seen to obey the perfect gas law, $p = \rho RT$. Absolute temperatures are used exclusively throughout the analysis for gases.

TABLE I

Description of Gases A, B, C, D, and E

Gas A	Gas B	Gas C	Gas D	Gas E
$p = \rho RT$	$p = \rho RT$	$p = \rho RT$	$p = \rho RT$	$p = \rho RT$
$k \sim T^{3/4}$	$k \sim T^{2/3}$	$\rho k = \text{const.}$	$k \sim \dfrac{T^{3/2}}{T + A_1}$	$k \sim \dfrac{T^{3/2}}{T + A_1}$
$\mu \sim T^{3/4}$	$\mu \sim T^{2/3}$	$\rho \mu = \text{const.}$	$\mu \sim \dfrac{T^{3/2}}{T + A_1}$	$\mu \sim \dfrac{T^{3/2}}{T + A_2}$
$c_p = \text{const.}$	$c_p = \text{const.}$	$c_p = \text{const.}$	$c_p = \text{const.}$	$c_p \begin{aligned} &= b_0 + b_1 T \\ &= b_2 + b_3 T \end{aligned}$
$Pr = \text{const.}$	$Pr = \text{const.}$	$Pr = \text{const.}$	$Pr = \text{const.}$	$Pr = \text{variable}$

Gases A, B, and C represent simple idealizations of real gas behavior. The power law variations for k and μ are commonly used

approximations. The assumption of constant specific heat and Prandtl number is included because, for real gases, the variation of these properties is small compared to those of k, μ, and ρ. The Sutherland type formula used to describe the conductivity and viscosity variations of gas D is closer to reality than are the simple power laws. Since c_p and Pr are maintained as constants for gas D, the same Sutherland formula is used for k and μ. Gas E is a close approximation to air. Hence, variations of c_p and Pr are included, and different Sutherland formulas are used for k and μ $(A_1 \neq A_2)$. Two linear equations are used to represent the specific heat variation over the temperature range studied.

HEAT TRANSFER RESULTS FOR GASES

Heat transfer results for gases A through E will be presented for a large number of special cases. Utilizing these results, a rapid and accurate shorthand method will be presented for calculating heat transfer to gases under variable property conditions. As described in the INTRODUCTION, the calculation procedure involves the use of results which have been derived for constant property fluids, and of a rule for extending these constant property results to variable property situations. This rule is commonly called a reference temperature.

The following approach will be used here in deriving and testing a reference temperature rule for heat transfer. First, using the numerous numerical calculations for gas A, a reference temperature rule will be derived. Then, tests of the wider applicability of this reference temperature result will be made using the less numerous solutions for gases B, C, D, and E.

Results for gas A. - The properties of gas A are given in Table I.

In order to compute the heat transfer, it is first necessary to solve the differential Eqs. (8a) subject to the boundary conditions (9). The gas properties appearing in Eqs. (8a) may be evaluated for gas A with the aid of Table I. So, it is seen that $c_p/c_{p_w} = 1$, $Pr = $ constant and

$$\frac{\rho\mu}{\rho_w\mu_w} = \frac{\rho k}{\rho_w k_w} = \left(\frac{T}{T_w}\right)^{-1/4} \tag{18a}$$

Introducing the dimensionless temperature $\theta = (T - T_\infty)/(T_w - T_\infty)$, it follows that

$$\frac{\rho\mu}{\rho_w\mu_w} = \frac{\rho k}{\rho_w k_w} = \left[\theta\left(1 - \frac{T_\infty}{T_w}\right) + \frac{T_\infty}{T_w}\right]^{-1/4} \tag{18b}$$

In light of this, it may be observed that before commencing with a solution of Eqs. (8a) to find F and θ, it is necessary first to specify the value of T_w/T_∞ (as well as of the Prandtl number). The appearance of this temperature ratio is associated with the variable property problem. But, the fact that only a temperature ratio appears is actually a considerable simplification; since for the general variable property fluid it is necessary to specify T_w and T_∞ separately.

Numerical solutions[*] of Eqs. (8a) for gas A were carried out for a wide range of values of T_w/T_∞ for $Pr = 0.7$ and for selected values of T_w/T_∞ for $Pr = 1.0$. The heat transfer results corresponding to these solutions are listed in Table II. The Nusselt and Grashof numbers, dimensionless parameters, are given by Eqs. (12) and (13b), respectively.

*All numerical integrations were carried out on an IBM Card Programmed Calculator using a technique presented in detail in appendix B of reference 2.

TABLE II

Heat Transfer Results for Gas A

T_w/T_∞	$Nu_{x,w}/Gr_{x,w}^{1/4}$ or $\overline{Nu}_{L,w}\Big/\frac{4}{3}\,Gr_{L,w}^{1/4}$	
	Pr = 0.7	Pr = 1.0
4	0.371	
3	0.368	0.418
5/2	0.366	
2	0.363	
3/4	0.348	
1/2	0.339	
1/3	0.330	0.375
1/4	0.323	

Now, we proceed to generalize these results. Attention is first focused on Pr = 0.7. From an analytical solution for the <u>constant property fluid</u>, it is found that the heat transfer (for Pr = 0.7) is

$$Nu_x/Gr_x^{1/4} = 0.353 = \overline{Nu}_L\Big/\frac{4}{3}\,Gr_L^{1/4} \qquad (19)^*$$

Here, the Grashof number is given by Eq. (13a). We are immediately led to ask whether there is some way by which the constant property result, Eq. (19), can be made to coincide with the variable property results appearing in Table II. It may be observed that the temperature for evaluating k, ν, and β in Eq. (19) is at our disposal. With regard to β, it has already been shown that it is proper to replace β by $1/T_\infty$ when perfect gases are involved. It is a matter of making a few trials

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*This result may also be thought of as applying to gas A when $T_w/T_\infty \to 1$.

to determine the temperature for evaluating k and ν of Eq. (19) which gives the best agreement between the constant property result and the variable property findings of Table II. This temperature, which may be termed a reference temperature T_r, is found to be

$$T_r = T_W - 0.38(T_W - T_\infty), \qquad \beta = 1/T_\infty \tag{20}$$

As shown by the dashed line of Fig. 2, the error in the heat transfer predicted from the constant property result by using this reference temperature is at most 0.6 percent over the entire range $1/4 \leq T_W/T_\infty \leq 4$. Figure 2 also shows the errors in the heat transfer predicted from the constant property result when k and y are evaluated at the wall temperature, at the ambient temperature, and at the film temperature T_f. β is taken as $1/T_\infty$ in all cases. It is noteworthy that the use of T_f as a reference temperature provides heat transfer results which would be adequate for almost all engineering purposes.

The results listed in Table II for Pr = 1 were obtained to check whether the reference temperatures found for Pr = 0.7 could be applied for the entire range of gas Prandtl numbers. When reference temperatures were computed for Pr = 1, they were found to be in excellent agreement with those for Pr = 0.7. So, it is felt that Eq. (20) is valid for the entire Prandtl number range of gases.

Having established a reference temperature rule for gas A, we now proceed to see if it can be used for predicting variable property heat transfer results for other gases.

Results for gas B. - As may be seen from Table II, gas B differs from gas A only by a lowering of the exponent in the conductivity and viscosity variations. The lower exponent becomes more appropriate in describing real gas behavior as the temperature increases.

Gas B will be treated in the following way: A special case will be selected and heat transfer results will be computed based on a solution of Eq. (8a). Then, for the same special case, an alternate heat transfer result is computed for gas B using the constant property formula and the reference temperature rule of Eq. (20). The comparison of the two heat transfer results provides a test of the reference temperature procedure.

The case selected for study is $Pr = 0.7$, $T_w/T_\infty = 3$. From a numerical solution of Eq. (8a), it is found that

$$Nu_{x,w}/Gr_{x,w}^{1/4} = 0.373 = \overline{Nu}_{L,w}\Big/\tfrac{4}{3}\,Gr_{L,w}^{1/4} \tag{21a}$$

The Grashof number is given by Eq. (13b). Then, using the constant property formula, Eq. (19), and the reference temperature relation, Eq. (20), an alternate heat transfer result for gas B is

$$Nu_{x,w}/Gr_{x,w}^{1/4} = 0.370 = \overline{Nu}_{L,w}\Big/\tfrac{4}{3}\,Gr_{L,w}^{1/4} \tag{21b}$$

It is seen that the prediction using the reference temperature procedure is very good.

Results for gas C. - The very special mathematical simplifications which are associated with gas C have already been pointed out; see Eq. (8b) and associated discussion. Inspection of Eq. (8b) shows that T_w/T_∞ need not be specified.

From a numerical solution of Eq. (8b), the heat transfer result for gas C for $Pr = 0.7$ is found to be

$$\frac{Nu_{x,w}}{Gr_{x,w}^{1/4}} = \frac{Nu_{x,\infty}}{Gr_{x,\infty}^{1/4}} = \frac{Nu_{x,r}}{Gr_{x,r}^{1/4}} = \ldots = 0.353 \tag{22}$$

where Gr_x is given by Eq. (13b). A similar finding is true for the overall heat transfer. It is thus seen that, as a consequence of the special properties of gas C, the ratio $Nu_x/Gr_x^{1/4}$ is independent of the second subscript indicating where the properties are evaluated.

Looking now at the constant property formula, Eq. (19), it is seen that no matter how the properties are evaluated, its heat transfer result coincides with the variable property result of Eq. (22). (That is, provided that β is evaluated as $1/T_\infty$.) So, the reference temperature of Eq. (20) is certainly satisfactory for gas C.

Results for gas D. - It has already been pointed out in the discussion of Table I that a Sutherland type representation for k and μ is generally closer to reality than are simple power laws. For gas D, the assumptions of constant c_p and Pr are maintained, so the same Sutherland relation is used for k and μ. The constant A_1 appearing in Table I is assigned the value 362° F abs. This number is from a correlation by Glassman and Bonilla (4) of thermal conductivity data for air.

With the aid of Table I, the property groupings $\rho\mu$ and ρk which appear in the differential Eqs. (8a) may be evaluated. When the dimensionless temperature $\theta = (T - T_\infty)/(T_w - T_\infty)$ is introduced, there results

$$\frac{\rho\mu}{\rho_w\mu_w} = \frac{\rho k}{\rho_w k_w} = \left[\frac{T_\infty}{T_w} + \theta\left(1 - \frac{T_\infty}{T_w}\right)\right]^{1/2}\left[\frac{1 + \frac{362}{T_w}}{\frac{T_\infty}{T_w} + \theta\left(1 - \frac{T_\infty}{T_w}\right) + \frac{362}{T_w}}\right]$$

Careful inspection of this expression shows that T_w and T_∞ must be separately specified before a solution of Eq. (8a) can be carried out. In other words, specific problems must be considered.

We proceed to test the reference temperature relation (20) in the same fashion described for gas B. Heat transfer results based on solutions of the differential equations are compared with alternate results determined using the reference temperature procedure. The following two special cases are considered:

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\left.\begin{array}{l}
Pr = 0.7 \\
T_w = 1800^\circ \text{ F abs} \\
T_\bullet = 600^\circ \text{ F abs}
\end{array}\right\} \text{case I}
\qquad
\left.\begin{array}{l}
Pr = 0.7 \\
T_w = 600^\circ \text{ F abs} \\
T_\bullet = 1800^\circ \text{ F abs}
\end{array}\right\} \text{case II}
$$

From numerical solution of Eq. (8a), the heat transfer results for these cases are

$$
Nu_{x,w}/Gr_{x,w}^{1/4} = \overline{Nu}_{L,w}\left/\frac{4}{3} Gr_{L,w}^{1/4}\right. =
\begin{cases}
0.371, \text{ case I} \\
0.335, \text{ case II}
\end{cases}
\tag{23a}
$$

Alternately, heat transfer results can be computed for gas D for these cases by using the constant property formula, Eq. (19), and the reference temperature rule, Eq. (20). The results are

$$
Nu_{x,w}/Gr_{x,w}^{1/4} = \overline{Nu}_{L,w}\left/\frac{4}{3} Gr_{L,w}^{1/4}\right. =
\begin{cases}
0.370, \text{ case I} \\
0.335, \text{ case II}
\end{cases}
\tag{23b}
$$

Again, it is seen that the predictions using the reference temperature procedure are good indeed.

Results for gas E. - As has already been pointed out, gas E is meant to be a close approximation to air. The k and μ variations are represented by Sutherland formulas (see table I) taken respectively from Glassman and Bonilla (4) and the NBS Tables (5). The constants A_1 and A_2 appearing in these representations are 362° F abs and 198.7° F abs (110.4° K), respectively. The specific heat data, taken from the NBS Tables, is represented by two straight lines over the temperature range

studied. The variation of the Prandtl number need not be specified,
since it is determined once c_p, k, and μ are given.

The property groupings $\rho\mu$, ρk, and c_p/c_{p_w} which appear in the
differential Eqs. (8a) may be evaluated with the aid of Table I. Once
this has been done, it is easy to see that it is necessary that T_w
and $T_\bullet$ be separately specified before proceeding with a solution
of Eqs. (8a). So again, as for gas D, specific problems must be
considered.

We now proceed to check the reference temperature procedure in the
same manner as was already used for gases B, C, and D. The special
cases selected here for this purpose are

$$\left.\begin{array}{l} Pr_w = 0.7 \\ T_w = 1800^\circ \text{ F abs} \\ T_\bullet = 600^\circ \text{ F abs} \end{array}\right\} \text{Case I} \qquad \left.\begin{array}{l} Pr_w = 0.7 \\ T_w = 600^\circ \text{ R} \\ T_\bullet = 1800^\circ \text{ R} \end{array}\right\} \text{Case II}$$

Numerical solutions of Eqs. (8a) provide the following heat transfer
results

$$Nu_{x,w}/Gr_{x,w}^{1/4} = \overline{Nu}_{L,w}\left/\frac{4}{3}\,Gr_{L,w}^{1/4}\right. = \begin{cases} 0.358, & \text{case I} \\ 0.346, & \text{case II} \end{cases} \tag{24a}$$

We now proceed to compute alternate heat transfer results based on
the constant property findings and the reference temperature relation,
Eq. (20). In connection with this computation, it is worthwhile noting
a particular feature associated with the fact that the gas under con-
sideration has a temperature-dependent Prandtl number. Over the tem-
perature range studied for gas E, the Prandtl number variation was 4.5
percent. First, it may be observed that the constant property analysis

yields

$$Nu_x/Gr_x^{1/4} = \overline{Nu}_L \Big/ \frac{4}{3} Gr_L^{1/4} = \text{function of } Pr \qquad (25)$$

where the Grashof number is given in Eq. (13a). This relation is plotted
in Fig. 3 for the Prandtl number range appropriate to gases. Now, con-
sider the application of Eq. (25) to a variable property fluid with
temperature-dependent Prandtl number. Under these conditions, not only
are k and ν evaluated at the reference temperature, but also Pr.

Using Eq. (25) and Fig. 3 in conjunction with the reference tem-
perature relation (20), alternate heat transfer results for gas E for
cases I and II are found to be

$$Nu_{x,w}/Gr_{x,w}^{1/4} = \overline{Nu}_{L,w} \Big/ \frac{4}{3} Gr_{L,w}^{1/4} = \begin{cases} 0.361, & \text{case I} \\ 0.343, & \text{case II} \end{cases} \qquad (24b)$$

Comparison of Eqs. (24a) and (24b) shows that even for this more com-
plicated (and more realistic) situation, the reference temperature pro-
cedure predicts very good heat transfer results.

FLUID PROPERTIES OF LIQUID MERCURY

As already noted in the INTRODUCTION, it was initially hoped to
study the variable property problem in several liquid metals. However,
it was observed that the property variations of the various liquid
metals of technical interest were sufficiently dissimilar that separate
investigations appeared to be necessary for each one. Moreover, the
computing time required to obtain numerical solutions for liquid metals
is an order of magnitude greater than that required for gases. So, it
was found necessary to limit the current study to one liquid metal,
namely, mercury.

When considering the property variations of liquid mercury, it might be well first to point out some qualitative trends and make comparisons with gases. For liquid mercury, the viscosity decreases with increasing temperature, while the thermal conductivity increases with temperature.[*] On the other hand, for gases, both the viscosity and conductivity increase with temperature. The Prandtl number of mercury shows a strong (percentage) decrease with increasing temperature, while the Prandtl number of most gases varies by only a few percent over a large temperature range. Also, the absolute magnitude of mercury's Prandtl number is 1 or 2 percent of that of gases. The above remarks apply to normal engineering conditions, e.g., gas dissociation is excluded.

Property data for liquid mercury was taken from the Liquid Metals Handbook (6). To facilitate numerical integrations, polynomial representations of the following form were fitted to the data

$$\sum_{n=0}^{3} a_n t^n \tag{26}$$

where t is in degrees Fahrenheit. The coefficients a_n are given in Table III.

---

[*]This is by no means the rule among liquid metals.

TABLE III

Coefficients For The Polynomical Representations

Of Properties Of Liquid Mercury

	k	μ	c_p	ρ
	$\dfrac{\text{Btu}}{\text{hr - ft - }^{\circ}\text{F}}$	$\dfrac{\text{lb}}{\text{ft - hr}}$	$\dfrac{\text{Btu}}{\text{lb - }^{\circ}\text{F}}$	$\dfrac{\text{lb}}{\text{cu ft}}$
a_0	4.47924	4.34620	$0.334620(10^{-1})$	851.514
a_1	$0.830958(10^{-2})$	$-0.991162(10^{-2})$	$-0.393353(10^{-5})$	$-0.864880(10^{-1})$
a_2	$-0.380163(10^{-5})$	$0.179060(10^{-4})$	$0.344649(10^{-8})$	$0.986194(10^{-5})$
a_3	0	$-0.127524(10^{-7})$	0	$-0.592566(10^{-8})$

HEAT TRANSFER RESULTS FOR LIQUID MERCURY

First, based on numerical solutions of Eqs. (8), heat transfer results are given for special cases. Then, these findings are utilized in determining reference temperatures.

Results for specific cases. - For liquid mercury, the buoyancy force appearing in the first of Eqs. (8) does not simplify as it did for gases. So, it is necessary to deal with Eqs. (8) as they stand. Further, the nature of the property variations of liquid mercury requires that t_w and t_∞ be specified separately, i.e., that specific problems be considered. The two cases chosen for study are described below. In both instances, $t_w > t_\infty$.

$$t_w = 600^\circ \text{ F}$$
$$t_\bullet = 100^\circ \text{ F}$$
$$Pr_w = 0.0083$$
$$Pr_\bullet = 0.022$$
case I

$$t_w = 450^\circ \text{ F}$$
$$t_\bullet = 150^\circ \text{ F}$$
$$Pr_w = 0.010$$
$$Pr_\bullet = 0.019$$
case II

On the basis of the rather lengthy numerical solutions of the differential Eqs. (8), the heat transfer results are found to be

$$Nu_{x,w}/Gr_{x,w}^{1/4} = \overline{Nu}_{L,w}/Gr_{L,w}^{1/4} = \begin{cases} 0.0501, & \text{case I} \\ 0.0556, & \text{case II} \end{cases} \tag{27}$$

where the Grashof number is given by Eq. (13).

Reference temperature relation. - Now, it may be inquired as to whether there is some reference temperature rule which will cause the constant property heat transfer results to coincide with those of Eq. (27).

It has already been noted that the constant property analysis yields heat transfer results which are given by Eq. (25). A plot of this relation is given on Fig. 4(a) for the Prandtl number range appropriate to liquid mercury. After making a few trials, it is found that by evaluating k, ν, β, and Pr at

$$T_r = T_w - 0.3(T_w - T_\bullet) \tag{28}$$

the constant property heat transfer results of Fig. 4(a) coincide with those of Eq. (27). It is emphasized that this same reference temperature is found for both variable property cases studied here.

It may next be inquired as to how well the film temperature, T_f, might serve as an alternate reference temperature. The heat transfer calculated for cases I and II by evaluating the constant property results of Fig. 4(a) at the film temperature T_f agree quite well with

Eq. (27), certainly within the range of most engineering requirements.
The same statement applies when all the properties (including β) are
evaluated at $T_r = T_w - 0.38(T_w - T_\infty)$, which was the reference tempera-
ture derived for gases.

EXPERIMENTAL VERIFICATION

A complete survey of available experimental heat transfer data for
free convection on a vertical plate is presented in Ref. 1. It is noted
there that the bulk of the experiments were carried out in air.

For air, the conditions of most tests were such that $1 < T_w/T_\infty \leq 1.5$.
These experimenters took no note of variable property effects. This
action is justified by the results of the present analysis; for example,
see Fig. 2. In one experiment, that of Weise (7), a value of T_w/T_∞ of
2.2 was achieved. Unfortunately, Weise's apparatus, constructed for .
horizontal plate tests, was not well suited for vertical plate
studies.

For liquid mercury, Saunders' (8) work represents the only experi-
ment on a vertical plate. His tests were carried out at very small
temperature differences, and hence variable property effects did not
enter.

So, it appears that there are currently no data available to check
the findings of the analysis presented here.

RESULTS FOR BOUNDARY LAYER THICKNESS
AND VELOCITY PARAMETERS

While the heat transfer is by far the result of greatest practical
importance, there are other quantities which may be of interest. A
brief description is given below of several quantities for which results

are to be reported. Following the descriptive paragraphs, numerical findings derived from the constant property analysis are given, and then tables of reference temperature are supplied for extending these constant property results to variable property situations.

(a) Boundary-layer thickness, δ_i: The thickness of the boundary layer is by no means a precise concept, and its definition is somewhat arbitrary. The distance from the plate surface at which $T - T_{\infty}$ has shrunk to a small fraction i of the overall temperature difference, $T_W - T_{\infty}$, is used here to define a boundary layer thickness. This definition, while satisfactory for Prandtl numbers below unity, should not be used for high Prandtl numbers.

(b) Maximum velocity, u_{max}: The vertical velocity u (parallel to the plate surface) takes on zero values both at the surface and at $y = \infty$. and hence achieves some maximum between.

(c) Location of the maximum velocity, y_u: The distance from the plate surface at which the velocity maximum occurs is denoted by y_u.

(d) Friction coefficient, c_f: The coefficient of friction provides a dimensionless presentation of the wall shear stress τ_w in the following manner.

$$c_f \equiv \frac{\tau_w}{\rho \left(\frac{v}{x}\right)^2} = \frac{\left[\mu \frac{\partial u}{\partial y}\right]_{y=0}}{\rho \left(\frac{v}{x}\right)^2} \tag{29}$$

The quotient, (v/x), plays the role of a characteristic velocity.

(e) Flow rate, W: The upward flow generated by the free convection forces is given by

$$W = b \int_0^{\infty} \rho u \, dy \tag{30}$$

where b is the plate width.

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Results from the constant property analysis. - Utilizing the tabulated constant property solutions of Ostrach (2) in conjuction with those of Ref. 1, results for the quantities described in paragraphs (a) through (e) have been computed. A presentation of these findings for the Prandtl number ranges appropriated to gases and to liquid mercury is made respectively on Figs. 3 and 4. Heat transfer results are also shown. The Grashof number is given by Eq. (13a).

It was originally decided to define the boundary-layer thickness as the distance from the plate where $(T - T_\infty)/(T_w - T_\infty) = 0.02$. For the Prandtl number range of gases, there was no difficulty evaluating such a definition from the available numerical solutions. But, for the low Prandtl number range, the numerical solutions were less precise at large distances from the wall, and it was necessary to use a thickness based on $(T - T_\infty)/(T_w - T_\infty) = 0.05$. It is estimated that for the Prandtl number range of mercury, $\delta_{0.02} \approx 1.25\,\delta_{0.05}$.

Variable property results; reference temperatures. - In a manner identical to that outlined for the heat transfer, the findings of the constant property analysis for the quantities described above may be extended to the variable property situation. It only remains to present appropriate reference temperatures which are needed to evaluate the results of Figs. 3 and 4.

Such reference temperatures, computed using the variable property solutions which have been obtained here for gases and for liquid mercury, are listed in Tables IV(a) and (b). The heat transfer reference temperatures, already given by Eqs. (20) and (28), are included for completeness.

TABLE IV(a)

Reference Temperature Relations For Gases
(For use with Fig. 3)

$$(T_r - T_w)/(T_\infty - T_w)$$

h	0.38	
$\delta_{0.02}$	0.67	
u_{max}	See below	$\beta = \dfrac{1}{T_\infty}$
y_u	0.24	
c_f	0.10	
W	0.85	

To find u_{max} for gases, evaluate the Prandtl number of Fig. 3 at the mean of the extreme values of Pr in the particular problem under consideration.

TABLE IV(b)

Reference Temperature Relations

For Liquid Mercury
(For use with Figs. 4(a) and (b))

$$(T_r - T_w')/(T_\infty - T_w)$$

h	0.30
$\delta_{0.05}$	0.60
u_{max}	0
y_u	0
c_f	0.10
W	0.40

CONCLUDING REMARKS

From the findings reported here, it appears that free convection heat transfer under variable property conditions can be computed quickly and accurately by using the constant property results in conjunction with reference temperature relations. For gases and liquid mercury, the reference temperature relations are given by Eqs. (20) and (28), respectively.

Further, it may be observed that the film temperature appears to serve as an adequate reference temperature (with $\beta = 1/T_\infty$ for gases) for most engineering purposes.

Reference temperature relations for use in computing boundary-layer thickness and velocity parameters for variable property conditions are given in Tables IV(a) and (b).

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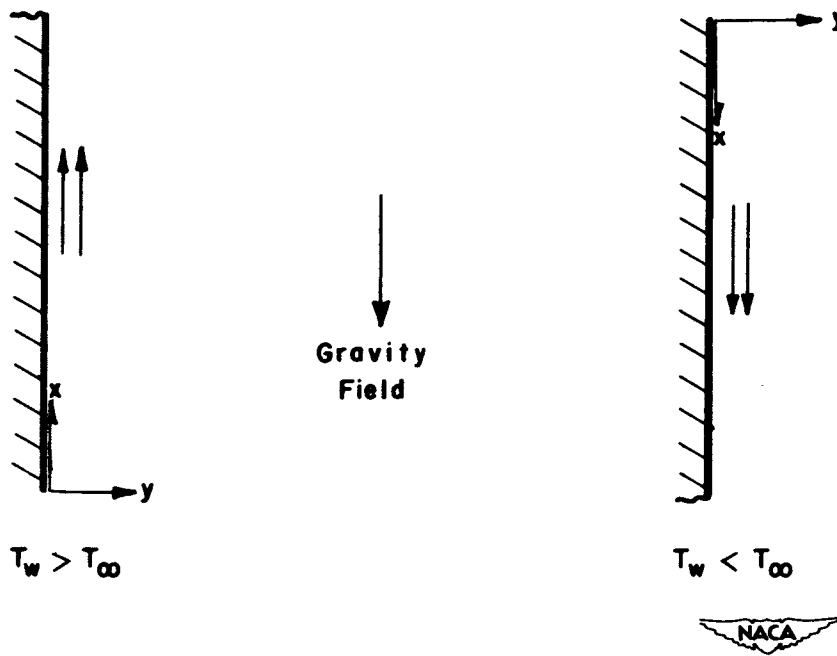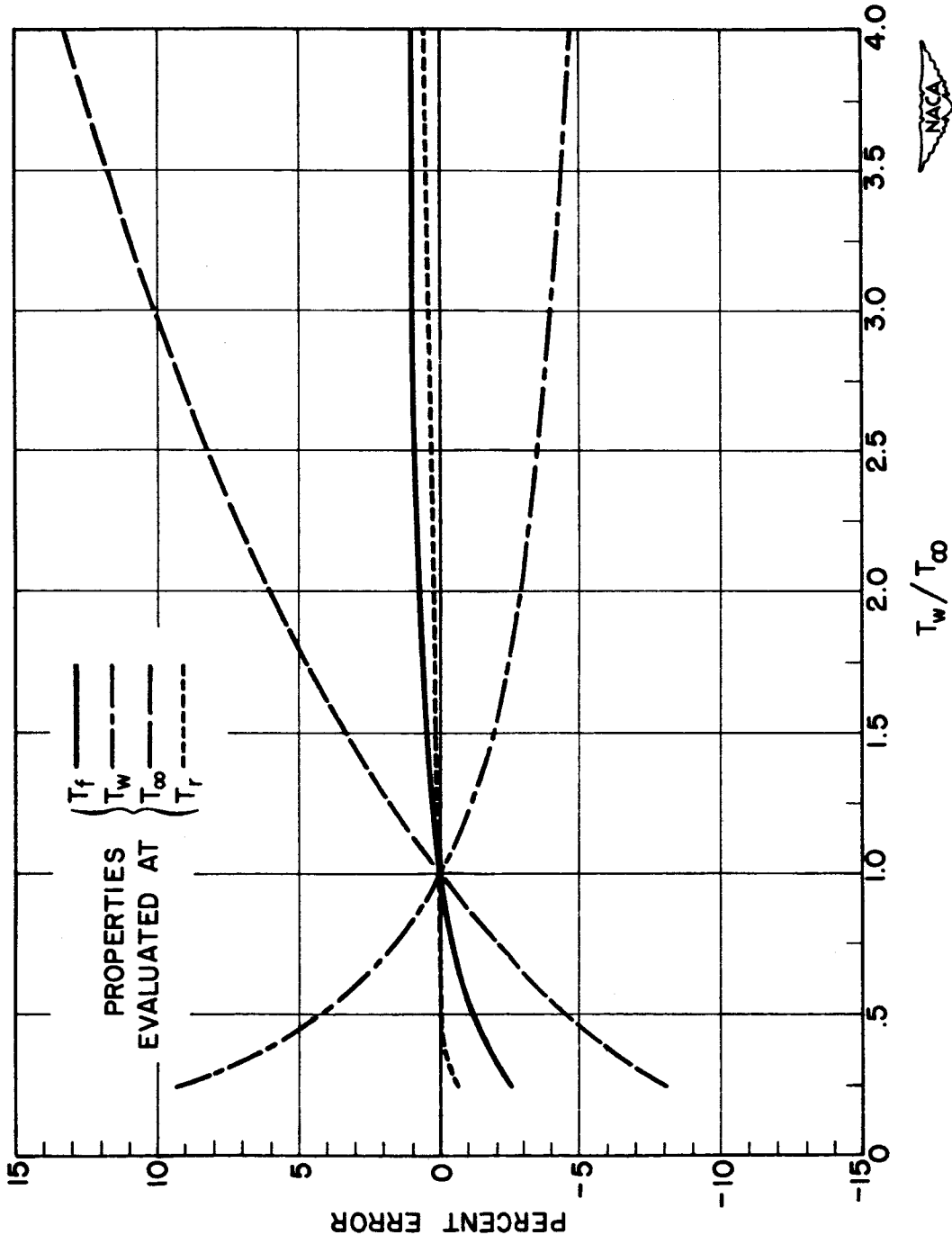
Figure 1. - Coordinate systems.

Figure 2. - Errors made in the prediction of the heat transfer for gas A by evaluating the constant property result at various temperatures.
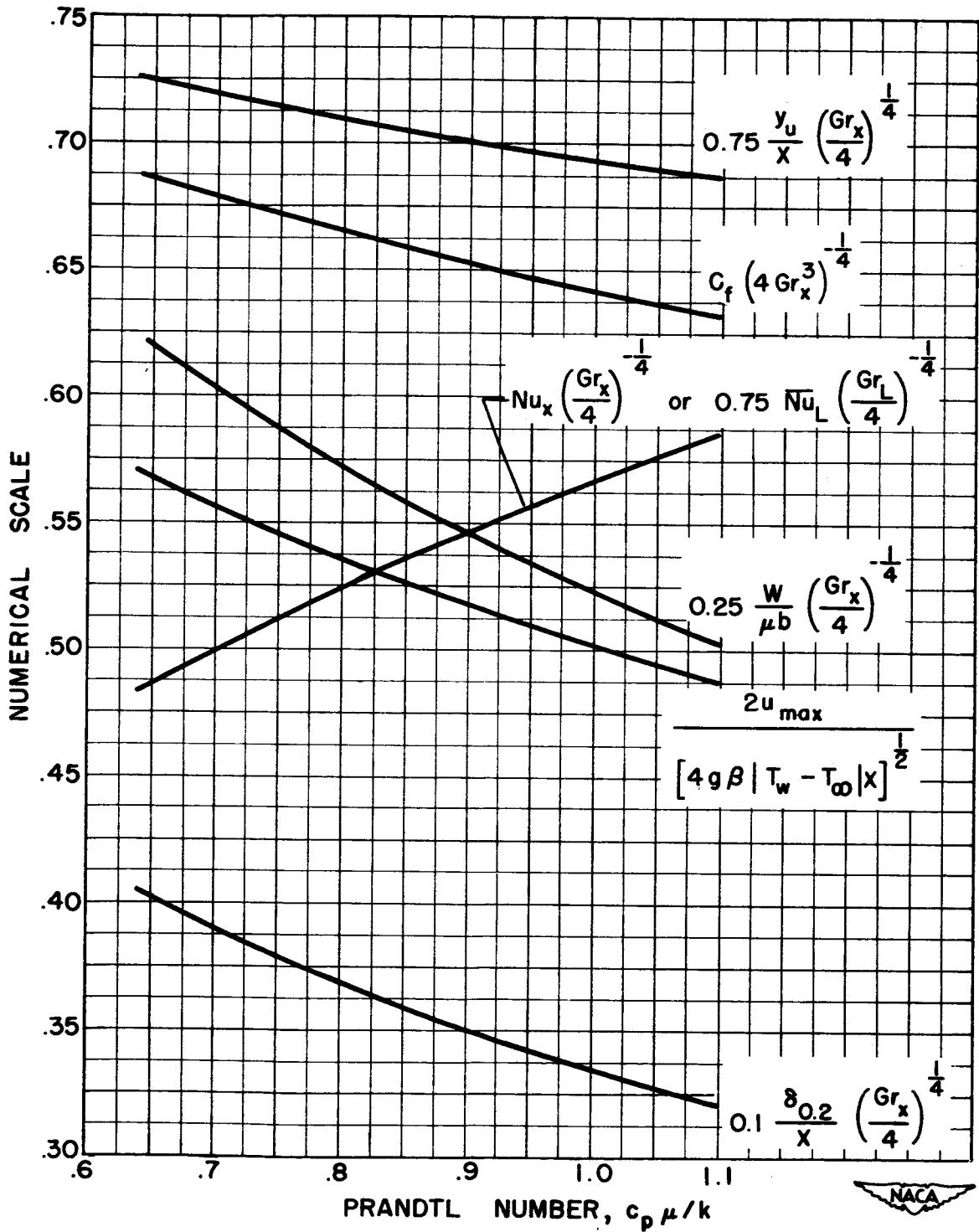
T_w / T_∞

PERCENT ERROR

PROPERTIES
EVALUATED AT

$\begin{cases} T_f \\ T_w \end{cases}$
$\begin{cases} T_\infty \\ T_r \end{cases}$

NACA

Figure 3. - Results of the constant property analysis for the Prandtl number range of gases.
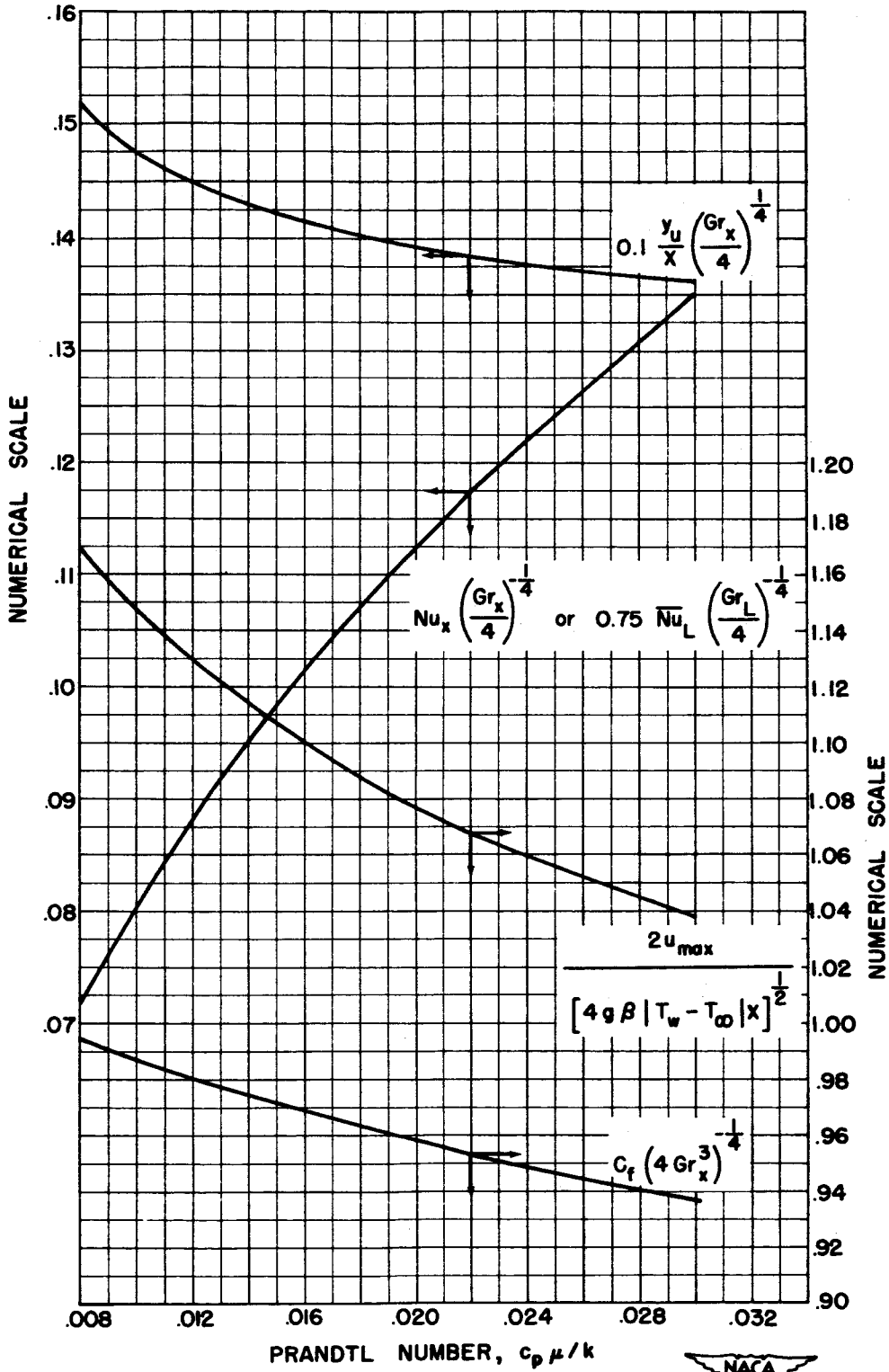
Figure 4(a) - Results of the constant property analysis for the Prandtl number
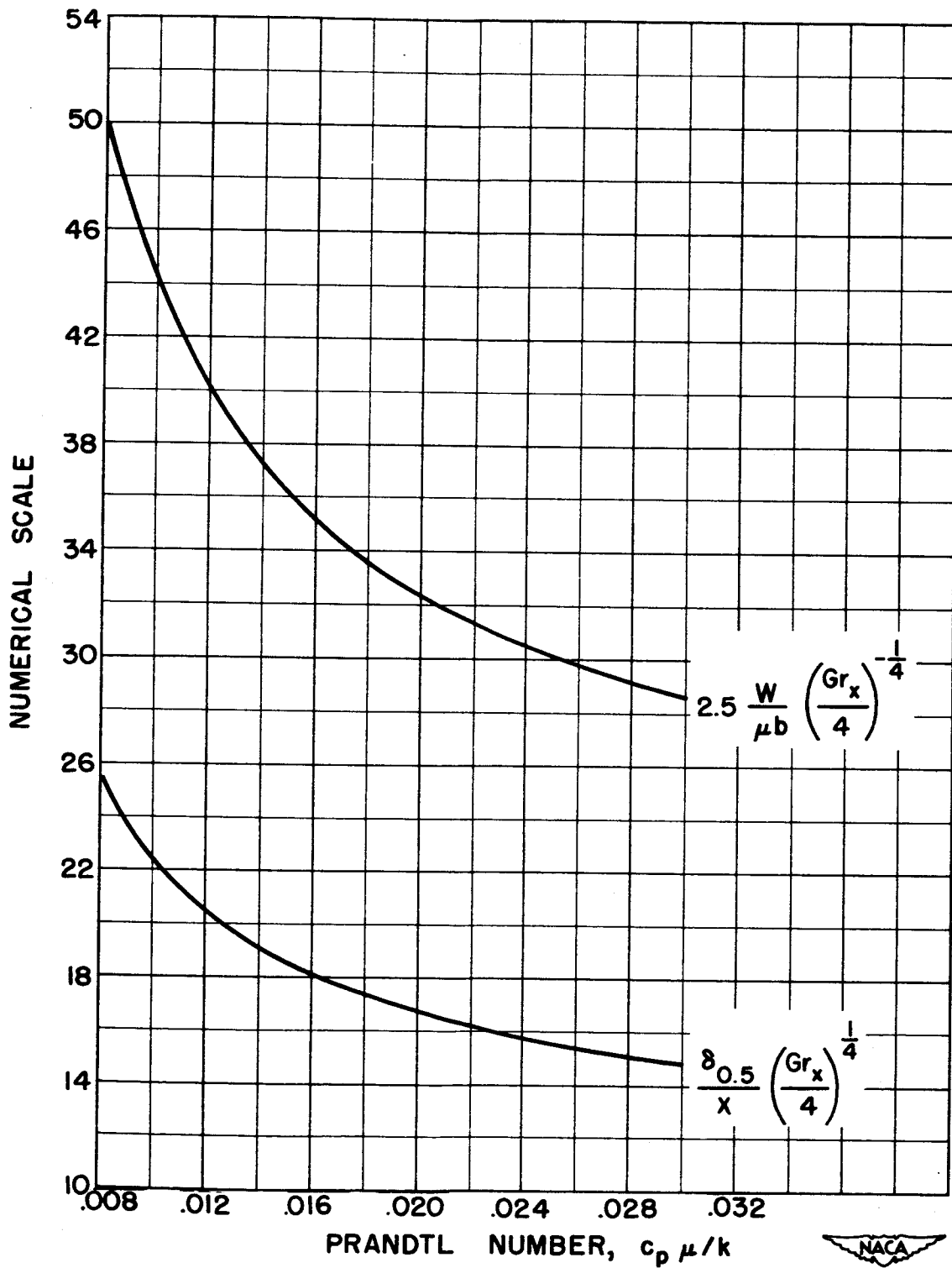range of liquid mercury.

Figure 4(b) - Results of the constant property analysis for the Prandtl number range of liquid mercury.