The Evans function for nonlocal equations

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Abstract

In recent studies of the master mode-locking equation, a model for solid-state cavity laser that includes nonlocal terms, bifurcations from stationary to seemingly time-periodic solitary waves have been observed. To decide whether the mechanism is a Hopf bifurcation or a bifurcation from the essential spectrum, a general framework for the Evans function for equations with nonlocal terms is developed and applied to the master mode-locking model.

1 Introduction

Our motivation for this paper was the desire to understand the dynamics of pulses in the master mode-locking equation

$$\mathrm{i}u_t + u_{xx} - \omega u + 4|u|^2 u = \mathrm{i}\epsilon \left[\frac{\Gamma_{\text{gain}}}{1 + \|u\|^2_{L^2(\mathbb{R})}/e_0} (\tau u_{xx} + u) - \Gamma_{\text{loss}} u + \beta |u|^2 u \right], \qquad x \in \mathbb{R}. \quad (1.1)$$

This equation describes a solid-state laser which is passively mode-locked, for instance, by a saturable absorber that attenuates weaker-intensity portions of individual pulses while preserving the total cavity energy. The complex function $u(x,t)$ represents the envelope of the electrical field. Note that (1.1) features the nonlocal term $\|u\|^2_{L^2(\mathbb{R})}$ which represents the total energy in the pulse. It models a bandwidth-limited amplification process in the laser cavity which saturates at a certain energy. We refer to Section 5 for more background information regarding (1.1) and the interpretation of the various constants that appear in (1.1).

Numerical simulations, see Section 5, indicate that (1.1) has stable stationary pulses that, beyond a certain parameter threshold, become time-periodic. Our goal in this paper is to show existence

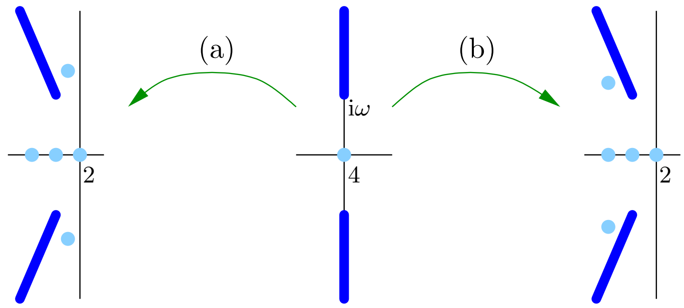

Figure 1: *The center plot shows the spectrum of waves for the nonlinear Schrödinger equation, i.e., for (1.1) with $\epsilon = 0$. Zero is an eigenvalue with multiplicity four. Upon making $\epsilon > 0$, the spectrum might look as shown in (a) and (b): In particular, an additional pair of eigenvalues may move off the essential spectrum. Assuming that the relative ordering of the real parts of eigenvalues and the essential spectrum does not change, the destabilizing bifurcation would be a Hopf bifurcation in (a) as the additional pair of eigenvalues is to the right of the essential spectrum, while the essential spectrum in (b) would destabilize prior to the point spectrum.*

and stability of the stationary pulses and to shed some light on the nature of the bifurcation scenario that generates time-periodic waves. The first guess is, of course, that the periodic waves bifurcate in a Hopf bifurcation. Thus, let us begin by discussing the spectrum of the linearization of (1.1) about a wave ϕ. For $\epsilon = 0$, (1.1) simply becomes the nonlinear Schrödinger equation for which the spectrum about the soliton solutions is well known [28]. In particular, zero is the only point spectrum, the rest is essential spectrum that occupies part of the imaginary axis as shown in Figure 1. We will show that, upon making $\epsilon > 0$, the point spectrum near zero will at most lead to saddle-node bifurcations but never to Hopf instabilities. The location of the essential spectrum for $\epsilon > 0$ is also readily computed with the result that it can destabilize the wave. As demonstrated in [23, 24, 27], such an essential instability may generate stable time-periodic waves. The issue is therefore whether there is any additional point spectrum present that may move into the right half-plane prior to the essential spectrum.

For the nonlinear Schrödinger equation, it has been shown in [11–13] that there is at most one pair of eigenvalues that can move off the essential spectrum upon adding a small local perturbation to it. Furthermore, this pair emerges from the edge of the essential spectrum located at $\lambda = \pm \mathrm{i}\omega$. The technique used to establish this result is the Evans function $E(\lambda)$, an analytic function of the eigenvalue parameter λ whose roots off the essential spectrum correspond to eigenvalues of the linearization about the wave [1]. Thus, if the Evans function can be extended into the essential spectrum in an analytic fashion, one would be able to locate and track its roots under perturbations. This extension was constructed in [6, 11], see also [9, 10, 12, 13] for further results and [22] for a recent survey.

The main theoretical result of the present paper is the construction of the Evans function and its extension across the essential spectrum for nonlocal eigenvalue problem such as the one arising from (1.1). More specifically, we will construct a function $E(\lambda)$ that counts with multiplicity all values of λ for which the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x) = A(x;\lambda)u(x) + [\mathcal{N}u](x)$$

has a localized solution. The nonlocal part $\mathcal{N}$ is quite general and includes two classes of terms that arise frequently in applications. Firstly, terms of the form

$$[\mathcal{N}u](x) = h_1(x)\langle h_2, u\rangle_{L^2(\mathbb{R})}$$

are allowed for functions $h_1, h_2 \in L^2(\mathbb{R})$. This is, in fact, the nonlocal term that will arise in the linearization of (1.1). Secondly, the nonlocal term can be of the form

$$[\mathcal{N}u](x) = u(x_0)h(x)$$

where again $h \in L^2(\mathbb{R})$, say, and $x_0 \in \mathbb{R}$ is a fixed number. Such terms arise when linearizing integro-differential equations such as

$$\begin{aligned} u_t(x,t) &= f(u(x,t)) - v(x,t) + \alpha \int_{-\infty}^{\infty} K(x-y)H(u(y,t)-\theta)\,\mathrm{d}y \\ v_t(x,t) &= \epsilon[u(x,t) - \gamma v(x,t)] \end{aligned} \tag{1.2}$$

where H denotes the Heaviside function and the kernel $K(x)$ is an even, non-negative function with normalized L^1-norm $\|K\|_{L^1(\mathbb{R})} = 1$. Indeed, assume that ϕ is the profile of the u-component of a travelling wave of (1.2) and that there are a finite number of positions $x_j \in \mathbb{R}$ for $j = 1, \ldots, m$, say, such that $\phi(x)$ is equal to the threshold θ precisely when $x = x_j$ for some j. Under this assumption, the results in [29, 30] show that the formal linearization of the nonlocal term about the wave ϕ is given by

$$u(\cdot) \longmapsto \sum_{j=1}^{m} \frac{1}{|\phi_x(x_j)|} K(x-x_j)u(x_j).$$

Integro-differential equations of the type (1.2) arise in models of neuronal networks in the brain.

We would like to mention that nonlocal eigenvalue problems have been studied in many other works such as [2–4], for instance. The focus of this work is on a systematic construction of the Evans function for nonlocal equations and its extension across the essential spectrum which has, to our knowledge, not been addressed before. To extend $E(\lambda)$ into the essential spectrum, we use the Gap Lemma [6, 11] and assume, in addition, that the nonlocal term localizes functions (in fact, we will assume that it maps weakly exponentially growing functions into weakly exponentially decaying functions). Section 3.5 contains an example, similar to the more complicated situation studied in [4], that illustrates the difficulties one faces when this hypothesis is not met. Nevertheless, we expect that our methods can be adapted to include this case as well. In any case, we emphasize that the localization hypothesis is not needed for the construction of the Evans function away from the essential spectrum. In passing, we also remark that it has recently been shown in [26] that the Evans function can also be extended across the absolute spectrum in cases where the Gap Lemma fails.

The paper is organized as follows. In Sections 2 and 3, we construct the Evans function for nonlocal equations and give its properties. In Section 4, we briefly recall transcritical bifurcations and apply these results to problems that arise when considering the existence and stability of pulses for perturbed nonlinear Schrödinger equations of the form of equation (1.1). Finally, in Section 5, our results are applied to equation (1.1).

2 The Evans function for nonlocal equations

2.1 The abstract framework

We consider a family $\mathcal{T}$ of bounded linear operators defined by

$$\mathcal{T}(\lambda)\ :\qquad \mathcal{D}\longrightarrow\mathcal{H},\qquad u\longmapsto\frac{\mathrm{d}u}{\mathrm{d}x}-A(\cdot;\lambda)u-\mathcal{K}\mathcal{J}u \tag{2.1}$$

for $\lambda\in\mathbb{C}$ where

$$\mathcal{D}=H^1(\mathbb{R},\mathbb{C}^n),\qquad \mathcal{H}=L^2(\mathbb{R},\mathbb{C}^n).$$

We shall explain the various different terms appearing in (2.1). We remark that all the results in this section are also true if we replace the spaces $\mathcal{D}$ and $\mathcal{H}$ by $C^1(\mathbb{R},\mathbb{C}^n)$ and $C^0(\mathbb{R},\mathbb{C}^n)$, respectively. For the sake of clarity, we make the following assumption throughout this section.

Hypothesis 1 *The matrix-valued function $A(x;\lambda)\in\mathbb{C}^{n\times n}$ is of the form*

$$A(x;\lambda)=\tilde{A}(x)+\lambda B(x)$$

where $\tilde{A}(\cdot)$ and $B(\cdot)$ are in $C^\infty_{\mathrm{bdd}}(\mathbb{R},\mathbb{R}^{n\times n})$.

In particular, throughout Section 2, we do not assume that the matrices A and B have limits as $x\to\pm\infty$.

The operators $\mathcal{K}$ and $\mathcal{J}$ are the nonlocal part of the operator $\mathcal{T}(\lambda)$.

Hypothesis 2 *There is an integer m such that*

$$\mathcal{J}\ :\ H^1(\mathbb{R}^-,\mathbb{C}^n)\times H^1(\mathbb{R}^+,\mathbb{C}^n)\hookrightarrow\mathcal{H}\longrightarrow\mathbb{C}^m,\qquad \mathcal{K}\ :\ \mathbb{C}^m\longrightarrow\mathcal{H}$$

are bounded linear operators.

In other words, we restrict ourselves to the situation where the nonlocal part factors through the finite-dimensional space $\mathbb{C}^m$ for some $m\in\mathbb{N}$. As a consequence, the nonlocal part $\mathcal{K}\mathcal{J}:\mathcal{D}\to\mathcal{H}$ is always compact.

Example 2.1 *Consider the nonlocal operator*

$$\mathcal{D}\longrightarrow\mathcal{H},\qquad u(\cdot)\longmapsto h(\cdot)\int_{-\infty}^{\infty}\langle g(y),u(y)\rangle\,\mathrm{d}y$$

where h and g are given functions in $L^2(\mathbb{R},\mathbb{C}^n)$. Upon using

$$\begin{aligned} \mathcal{J} &: \quad \mathcal{D} \longrightarrow \mathbb{C}, \qquad u \longmapsto \int_{-\infty}^{\infty} \langle g(y), u(y)\rangle \, \mathrm{d}y \\ \mathcal{K} &: \quad \mathbb{C} \longrightarrow \mathcal{H}, \qquad a \longmapsto ah, \end{aligned}$$

we see that this operator fits into the framework described above.

Example 2.2 *Consider the nonlocal operator*

$$\mathcal{D} \longrightarrow \mathcal{H}, \qquad u(\cdot) \longmapsto \sum_{j=1}^{m} h_j(\cdot)u(x_j)$$

where $x_j \in \mathbb{R}$ and $h_j \in L^2(\mathbb{R},\mathbb{C}^{n\times n})$ are given. We define

$$\begin{aligned} \mathcal{J} &: \quad \mathcal{D} \longrightarrow \mathbb{C}^{nm}, \qquad u \longmapsto (u(x_1),\dots,u(x_m)), \qquad u(x_j) \in \mathbb{C}^n \\ \mathcal{K} &: \quad \mathbb{C}^{nm} \longrightarrow \mathcal{H}, \qquad a \longmapsto \sum_{j=1}^{m} h_j a_j, \qquad a_j \in \mathbb{C}^n \end{aligned}$$

to see that the above nonlocal operator can be cast within our framework.

With the assumptions made above, the operators $\mathcal{T}(\lambda)$ are closed and densely defined operators in $\mathcal{H}$ with domain $\mathcal{D}$. We are interested in the set of λ for which $\mathcal{T}(\lambda)$ is not invertible.

Recall that a bounded linear operator $\mathcal{L} : X \to Y$ is said to be a Fredholm operator if its range $\mathrm{R}(\mathcal{L})$ is closed in Y, and both the dimension of its null space $\mathrm{N}(\mathcal{L})$ and the codimension of its range $\mathrm{R}(\mathcal{L})$ are finite. The difference $\dim \mathrm{N}(\mathcal{L}) - \operatorname{codim} \mathrm{R}(\mathcal{L})$ is called the Fredholm index of $\mathcal{L}$.

Definition 2.3 (Spectrum) *We say that λ is in the spectrum Σ of $\mathcal{T}$ if $\mathcal{T}(\lambda)$ is not invertible. We say that $\lambda \in \Sigma$ is in the point spectrum Σ_{pt} of $\mathcal{T}$, or alternatively that $\lambda \in \Sigma$ is an eigenvalue of $\mathcal{T}$, if $\mathcal{T}(\lambda)$ is a Fredholm operator with index zero and* $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$. *The complement $\Sigma \setminus \Sigma_{\mathrm{pt}} =: \Sigma_{\mathrm{ess}}$ is called the essential spectrum.*

We emphasize that the spectrum of the individual operators $\mathcal{T}(\lambda) : \mathcal{D} \to \mathcal{H}$, for fixed λ, is of no interest to us, although we will use information about the spectrum of the matrix $A(x;\lambda)$ in Section 3.

For any λ in the point spectrum, we define its multiplicity as follows. Recall that $A(x;\lambda)$ is of the form $A(x;\lambda) = \tilde{A}(x) + \lambda B(x)$. We begin with geometrically simple eigenvalues λ, that is, we assume that λ is in the point spectrum of $\mathcal{T}$, with

$$\mathcal{T}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}x} - \tilde{A}(\cdot) - \lambda B(\cdot) - \mathcal{K}\mathcal{J},$$

such that $\mathrm{N}(\mathcal{T}(\lambda))$ is one-dimensional and spanned by $u_1(\cdot)$. We say that λ has (algebraic) multiplicity ℓ if there are functions $u_j \in \mathcal{D}$ for $j = 2,\dots,\ell$ such that

$$\mathcal{T}(\lambda)u_j = Bu_{j-1}$$

for $j = 2, \ldots, \ell$, but so that there is no solution $u \in \mathcal{D}$ to

$$\mathcal{T}(\lambda)u = Bu_\ell.$$

Finally, for an eigenvalue λ of $\mathcal{T}$ with arbitrary geometric multiplicity, we define the algebraic multiplicity ℓ as the sum of the multiplicities of a maximal set of linearly independent elements in $\mathrm{N}(\mathcal{T}(\lambda))$.

Define the local part $\mathcal{T}_{\mathrm{loc}}$ of $\mathcal{T}$ by

$$\mathcal{T}_{\mathrm{loc}}(\lambda) = \frac{\mathrm{d}}{\mathrm{d}x} - A(\cdot;\lambda).$$

Since the operators $\mathcal{T}$ and $\mathcal{T}_{\mathrm{loc}}$ differ by the compact operator $\mathcal{K}\mathcal{J}$, which does not affect Fredholm properties and indices [15], we have the following result.

Lemma 2.4 *Assume that Hypotheses 1 and 2 are met, then the essential spectra of $\mathcal{T}$ and $\mathcal{T}_{\mathrm{loc}}$ are the same.*

Since $\mathcal{T}$ and $\mathcal{T}_{\mathrm{loc}}$ have the same essential spectrum, we concentrate on locating the point spectrum of $\mathcal{T}$. To locate those λ for which $\dim \mathrm{N}(\mathcal{T}(\lambda)) > 0$, we shall investigate the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = A(x;\lambda)u(x) + [\mathcal{K}\mathcal{J}u](x)$$

for $u \in \mathcal{D}$, and the associated ordinary differential equation (ODE)

$$\frac{\mathrm{d}}{\mathrm{d}x}u = A(x;\lambda)u \tag{2.2}$$

for $u \in \mathbb{C}^n$. We denote by $\Phi(x, y; \lambda)$ the evolution associated with (2.2). A particularly useful notion pertaining to the ODE (2.2) is exponential dichotomies:

Definition 2.5 (Exponential dichotomies) *Let $I = \mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$, and fix $\lambda_* \in \mathbb{C}$. We say that (2.2), with $\lambda = \lambda_*$ fixed, has an exponential dichotomy on I if there exist constants $K > 0$ and $\kappa^{\mathrm{s}} < 0 < \kappa^{\mathrm{u}}$ as well as a family of projections $P(x)$, defined and continuous for $x \in I$, such that the following is true.*

- *For any fixed $y \in I$ and $u_0 \in \mathbb{C}^n$, there exists a solution $\Phi^{\mathrm{s}}(x, y)u_0$ of (2.2) with initial condition $\Phi^{\mathrm{s}}(y, y)u_0 = P(y)u_0$ for $x = y$, and*

  $$|\Phi^{\mathrm{s}}(x, y)| \le K\mathrm{e}^{\kappa^{\mathrm{s}}(x-y)}$$

  for all $x \ge y$ with $x, y \in I$.

- *For any fixed $y \in I$ and $u_0 \in \mathbb{C}^n$, there exists a solution $\Phi^{\mathrm{u}}(x, y)u_0$ of (2.2) with initial condition $\Phi^{\mathrm{u}}(y, y)u_0 = (\mathrm{id} - P(y))u_0$ for $x = y$, and*

  $$|\Phi^{\mathrm{u}}(x, y)| \le K\mathrm{e}^{\kappa^{\mathrm{u}}(x-y)}$$

  for all $x \le y$ with $x, y \in I$.

- *The solutions $\Phi^{\rm s}(x,y)u_0$ and $\Phi^{\rm u}(x,y)u_0$ satisfy*

$$\begin{aligned} \Phi^{\rm s}(x,y)u_0 &\in \mathrm{R}(P(x)) && \textit{for all } x \geq y \textit{ with } x,y \in I \\ \Phi^{\rm u}(x,y)u_0 &\in \mathrm{N}(P(x)) && \textit{for all } x \leq y \textit{ with } x,y \in I. \end{aligned}$$

The x-independent dimension of $\mathrm{N}(P(x))$ is referred to as the Morse index $i(\lambda_*)$ of the exponential dichotomy on I. If (2.2) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$, the associated Morse indices are denoted by $i_+(\lambda_*)$ and $i_-(\lambda_*)$, respectively.

The following result due to Palmer relates spectral properties of $\mathcal{T}_{\rm loc}$ and exponential dichotomies of (2.2).

Lemma 2.6 ([18, 19]) *Assume that Hypotheses 1 and 2 are met. We have that $\lambda \notin \Sigma_{\rm ess}$ if, and only if, (2.2) has exponential dichotomies on $\mathbb{R}^+$ and on $\mathbb{R}^-$, and the associated Morse indices $i_+(\lambda) = i_-(\lambda)$ are the same.*

In the situation described in Lemma 2.6, we denote by $\Phi^{\rm s}_\pm(x,y;\lambda)$ and $\Phi^{\rm u}_\pm(x,y;\lambda)$ the exponential dichotomies of (2.2) on $\mathbb{R}^\pm$. We also define the complementary projections

$$P^{\rm s}_\pm(x;\lambda) := \Phi^{\rm s}_\pm(x,x;\lambda), \qquad P^{\rm u}_\pm(x;\lambda) := \Phi^{\rm u}_\pm(x,x;\lambda) \tag{2.3}$$

for $x \in \mathbb{R}^\pm$. The above definition then implies that

$$\Phi^{\rm s}_\pm(x,x;\lambda) = \Phi(x,y;\lambda)P^{\rm s}_\pm(y;\lambda), \qquad \Phi^{\rm u}_\pm(x,x;\lambda) = \Phi(x,y;\lambda)P^{\rm u}_\pm(y;\lambda)$$

which we shall use frequently in the analysis below. On account of the results presented in [20], the projections and evolutions can be chosen in such a fashion that they depend analytically on λ for λ in any simply-connected domain in $\mathbb{C}$ that has empty intersection with $\Sigma_{\rm ess}$.

2.2 Definition of the Evans function

Throughout this section, we assume that λ varies in a fixed simply-connected open subset of $\mathbb{C}\setminus\Sigma_{\rm ess}$. We have that $u \in \mathrm{N}(\mathcal{T}(\lambda))$ if, and only if,

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = A(x;\lambda)u(x) + [\mathcal{K}\mathcal{J}u](x), \qquad x \in \mathbb{R}. \tag{2.4}$$

We write this equation as the equivalent system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}u(x) &= A(x;\lambda)u(x) + [\mathcal{K}a](x), \qquad x \in \mathbb{R} && (2.5) \\ a &= \mathcal{J}u && (2.6) \end{aligned}$$

where $a \in \mathbb{C}^m$.

We begin by solving (2.5). Using the exponential dichotomies for (2.2), the general bounded solution of (2.5) on $\mathbb{R}^-$ and $\mathbb{R}^+$ is given by

$$\begin{aligned} u^-(x) &= -\Phi(x,0;\lambda)b^- + \int_0^x \Phi_-^{\mathrm{u}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_{-\infty}^x \Phi_-^{\mathrm{s}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y \qquad (2.7)\\ u^+(x) &= \Phi(x,0;\lambda)b^+ + \int_0^x \Phi_+^{\mathrm{s}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_{\infty}^x \Phi_+^{\mathrm{u}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y \end{aligned}$$

where $x \in \mathbb{R}^-$ in the first and $x \in \mathbb{R}^+$ in the second equation. The vectors a and (b^-, b^+) are arbitrary subject to

$$(a, b^-, b^+) \in \mathbb{C}^m \times \mathrm{R}(P_-^{\mathrm{u}}(0;\lambda)) \times \mathrm{R}(P_+^{\mathrm{s}}(0;\lambda)). \qquad (2.8)$$

For given a and (b^-, b^+) as above, we can write

$$(u^-, u^+) = V(\lambda)a + W(\lambda)(b^-, b^+) \in H^1(\mathbb{R}^-, \mathbb{C}^n) \oplus H^1(\mathbb{R}^+, \mathbb{C}^n), \qquad (2.9)$$

where $V(\lambda)a$ and $W(\lambda)b$ are given by the right-hand side of (2.7), i.e., via

$$\begin{aligned} [V(\lambda)a](x) &= \Bigg(\int_0^x \Phi_-^{\mathrm{u}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_{-\infty}^x \Phi_-^{\mathrm{s}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y, \\ &\qquad \int_0^x \Phi_+^{\mathrm{s}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_{\infty}^x \Phi_+^{\mathrm{u}}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y \Bigg) \\ [W(\lambda)b](x) &= (-\Phi(x,0;\lambda)b^-, \Phi(x,0;\lambda)b^+). \end{aligned}$$

Note that the terms on the right-hand side are considered as functions of x. Thus, $W(\lambda)$ is the general bounded solution on $\mathbb{R}^-$ and $\mathbb{R}^+$ of the homogeneous part (2.2) of (2.5), while $V(\lambda)$ is a particular solution of the inhomogeneous equation (2.5).

Thus, we see that there is a solution $u \in \mathcal{D}$ to (2.4) if, and only if,

$$(u^-, u^+) = V(\lambda)a + W(\lambda)(b^-, b^+)$$

and

$$u^+(0) - u^-(0) = 0, \qquad a = \mathcal{J}(u^-, u^+).$$

Using (2.3), (2.7) and (2.9), the last two equations can be written as

$$\begin{aligned} P_-^{\mathrm{u}}(0;\lambda)b^- + P_+^{\mathrm{s}}(0;\lambda)b^+ - G(\lambda)a &= 0 \qquad (2.10)\\ (\mathrm{id} - \mathcal{J}V(\lambda))a - \mathcal{J}W(\lambda)(b^-, b^+) &= 0 \end{aligned}$$

where we denote by $G(\lambda) \in \mathbb{C}^{n\times m}$ the matrix defined by

$$G(\lambda)a = \int_{-\infty}^0 \Phi_-^{\mathrm{s}}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_0^\infty \Phi_+^{\mathrm{u}}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y.$$

In summary, the analysis above shows that λ is in the point spectrum of $\mathcal{T}$ if, and only if, the system

$$\begin{aligned} b^- + b^+ - G(\lambda)a &= 0 \\ (\mathrm{id} - \mathcal{J}V(\lambda))a - \mathcal{J}W(\lambda)(b^-, b^+) &= 0 \end{aligned}$$

has a non-zero solution (a, b^-, b^+) as in (2.8). Note that we can drop the projections in (2.10) because b^- and b^+ lie in their ranges by definition (2.8).

To measure to what extent (2.10) can be solved, we choose analytic bases $\{b_1(\lambda), \dots, b_k(\lambda)\}$ and $\{b_{k+1}(\lambda), \dots, b_n(\lambda)\}$ of $\mathrm{R}(P^{\mathrm{u}}_-(0;\lambda))$ and $\mathrm{R}(P^{\mathrm{s}}_+(0;\lambda))$, respectively. Such a choice is possible due to results presented in [15, Chapter II.4.2] since the projections $P^{\mathrm{u}}_-(0;\lambda)$ and $P^{\mathrm{s}}_+(0;\lambda)$ depend analytically on λ. Note also that

$$\dim \mathrm{R}(P^{\mathrm{u}}_-(0;\lambda)) + \dim \mathrm{R}(P^{\mathrm{s}}_+(0;\lambda)) = n$$

on account of Lemma 2.6 so that the numbering of the basis vectors is consistent. It is convenient to use the coordinates provided by these two bases. Thus, we define the linear isomorphism

$$C(\lambda): \qquad \mathbb{C}^n \longrightarrow \mathrm{R}(P^{\mathrm{u}}_-(0;\lambda)) \times \mathrm{R}(P^{\mathrm{s}}_+(0;\lambda)), \qquad d \longmapsto \left(\sum_{j=1}^{k} b_j(\lambda) d_j, \sum_{j=k+1}^{n} b_j(\lambda) d_j \right),$$

which we also view as

$$C(\lambda): \qquad \mathbb{C}^n \longrightarrow \mathbb{C}^n, \qquad d \longmapsto \sum_{j=1}^{n} b_j(\lambda) d_j,$$

and write (2.9) equivalently as

$$(u^-, u^+) = V(\lambda)a + W(\lambda)C(\lambda)d$$

for arbitrary $(a, d) \in \mathbb{C}^m \times \mathbb{C}^n$, where

$$(u^-, u^+) = V(\lambda)a + W(\lambda)(b^-, b^+) \in H^1(\mathbb{R}^-, \mathbb{C}^n) \oplus H^1(\mathbb{R}^+, \mathbb{C}^n)$$

is defined by the right-hand side of (2.7). With these conventions, we see that (2.10) is equivalent to

$$\begin{aligned} C(\lambda)d - G(\lambda)a &= 0 \\ -\mathcal{J}W(\lambda)C(\lambda)d + (\mathrm{id} - \mathcal{J}V(\lambda))a &= 0 \end{aligned}$$

which has a non-trivial solution $(d, a) \neq (0, 0)$ if, and only if,

$$\det S(\lambda) = 0$$

where

$$S(\lambda) = \begin{pmatrix} C(\lambda) & -G(\lambda) \\ -\mathcal{J}W(\lambda)C(\lambda) & \mathrm{id} - \mathcal{J}V(\lambda) \end{pmatrix}. \tag{2.11}$$

Hence, we define

$$E(\lambda) = \det S(\lambda) \tag{2.12}$$

and say that $E(\lambda)$ is an Evans function for the operator $\mathcal{T}$. Note that

$$E_{\mathrm{loc}}(\lambda) = \det C(\lambda)$$

is the ordinary Evans function for the local part of (2.4).

Theorem 1 *The Evans function $E(\lambda)$ is defined and analytic on simply-connected open subsets of $\mathbb{C} \setminus \Sigma_{\text{ess}}$. It has the following properties:*

1. *$E(\lambda) = 0$ if, and only if, λ is in the point spectrum of $\mathcal{T}$.*
2. *The order of λ as a root of $E(\lambda)$ is equal to the algebraic multiplicity of λ as an eigenvalue of $\mathcal{T}$.*

The statement (i) is an immediate consequence of the above discussion. Statement (ii) is proved in the next section.

2.3 Multiplicity properties of the Evans function

In this section, we prove that the order of λ_* as a root of $E(\lambda)$ is equal to the algebraic multiplicity of λ_* as an eigenvalue of $\mathcal{T}$. The idea of the proof is taken from [21]. We remark that the proof given in [5] does not appear to carry over to nonlocal equations. Throughout this section, we fix an eigenvalue λ_* with arbitrary geometric and algebraic multiplicity in the point spectrum and consider λ near λ_*.

We define

$$\tilde{W}(\lambda) = W(\lambda)C(\lambda).$$

For any analytic function $w(\lambda)$, we set

$$w^{(j)}(\lambda) := \frac{\mathrm{d}^j}{\mathrm{d}\lambda^j} w(\lambda).$$

Lemma 2.7 *Fix (a, d) and define*

$$w(\lambda) := V(\lambda)a + \tilde{W}(\lambda)d \in H^1(\mathbb{R}^-, \mathbb{C}^n) \oplus H^1(\mathbb{R}^+, \mathbb{C}^n).$$

We then have

$$\mathcal{T}_{\text{loc}}(\lambda)w^{(j)}(\lambda) = jBw^{(j-1)}(\lambda)$$

for any $j \geq 1$ and every λ.

Proof. The lemma follows immediately from taking derivatives of the equation

$$\mathcal{T}_{\text{loc}}(\lambda)w(\lambda) = \mathcal{K}\mathcal{J}a$$

which is satisfied for all λ due to the definition of $w(\lambda)$. Note that the right-hand side does not depend on λ. We omit the details. ∎

For certain fixed $(a_j, d_j) \in \mathbb{C}^m \times \mathbb{C}^n$ that will be chosen below, we define

$$v_\ell(\lambda) = \sum_{j=1}^{\ell} \frac{1}{(\ell - j)!} \partial_\lambda^{\ell-j} [V(\lambda)a_j + \tilde{W}(\lambda)d_j],$$

and note that

$$\mathcal{T}_{\mathrm{loc}}(\lambda)v_\ell(\lambda) = \mathcal{K}\mathcal{J}a_\ell + Bv_{\ell-1}(\lambda)$$

so that

$$\mathcal{T}(\lambda)v_\ell(\lambda) = Bv_{\ell-1}(\lambda).$$

Using the notation $[v(\lambda)](x) =: v(x;\lambda)$, and writing $v = (v^-, v^+)$, the jump of $v_\ell(\lambda)$ at $x = 0$ is given by

$$\begin{aligned} v_\ell^-(0;\lambda) - v_\ell^+(0;\lambda) &= \sum_{j=1}^{\ell} \frac{1}{(\ell-j)!}\partial_\lambda^{\ell-j}\Big([V_0^-(\lambda) - V_0^+(\lambda)]a_j + C(\lambda)[P_- - P_+]d_j\Big) \\ &= \sum_{j=1}^{\ell} \frac{1}{(\ell-j)!}\partial_\lambda^{\ell-j}\Big(G(\lambda)a_j + C(\lambda)[P_- - P_+]d_j\Big) \end{aligned}$$

where

$$\begin{aligned} V_0^-(\lambda)a &= \int_{-\infty}^{0} \Phi_-^{\mathrm{s}}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y \\ V_0^+(\lambda)a &= \int_{\infty}^{0} \Phi_+^{\mathrm{u}}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y. \end{aligned}$$

Furthermore, we have

$$a_\ell - \mathcal{J}v_\ell(\lambda) = a_\ell - \sum_{j=1}^{\ell}\frac{1}{(\ell-j)!}\partial_\lambda^{\ell-j}[\mathcal{J}V(\lambda)a_j + \mathcal{J}\tilde{W}(\lambda)d_j].$$

In summary, we see that

$$v_\ell^-(0;\lambda) - v_\ell^+(0;\lambda) = 0, \qquad a_\ell - \mathcal{J}v_\ell(\lambda) = 0$$

if, and only if,

$$\sum_{j=1}^{\ell}\frac{1}{(\ell-j)!}\partial_\lambda^{\ell-j}S(\lambda)(d_j,a_j) = 0.$$

In other words, $\mathcal{T}(\lambda_*)$ has a Jordan chain of length ℓ if, and only if, there are coefficients $(a_j,d_j) \in \mathbb{C}^m \times \mathbb{C}^n$ for $j = 1,\dots,\ell$ such that

$$\sum_{j=1}^{l}\frac{1}{(l-j)!}\partial_\lambda^{l-j}S(\lambda_*)(d_j,a_j) = 0$$

for $l = 1,\dots,\ell$. The solvability of the above equation is linked to the order of λ_* as a root of the Evans function $\det S(\lambda)$ by [21, Lemma 4.1]. This completes the proof of Theorem 1.

3 Extension of the Evans function across the essential spectrum

The Evans function we constructed in the preceding section is only defined for λ in a complement of the essential spectrum. In conservative partial differential equations (PDEs), in particular for

Hamiltonian PDEs, the essential spectrum lies on the imaginary axis. If perturbations are added to the PDE, eigenvalues may emerge from the essential spectrum and therefore may cause instabilities. The Evans function has been used to detect these eigenvalues [11–13]. In this section, we show that the Evans function for nonlocal equations can also be extended across the essential spectrum.

First, however, consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u = A(x;\lambda)u. \tag{3.1}$$

We assume that the limits of the coefficient matrix for $x \to \pm\infty$ exist and are approached exponentially. For the sake of clarity, we consider only the case where these limits are equal, but we remark that the results presented below are also true if the limits are different.

Hypothesis 3 *For each $\lambda \in \mathbb{C}$, there are positive constants ρ_0 and C and a matrix $A_0(\lambda)$, analytic in λ, such that*

$$|A(x;\lambda) - A_0(\lambda)| \leq C\mathrm{e}^{-\rho_0|x|}$$

for all $x \in \mathbb{R}$.

The eigenvalues of the asymptotic matrix $A_0(\lambda)$ play an important role in the following analysis. To distinguish them from the temporal eigenvalues λ, the eigenvalues of $A_0(\lambda)$ for fixed λ are referred to as spatial eigenvalues. The essential spectrum of $\mathcal{T}$ is given by

$$\Sigma_{\mathrm{ess}} = \{\lambda \in \mathbb{C};\ \mathrm{spec}(A_0(\lambda)) \cap \mathrm{i}\mathbb{R} \neq \emptyset\}.$$

For any element $\lambda_* \in \Sigma_{\mathrm{ess}}$, there exists a $\kappa > 0$ with the following property. Choose any $\eta \in (0,\kappa)$, then, for any λ close to λ_*, the spectrum of $A_0(\lambda)$ is the union of three disjoint sets $\sigma_{\mathrm{c}}(\lambda)$, $\sigma_{\mathrm{uu}}(\lambda)$ and $\sigma_{\mathrm{ss}}(\lambda)$ that depends continuously on λ and that satisfy

$$\begin{aligned}
\sigma_{\mathrm{c}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ |\mathrm{Re}\,\nu| < \eta\} \\
\sigma_{\mathrm{uu}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \mathrm{Re}\,\nu > \kappa\} \\
\sigma_{\mathrm{ss}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \mathrm{Re}\,\nu < -\kappa\}
\end{aligned} \tag{3.2}$$

so that $\sigma_{\mathrm{c}}(\lambda_*)$ consists of all purely imaginary eigenvalues $\nu \in \mathrm{i}\mathbb{R}$ of $A_0(\lambda_*)$. Thus, the strong stable and strong unstable spectral sets $\sigma_{\mathrm{ss}}(\lambda)$ and $\sigma_{\mathrm{uu}}(\lambda)$ consist of all eigenvalues that keep a uniform distance from the imaginary axis as we vary λ near the essential spectrum.

We extend the nonlocal part of the Evans function, i.e., the particular solution $V(\lambda)$ of (2.5), in Section 3.1. Afterwards, in Sections 3.2 and 3.3, we review the extension of the local Evans function across the essential spectrum.

3.1 Extending the nonlocal part of the Evans function

We pick $\lambda_* \in \Sigma_{\mathrm{ess}}$ and recall the spectral decomposition (3.2)

$$\begin{aligned}
\sigma_{\mathrm{c}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ |\mathrm{Re}\,\nu| < \eta\} \\
\sigma_{\mathrm{uu}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \mathrm{Re}\,\nu > \kappa\} \\
\sigma_{\mathrm{ss}}(\lambda) &:= \mathrm{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \mathrm{Re}\,\nu < -\kappa\}
\end{aligned}$$

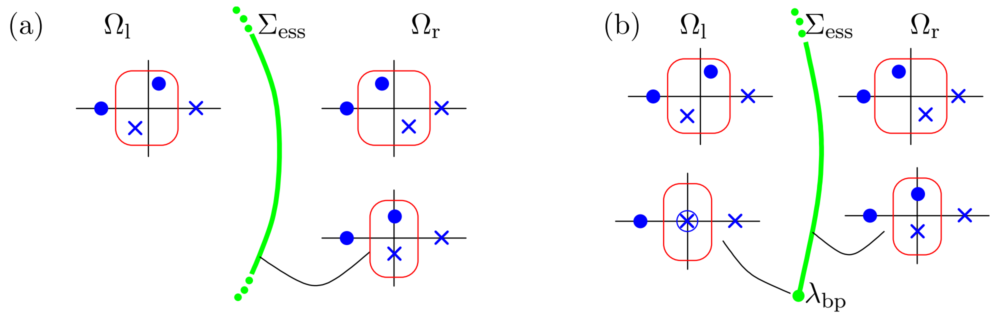

Figure 2: *We plot the essential spectrum $\Sigma_{\rm ess}$ and the two regions $\Omega_{\rm r}$ and $\Omega_{\rm l}$ to the right and left of $\Sigma_{\rm ess}$. The insets show the spatial spectrum of $A_0(\lambda)$: the* crosses *form the unstable spectral set $\sigma_{\rm u}(\lambda)$, while the* bullets *form the stable spectral set $\sigma_{\rm s}(\lambda)$. The spatial eigenvalues inside the boxes belong to the center spectrum $\sigma_{\rm c}(\lambda)$, while the remaining spatial eigenvalues to the left and right of the boxes belong to the strong-stable and the strong-unstable spectra denoted by $\sigma_{\rm ss}(\lambda)$ and $\sigma_{\rm uu}(\lambda)$, respectively. In (b), the point $\lambda_{\rm bp}$ is a simple branch point where two spatial eigenvalues collide with geometric multiplicity one.*

from the preceding section, where λ is in a small neighborhood $U_\delta(\lambda_*)$ of λ_* and $\kappa > \eta > 0$. Using Dunford's integral [15, Chapter I.5.6], we obtain spectral projections $P_0^{\rm c}(\lambda)$, $P_0^{\rm ss}(\lambda)$ and $P_0^{\rm uu}(\lambda)$ belonging to this spectral decomposition of $A_0(\lambda)$.

In the next step, we construct projections and evolution operators for the full equation (3.1):

$$\frac{\rm d}{{\rm d}x}u = A(x;\lambda)u. \tag{3.3}$$

Denote the evolution of (3.3) by $\Phi(x,y;\lambda)$. It follows from the results proved in [20] that there are complementary projections $P_+^{\rm ss}(x;\lambda)$ and $P_+^{\rm cu}(x;\lambda)$, defined for $x \geq 0$, that vary analytically in λ for $\lambda \in U_\delta(\lambda_*)$ such that

$$\begin{aligned}
&|P_+^{\rm ss}(x;\lambda) - P_0^{\rm ss}(\lambda)| \to 0 && x \to \infty \\
&\Phi(x,y;\lambda)P_+^{\rm ss}(y;\lambda) = P_+^{\rm ss}(x;\lambda)\Phi(x,y;\lambda) && \forall x,y \geq 0 \\
&|\Phi(x,y;\lambda)P_+^{\rm ss}(y;\lambda)| \leq K{\rm e}^{-\kappa|x-y|} && x \geq y \geq 0 \\
&|\Phi(x,y;\lambda)P_+^{\rm cu}(y;\lambda)| \leq K{\rm e}^{\eta|x-y|} && y \geq x \geq 0.
\end{aligned}$$

In fact, the last two estimates are also true for derivatives with respect to λ. Analogous statements are true on $\mathbb{R}^-$ for certain projections $P_-^{\rm cs}(x;\lambda)$ and $P_-^{\rm uu}(x;\lambda)$.

The extension of the nonlocal part of the Evans function can now be defined provided the nonlocal operator is well defined on spaces of functions that grow exponentially with a small rate. We define

$$X_\rho = \left\{ u \in H^1_{\rm loc}(\mathbb{R},\mathbb{C}^n);\ \left\|{\rm e}^{\rho|x|}u(x)\right\|_{L^2(\mathbb{R})} + \left\|{\rm e}^{\rho|x|}u_x(x)\right\|_{L^2(\mathbb{R})} =: \|u\|_\rho < \infty \right\}.$$

Hypothesis 4 *Assume that there exists a $\rho_1 > 0$ such that $\mathcal{J} : X_{-\rho_1} \longrightarrow \mathbb{C}^m$ and $\mathcal{K} : \mathbb{C}^m \to X_{\rho_1}$ are bounded. Furthermore, we assume that $\delta > 0$ is so small that the gap η in (3.2) is smaller than ρ_1 for all $\lambda \in U_\delta(\lambda_*)$.*

It is now straightforward to show that the functions

$$\begin{aligned}
V(\lambda)a &:= \Big(\int_0^x \Phi_-^{\rm uu}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_{-\infty}^x \Phi_-^{\rm cs}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y, \\
&\qquad \int_0^x \Phi_+^{\rm ss}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_\infty^x \Phi_+^{\rm cu}(x,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y\Big) \\
G(\lambda)a &:= \int_{-\infty}^0 \Phi_-^{\rm cs}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y + \int_0^\infty \Phi_+^{\rm cu}(0,y;\lambda)[\mathcal{K}a](y)\,\mathrm{d}y
\end{aligned}$$

are well defined and analytic in λ for $\lambda \in U_\delta(\lambda_*)$ and $a \in \mathbb{C}^m$. Furthermore, the two components of $[V(\lambda)a](x)$ are particular solutions to

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x) = A(x;\lambda)u(x) + [\mathcal{K}a](x)$$

on $\mathbb{R}^-$ and $\mathbb{R}^+$ that decay exponentially as $|x| \to \infty$.

3.2 Extending the local Evans function away from branch points

Assume that, for some $\delta > 0$, $\Sigma_{\rm ess} \cap U_\delta(\lambda_*)$ is a curve as shown in Figure 2(a). As a consequence, within $U_\delta(\lambda_*)$, we can distinguish the regions $\Omega_{\rm l}$ and $\Omega_{\rm r}$ to the left and right, respectively, of $\Sigma_{\rm ess}$. Suppose that a local Evans function for the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}u = A(x;\lambda)u$$

has been defined in the region $\Omega_{\rm r}$. Our goal is to extend this Evans function analytically, locally near λ_*, into the region $\Omega_{\rm l}$. For $\lambda \in \Omega_{\rm r}$, we define

$$\begin{aligned}
\sigma_{\rm u}(\lambda) &:= \operatorname{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \operatorname{Re}\nu > 0\} \\
\sigma_{\rm s}(\lambda) &:= \operatorname{spec}(A_0(\lambda)) \cap \{\nu \in \mathbb{C};\ \operatorname{Re}\nu < 0\}
\end{aligned}$$

Since $\Omega_{\rm r}$ does not intersect the essential spectrum, there is an integer k such that $\#\sigma_{\rm u}(\lambda) = k$ and $\#\sigma_{\rm s}(\lambda) = n - k$ for all $\lambda \in \Omega_{\rm r}$, where the spatial eigenvalues are counted with their algebraic multiplicity.

Hypothesis 5 *We assume that there are disjoint subsets $\sigma_{\rm u}(\lambda)$ and $\sigma_{\rm s}(\lambda)$ of $\mathbb{C}$, defined and continuous for $\lambda \in \Sigma_{\rm ess} \cap U_\delta(\lambda_*)$, such that*

$$\sigma_{\rm u}(\lambda) \cup \sigma_{\rm s}(\lambda) = \operatorname{spec}(A_0(\lambda)), \qquad \lambda \in \Sigma_{\rm ess} \cap U_\delta(\lambda_*)$$

and such that the spectral sets $\sigma_{\rm u}(\lambda)$ and $\sigma_{\rm s}(\lambda)$ are the limits of the spectral sets $\sigma_{\rm u}(\tilde\lambda)$ and $\sigma_{\rm s}(\tilde\lambda)$, respectively, as $\tilde\lambda$ approaches λ with $\tilde\lambda \in \Omega_{\rm r}$ (see Figure 2(a)). Furthermore, we assume that $\delta > 0$ is so small that the overlap of the stable and unstable spectra is small,

$$\sup\{\operatorname{Re}\nu;\ \nu \in \sigma_{\rm s}(\lambda)\} - \inf\{\operatorname{Re}\nu;\ \nu \in \sigma_{\rm u}(\lambda)\} < \min\{\rho_0, \rho_1\},$$

where ρ_0 and ρ_1 have been defined in Hypotheses 3 and 4, respectively.

As a consequence, possibly after making the neighborhood of λ_* a bit smaller, we can uniquely divide the spatial spectrum of $A_0(\lambda)$ into two disjoint sets, $\sigma_{\rm u}(\lambda)$ and $\sigma_{\rm s}(\lambda)$, that continue the unstable and stable spectral sets for $\lambda \in \Omega_{\rm r}$ into $\Sigma_{\rm ess}$ and into the region $\Omega_{\rm l}$.

Using the Gap Lemma [6, 11], we can find a set of analytic vectors $\{b_1(\lambda), \dots, b_n(\lambda)\}$ such that $\{b_1(\lambda), \dots, b_k(\lambda)\}$ and $\{b_{k+1}(\lambda), \dots, b_n(\lambda)\}$ are bases of $\mathrm{R}(P_-^{\rm u}(0;\lambda))$ and $\mathrm{R}(P_+^{\rm s}(0;\lambda))$, respectively, for $\lambda \in \Omega_{\rm r}$. Furthermore, we have that

$$|\Phi(x,0;\lambda)b_j(\lambda)| \le K\mathrm{e}^{\eta|x|} \begin{cases} x \le 0 & j = 1, \dots, k \\ x \ge 0 & j = k+1, \dots, n \end{cases} \tag{3.4}$$

where $\eta > 0$ is as small as we wish provided we make $\delta > 0$ smaller. We let

$$C(\lambda) = [b_1(\lambda), \dots, b_k(\lambda), b_{k+1}(\lambda), \dots, b_n(\lambda)]$$

and define

$$W(\lambda)C(\lambda) := (-\Phi(x,0;\lambda)C(\lambda), \Phi(x,0;\lambda)C(\lambda))$$

where $x < 0$ in the first and $x > 0$ in the second component. Thus, using the estimates (3.4) and Hypothesis 4, we see that $\mathcal{J}W(\lambda)C(\lambda)$ is bounded on $\mathbb{C}^n$ and analytic in λ for $\lambda \in U_\delta(\lambda_*)$. Hence, the expressions (2.11) and (2.12) for the Evans function make sense for $\lambda \in U_\delta(\lambda_*)$, and the resulting function is analytic in λ.

Theorem 2 *Assume that Hypotheses 1–5 are satisfied. The Evans function (2.12) is then well defined and analytic in λ for $\lambda \in U_\delta(\lambda_*)$.*

3.3 Extending the local Evans function through branch points

In this section, we extend the Evans function analytically near branch points of the essential spectrum where two spatial eigenvalues ν of the asymptotic matrix $A_0(\lambda)$ collide on the imaginary axis to form a Jordan block as illustrated in Figure 2(b). Let

$$d(\lambda, \nu) := \det[A_0(\lambda) - \nu] \tag{3.5}$$

denote the characteristic polynomial associated with the asymptotic matrix $A_0(\lambda)$. We impose the following non-degeneracy conditions.

Hypothesis 6 *We assume that the matrix $A_0(\lambda)$ has a simple branch point at $(\lambda, \nu) = (\lambda_{\rm bp}, \nu_{\rm bp})$ where $\nu_{\rm bp} \in \mathrm{i}\mathbb{R}$, that is, we assume that $(d, d_\nu)(\lambda_{\rm bp}, \nu_{\rm bp}) = 0$,*

$$\det \begin{bmatrix} d_\lambda & d_\nu \\ d_{\lambda\nu} & d_{\nu\nu} \end{bmatrix}_{(\lambda_{\rm bp},\nu_{\rm bp})} = \det \begin{bmatrix} d_\lambda & 0 \\ d_{\lambda\nu} & d_{\nu\nu} \end{bmatrix}_{(\lambda_{\rm bp},\nu_{\rm bp})} \neq 0,$$

and that there are no other purely imaginary roots $\nu \in \mathrm{i}\mathbb{R}$ of $d(\lambda_{\rm bp}, \nu) = 0$ besides $\nu = \nu_{\rm bp}$.

Hypothesis 6 implies in particular that simple branch points depend smoothly on additional parameters ϵ if the asymptotic matrix depends smoothly on ϵ.

Upon making the change of coordinates

$$\lambda = \lambda_{\rm bp} + \gamma^2, \tag{3.6}$$

we obtain n solutions $\nu_j(\gamma)$ with $j = 1, \ldots, n$ of $d(\lambda, \nu) = 0$. We can order these solutions so that $\mathrm{Re}\, \nu_j(\gamma) > \kappa$ for $j = 1, \ldots, k-1$ and $\mathrm{Re}\, \nu_j(\gamma) < -\kappa$ for $j = k+2, \ldots, n$, while the remaining two solutions, $\nu_k(\gamma)$ and $\nu_{k+1}(\gamma)$, are close to $\nu_{\rm bp}$ and are given by

$$\nu_k(\gamma) = \nu_{\rm bp} + \check{d}\gamma + \mathrm{O}(\gamma^2), \qquad \nu_{k+1}(\gamma) = \nu_{\rm bp} - \check{d}\gamma + \mathrm{O}(\gamma^2)$$

where $\check{d} = \sqrt{2d_{\nu\nu}/d_\lambda}$ (evaluated at the branch point). In particular, the two spatial eigenvalues $\nu_k(\gamma)$ and $\nu_{k+1}(\gamma)$ depend analytically on γ. Using again the Gap Lemma [6, 11], we can find a set of analytic vectors $\{b_1(\gamma), \ldots, b_n(\gamma)\}$ such that $\{b_1(\gamma), \ldots, b_k(\gamma)\}$ and $\{b_{k+1}(\gamma), \ldots, b_n(\gamma)\}$ are bases of $\mathrm{R}(P_-^{\rm u}(0;\gamma))$ and $\mathrm{R}(P_+^{\rm s}(0;\gamma))$, respectively, for all γ. Note that $\mathrm{Re}\, \nu_k(\gamma) > 0 > \mathrm{Re}\, \nu_{k+1}(\gamma)$ precisely when $\mathrm{Re}(\check{d}\gamma) > 0$. Furthermore, we have that

$$\left|\Phi(x, 0; \lambda_{\rm bp} + \gamma^2) b_j(\gamma)\right| \le K \mathrm{e}^{\eta|x|} \begin{cases} x \le 0 & j = 1, \ldots, k \\ x \ge 0 & j = k+1, \ldots, n \end{cases}$$

where $\eta > 0$ is as small as we wish. Proceeding as in the preceding section, we see that the Evans function can be extended as a function of γ into a neighborhood $U_\delta(0)$ of $\gamma = 0$. The extended Evans function is analytic in γ for all γ near zero since it is analytic for all $\gamma \neq 0$ and uniformly bounded in γ. However, a root γ of the Evans function corresponds to an eigenvalue if, and only if, $\mathrm{Re}(\check{d}\gamma) > 0$ since only then is the spatial eigenvalue $\nu_k(\gamma)$ contained in the unstable spectrum of the asymptotic matrix.

Theorem 3 *Assume that Hypotheses 1–4 and 6 are satisfied. The Evans function (2.12) is then well defined and analytic as a function $E_{\rm bp}(\gamma)$ of γ for γ close to zero, where γ and λ are related via (3.6). A root γ of the Evans function $E_{\rm bp}(\gamma)$ corresponds to an eigenvalue $\lambda = \lambda_{\rm bp} + \gamma^2$ if, and only if,* $\mathrm{Re}(\check{d}\gamma) > 0$*, where $\check{d} = \sqrt{2d_{\nu\nu}/d_\lambda}$ is evaluated at the branch point with $d(\lambda, \nu)$ as in (3.5).*

3.4 Special case: Small nonlocal perturbations

Lastly, we consider the special case where the nonlocal term is small. Thus, consider the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x) = A_0(x;\lambda)u(x) + \epsilon A_1(x)u(x) + \epsilon[\mathcal{K}\mathcal{J}u](x) \tag{3.7}$$

or the equivalent system

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x}(x) &= A_0(x;\lambda)u(x) + \epsilon A_1(x)u(x) + \epsilon[\mathcal{K}a](x) \\ a &= \mathcal{J}u \end{aligned}$$

where $a \in \mathbb{C}^m$, and ϵ is close to zero. The Evans function associated with (3.7) is then of the form

$$E(\lambda,\epsilon) = \det \begin{pmatrix} C(\lambda,\epsilon) & -\epsilon G(\lambda,\epsilon) \\ -\mathcal{J}W(\lambda,\epsilon)C(\lambda,\epsilon) & \mathrm{id} - \epsilon \mathcal{J} V(\lambda,\epsilon) \end{pmatrix}.$$

In particular, the Evans function $E(\lambda,\epsilon)$ is smooth in ϵ and we have

$$E(\lambda,0) = \det C(\lambda,0)$$

which is the Evans function associated with the local part

$$\frac{\mathrm{d}u}{\mathrm{d}x} = A_0(x;\lambda)u \tag{3.8}$$

of (3.7). Thus, to study the perturbed problem for small $\epsilon \neq 0$, it suffices to investigate $E(\lambda,\epsilon)$ near roots λ of $E(\lambda,0)$. Expansions of $E(\lambda,\epsilon)$ away from branch points can be obtained as in [8]. Alternatively, if λ is not in the essential spectrum, we may use Liapunov-Schmidt reduction to calculate the perturbed eigenvalues.

We therefore concentrate on the case where $\lambda_* \in \Sigma_{\mathrm{ess}}$ is a simple branch point for $\epsilon = 0$ that satisfies $E(\lambda_*,0) = 0$. Since the branch point is simple, it persists as a smooth function $\lambda_*(\epsilon)$, and we let $\lambda = \lambda_*(\epsilon) + \gamma^2$. We assume that vectors

$$b_1(\gamma,\epsilon),\ldots,b_k(\gamma,\epsilon),b_{k+1}(\gamma,\epsilon),\ldots,b_n(\gamma,\epsilon),$$

analytic in γ and smooth in ϵ, have been constructed. Furthermore, we assume that the local Evans function for $\epsilon = 0$ has a simple root at $\gamma = 0$. We may then assume that

$$b_k(0,0) = b_{k+1}(0,0)$$

and denote by $\varphi^{\mathrm{c}}(x)$ the solution of (3.8) with initial condition $\varphi^{\mathrm{c}}(0) = b_k(0,0) = b_{k+1}(0,0)$. We also choose a non-zero vector $\psi(0)$ with $\psi(0) \perp b_j(0,0)$ for all j. Such a choice is possible since $b_k = b_{k+1}$ so that the codimension of $\mathrm{span}\{b_j(0,0)\}$ is one. We denote the solution to

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -A_0(x;\lambda_*)^* w, \qquad w(0) = \psi(0)$$

by $\psi(x)$. An adaption of the results in [11, Section 4] shows that the Evans function $E_{\mathrm{bp}}(\gamma,\epsilon)$, defined in Theorem 3, has the expansion

$$E_{\mathrm{bp}}(\gamma,\epsilon) = \check{c}_1\gamma + \check{c}_2\epsilon + \mathrm{O}(\gamma^2+\epsilon^2) \tag{3.9}$$

near $(\gamma,\epsilon) = (0,0)$ where

$$\begin{aligned} \check{c}_1 &= \langle \psi(0), \partial_\gamma [b_k(0,0) - b_{k+1}(0,0)] \rangle \\ \check{c}_2 &= \int_{-\infty}^{\infty} \Big\langle \psi(x), \partial_\epsilon [A_0(x;\lambda_*(\epsilon))]\Big|_{\epsilon=0} \varphi^{\mathrm{c}}(x) + A_1(x)\varphi^{\mathrm{c}}(x) + [\mathcal{K}\mathcal{J}\varphi^{\mathrm{c}}](x) \Big\rangle \,\mathrm{d}x \end{aligned}$$

which is the result one would expect from a regular perturbation analysis if it could be applied.

3.5 A simple example

Hypothesis 4 appears to be necessary to extend the Evans function smoothly into a branch point. To illustrate this, consider the equation

$$u_{xx} + \epsilon\delta_1 u = \lambda u \tag{3.10}$$

on the space $C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{C})$, where $\delta_a u = u(a)$ is the delta-function. We rewrite this equation as

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 \\ \epsilon a \end{pmatrix} \tag{3.11}$$

$$a = \delta_1 u. \tag{3.12}$$

We see that $\lambda = 0$ is a branch point of the dispersion relation of the local part

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$$

and we therefore define $\lambda = \gamma^2$. Solving the local part, we obtain the solutions

$$\mathrm{e}^{Ax} = \frac{1}{2\gamma} \mathrm{e}^{\gamma x} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} (\gamma, 1) + \frac{1}{2\gamma} \mathrm{e}^{-\gamma x} \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} (\gamma, -1).$$

In particular, for $\mathrm{Re}\,\gamma > 0$, we have

$$C(\lambda) = \begin{pmatrix} 1 & -1 \\ \gamma & \gamma \end{pmatrix}$$

which then also defines the extended local Evans function

$$E_{\mathrm{local}}(\gamma) = 2\gamma$$

for $\mathrm{Re}\,\gamma < 0$. Note that the local Evans function has a zero at the branch point $\gamma = 0$. We remark that the local Evans function can be interpreted as a transmission coefficient (see [14] for a detailed account).

Let us now construct the Evans function for the full problem. The general bounded solutions of (3.11) for $\mathrm{Re}\,\gamma > 0$ are given by

$$\begin{aligned} & b^- \mathrm{e}^{\gamma x} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \left(1 - \mathrm{e}^{\gamma x}\right) \begin{pmatrix} 1 \\ \gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} && x \le 0 \\ & b^+ \mathrm{e}^{-\gamma x} \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \left(1 - \mathrm{e}^{-\gamma x}\right) \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} && x \ge 0 \end{aligned}$$

We need to solve the continuity equation at $x = 0$,

$$b^- \begin{pmatrix} 1 \\ \gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} = b^+ \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} + \frac{\epsilon a}{2\gamma^2} \begin{pmatrix} 1 \\ \gamma \end{pmatrix},$$

and the equation $a = u(1)$ which is given by

$$a = b^+ \mathrm{e}^{-\gamma} + \frac{\epsilon a}{2\gamma^2} \left(2 - \mathrm{e}^{-\gamma}\right).$$

In matrix form using (b^-, b^+, a) as coordinates, we have to solve

$$\det \begin{pmatrix} 1 & -1 & 0 \\ \gamma & \gamma & \frac{-\epsilon}{\gamma} \\ 0 & -\mathrm{e}^{-\gamma} & 1 - \frac{\epsilon(2-\mathrm{e}^{-\gamma})}{2\gamma^2} \end{pmatrix} = 2\gamma - \frac{2\epsilon}{\gamma} = 0$$

so that $\lambda = \gamma^2 = \epsilon$ is the bifurcating eigenvalue. It is obvious from (3.10) that the associated eigenfunction in $C^0_{\mathrm{unif}}(\mathbb{R}, \mathbb{C})$ is $u(x) = 1$.

Note that the Evans function for the full problem is *not* a small perturbation of the local Evans function. Instead a pole at $\gamma = 0$ appears in the nonlocal part so that the bifurcating eigenvalue is of order $\mathrm{O}(\epsilon)$ and not of order $\mathrm{O}(\epsilon^2)$ as suggested by the local Evans function which has a simple zero at $\gamma = 0$. This scenario was labelled the NLEP paradox in [4]. Though we will not study this case in generality here, we believe that our methods can be adapted to extend the Evans function across the essential spectrum even if Hypothesis 4 is not met.

4 Regular perturbation theory

4.1 Transcritical bifurcations

Suppose that $\mathcal{F} : X \times \mathbb{R} \to Y$ is C^2 and satisfies $\mathcal{F}(0, \epsilon) = 0$ for $\epsilon \in \mathbb{R}$. Here, X and Y represent two Hilbert spaces. Suppose that $\mathcal{F}_U(0, 0)$ is a Fredholm operator of index zero and that $\mathrm{N}(\mathcal{F}_U(0, 0)) = \mathrm{span}\{U_*\} \neq \{0\}$. Choose $U_*^{\mathrm{ad}} \in Y$ so that

$$Y = \mathrm{span}\{U_*^{\mathrm{ad}}\} \oplus \mathrm{R}(\mathcal{F}_U(0, 0)), \qquad U_*^{\mathrm{ad}} \perp \mathrm{R}(\mathcal{F}_U(0, 0)).$$

Liapunov-Schmidt reduction (see, for instance, [7, Ch. VII, §1(a)-(d)]) shows that $\mathcal{F}(U, \epsilon) = 0$ for (U, ϵ) close to $(0, 0)$ if, and only if,

$$g(z, \epsilon) = z \left[\frac{c_1 z}{2} + c_2 \epsilon + \mathrm{O}(\epsilon^2 + z^2) \right] = 0,$$

where

$$U = z U_* + \mathrm{O}(\epsilon^2 + z^2),$$

and the coefficients c_1 and c_2 are given by

$$c_1 = \langle U_*^{\mathrm{ad}}, \mathcal{F}_{UU}(0, 0)[U_*, U_*] \rangle_Y, \qquad c_2 = \langle U_*^{\mathrm{ad}}, \mathcal{F}_{U\epsilon}(0, 0)[U_*] \rangle_Y.$$

4.2 Application I: Existence of pulses for perturbed NLS equations

As the first application, consider the perturbed nonlinear Schrödinger equation

$$\mathrm{i} u_t + u_{xx} - \omega u + 4|u|^2 u = \mathrm{i}\epsilon G(u) \tag{4.1}$$

reminiscent of (1.1). Motivated by the properties of the right-hand side of (1.1), we assume that the perturbation $G : H^2(\mathbb{R},\mathbb{R}) \to L^2(\mathbb{R},\mathbb{R})$ can be extended to a function $G : H^2(\mathbb{R},\mathbb{C}) \to L^2(\mathbb{R},\mathbb{C})$ that is invariant under gauge transformations,

$$G(\mathrm{e}^{\mathrm{i}\gamma}u) = \mathrm{e}^{\mathrm{i}\gamma}G(u),$$

and reversible

$$G(Ru) = RG(u), \qquad [Ru](x) = u(-x).$$

We remark, however, that the results given below can be generalized to perturbations that are not gauge invariant or not reversible.

The unperturbed steady-state equation

$$F(u,\omega) := u_{xx} - \omega u + 4|u|^2u = 0 \tag{4.2}$$

admits the soliton solutions

$$u = \phi_0(\omega) = \sqrt{\frac{\omega}{2}}\,\mathrm{sech}(\sqrt{\omega}x) \tag{4.3}$$

where $\omega > 0$ is arbitrary.

Proposition 4.1 *Assume that the function*

$$g(\omega) = \langle \phi_0(\omega), G(\phi_0(\omega))\rangle_{L^2(\mathbb{R})}$$

has a simple root at $\omega = \omega_*$, then there are unique smooth functions $\omega(\epsilon)$ and $\phi(\epsilon)$ with $\omega(0) = \omega_*$ and $\phi(0) = \phi_0(\omega_*)$ such that (4.1) has the unique steady state $\phi(\epsilon)$ near $\phi_0(\omega_*)$ for each ϵ close to zero. Here, uniqueness is up to translation and gauge symmetry. If $g(\omega) \neq 0$, then the steady state (4.3) does not persist for $\epsilon \neq 0$.

To prove the proposition, we consider the function

$$\begin{aligned}\mathcal{F} \quad : \quad & H_0^2(\mathbb{R}^+,\mathbb{C}) \times \mathbb{R} \times \mathbb{R} \longrightarrow L^2(\mathbb{R}^+,\mathbb{C}) \times \mathbb{R} \\ & (v,\epsilon,\omega) \longmapsto \begin{pmatrix} F(\phi_0(\omega)+v,\omega) - \mathrm{i}\epsilon G(\phi_0(\omega)+v) \\ \mathrm{Im}\, v(0) \end{pmatrix}\end{aligned}$$

where F has been defined in (4.2). Here, $H_0^2(\mathbb{R}^+)$ denotes the space of H^2-functions on $\mathbb{R}^+$ with $u_x(0) = 0$. We split the function v into real and imaginary part so that $v = v_1 + \mathrm{i}v_2$ and apply the result from Section 4.1 with $X = H_0^2(\mathbb{R}^+,\mathbb{R}^2) \times \mathbb{R}$ and $Y = L^2(\mathbb{R}^+,\mathbb{R}^2) \times \mathbb{R}$, where ω will play the role of the ϵ of Section 4.1.

First, we see that $\mathcal{F}(0,0,\omega) = 0$ for all $\omega > 0$. Second, we have that

$$\mathcal{F}_{(v,\epsilon)}(0,0,\omega) = \begin{pmatrix} L_{\mathrm{R}}(\omega) & 0 & 0 \\ 0 & L_{\mathrm{I}}(\omega) & -G(\phi_0(\omega)) \\ 0 & \delta_0 & 0 \end{pmatrix}$$

where δ_0 is the delta function and

$$L_{\mathrm{R}}(\omega) = \partial_{xx} - \omega + 12\phi_0^2(\omega), \qquad L_{\mathrm{I}}(\omega) = \partial_{xx} - \omega + 4\phi_0^2(\omega).$$

It follows from [28] that $L_{\mathrm{R}}(\omega)$ is invertible on $H_0^2(\mathbb{R}^+, \mathbb{R})$, while $L_{\mathrm{I}}(\omega)$ has a one-dimensional null space spanned by $\phi_0(\omega)$. Therefore, since $L_{\mathrm{I}}(\omega)$ is self-adjoint, $G(\phi_0(\omega)) \in \mathrm{R}(L_{\mathrm{I}}(\omega))$ if, and only if, $g(\omega) = 0$. In particular, $\mathcal{F}_{(v,\epsilon)}(0,0,\omega)$ is invertible if $g(\omega) \neq 0$ which proves the second part of the proposition. Hence, we assume from now on that $g(\omega) = 0$ so that $G(\phi_0(\omega)) = L_{\mathrm{I}}(\omega)v_2$ for some $v_2 \in H_0^2(\mathbb{R}^+, \mathbb{R})$. We can assume that $v_2(0) = 0$ upon adding a multiple of the kernel $\phi_0(\omega)(x)$. Thus, the null space of $\mathcal{F}_{(v,\epsilon)}(0,0,\omega)$ is spanned by $U_* = (0, v_2, 1)$, while $U_*^{\mathrm{ad}} = (0, \phi_0(\omega), 0)$ spans the complement of the range of $\mathcal{F}_{(v,\epsilon)}(0,0,\omega)$.

It remains to compute the coefficient c_2 from Section 4.1:

$$\begin{aligned} c_2 &= \langle U_*^{\mathrm{ad}}, \mathrm{D}_\omega \mathcal{F}_{(v,\epsilon)}(0,0,\omega)[U_*]\rangle_Y \;=\; \langle \phi_0(\omega), [\partial_\omega L_{\mathrm{I}}(\omega)]v_2 - \partial_\omega[G(\phi_0(\omega))]\rangle \\ &= \langle [\partial_\omega L_{\mathrm{I}}(\omega)]\phi_0(\omega), v_2\rangle - \langle \phi_0(\omega), \partial_\omega[G(\phi_0(\omega))]\rangle \;=\; -\langle L_{\mathrm{I}}\partial_\omega\phi_0(\omega), v_2\rangle - \langle \phi_0(\omega), \partial_\omega[G(\phi_0(\omega))]\rangle \\ &= -\langle \partial_\omega\phi_0(\omega), L_{\mathrm{I}}v_2\rangle - \langle \phi_0(\omega), \partial_\omega[G(\phi_0(\omega))]\rangle \;=\; -\frac{\mathrm{d}}{\mathrm{d}\omega}\langle \phi_0(\omega), G(\phi_0(\omega))\rangle, \end{aligned}$$

where we used that

$$[\partial_\omega L_{\mathrm{I}}(\omega)]\phi_0(\omega) = -L_{\mathrm{I}}(\omega)\partial_\omega\phi_0(\omega)$$

since $L_{\mathrm{I}}(\omega)\phi_0(\omega) \equiv 0$. The scalar products in the above calculation of c_2 are taken in $L^2(\mathbb{R}^+)$ excepted when indicated otherwise. This completes the proof of Proposition 4.1.

4.3 Application II: Splitting of discrete eigenvalues

As another application, we consider the unfolding of a Jordan block with a persisting null space. This situation occurs quite naturally when studying non-conservative perturbations of conservative systems. Assume that X and Y are two Hilbert spaces with $X \hookrightarrow Y$. Let $\mathcal{A} : \mathbb{R} \to \mathrm{L}(X, Y)$ be a differentiable map and suppose that $\mathcal{A}(0)$ is Fredholm with index zero. We consider the eigenvalue problem

$$\mathcal{A}(\epsilon)u = \lambda u. \tag{4.4}$$

We assume that $\mathrm{N}(\mathcal{A}(\epsilon))$ is one-dimensional and spanned by $v(\epsilon)$ with $|v(\epsilon)| = 1$ for all ϵ. At $\epsilon = 0$, we assume that $\lambda = 0$ has algebraic multiplicity two, and we denote the generalized eigenvector by w_0 so that $\mathcal{A}(0)w_0 = v(0)$. There exists then a vector w_0^{ad} with $\langle w_0^{\mathrm{ad}}, w_0\rangle_Y \neq 0$ such that $\langle w_0^{\mathrm{ad}}, \mathcal{A}(0)u\rangle_Y = 0$ for all u.

An application of the results from Section 4.1 to the mapping

$$\mathcal{F}(w, \lambda, \epsilon) = \begin{pmatrix} [\mathcal{A}(\epsilon) - \lambda]w - \lambda v(\epsilon) \\ |v(\epsilon) + w|_Y^2 - 1 \end{pmatrix}$$

gives that solutions to (4.4) near $(\lambda, \epsilon) = 0$ are in one-to-one correspondence to solutions of

$$\lambda[c_1\lambda + c_2\epsilon + \mathrm{O}(\lambda^2 + \epsilon^2)] = 0$$

where

$$c_1 = -\langle w_0^{\mathrm{ad}}, w_0 \rangle_Y, \qquad c_2 = \langle w_0^{\mathrm{ad}}, \mathcal{A}_\epsilon(0) w_0 - v_\epsilon(0) \rangle_Y.$$

In particular, the bifurcating eigenvalue is given by

$$\lambda = \frac{\langle w_0^{\mathrm{ad}}, \mathcal{A}_\epsilon(0) w_0 - v_\epsilon(0) \rangle_Y}{\langle w_0^{\mathrm{ad}}, w_0 \rangle_Y} \, \epsilon + \mathrm{O}(\epsilon^2). \tag{4.5}$$

5 The master mode-locking equation

In this section, we consider the model that motivated the results given in the previous sections. The master mode-locking equation is given by

$$\mathrm{i} u_t + u_{xx} - \omega u + 4|u|^2 u = \mathrm{i}\epsilon \left[\frac{\Gamma_{\mathrm{gain}}}{1 + \|u\|^2/e_0} (\tau u_{xx} + u) - \Gamma_{\mathrm{loss}} u + \beta |u|^2 u \right] \tag{5.1}$$

where $u(x,t) \in \mathbb{C}$, and where we used the notation

$$\|u\|^2 = \int_{-\infty}^{\infty} |u(x)|^2 \, \mathrm{d}x, \qquad \langle u, v \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} u(x) \overline{v(x)} \, \mathrm{d}x.$$

As mentioned in the introduction, equation (5.1) serves as a model for cavity lasers with bandwidth-limited saturated gain and intensity-dependent loss or gain provided by saturable absorbers in the cavity. More specifically, the left-hand side of (5.1) is the focusing nonlinear Schrödinger equation which models the propagation of pulses in ideal nonlinear optical fibers. The nonlocal term on the right-hand side accounts for bandwidth-limited gain with bandwidth $1/\sqrt{\tau}$ whose saturation energy is equal to e_0. The term $-\mathrm{i}\epsilon\Gamma_{\mathrm{loss}} u$ accounts for loss in the fiber, while the cubic term on the right-hand side models a saturable absorber that introduces intensity-dependent loss (for $\beta < 0$) or gain (for $\beta > 0$). We refer to [9, 16, 17] for references and more background information regarding (1.1).

Consistent with the underlying physical motivation, we assume that the constants $\Gamma_{\mathrm{gain}}, \tau, \epsilon$ are non-negative so that (5.1) is well posed on $H^1(\mathbb{R})$. We refer to [17] for existence and uniqueness results for the time evolution of (5.1).

For $\epsilon = 0$, equation (5.1) coincides with the nonlinear Schrödinger equation

$$\mathrm{i} u_t + u_{xx} - \omega u + 4|u|^2 u = 0 \tag{5.2}$$

which admits the solitons

$$\phi_0(x;\omega) = \sqrt{\frac{\omega}{2}} \, \mathrm{sech}(\sqrt{\omega} x) \tag{5.3}$$

for $\omega > 0$. We are interested in studying which of the solitons persist for $\epsilon > 0$ and what their stability properties are.

5.1 Existence of solitary waves

It is straightforward to check that we are in the setup of Section 4.2 so that Proposition 4.1 applies. Thus, we need to compute

$$g(\omega) = \langle \phi_0(\cdot;\omega), G(\phi_0(\cdot;\omega)) \rangle_{L^2(\mathbb{R})}$$

where

$$G(u) = \frac{\Gamma_{\text{gain}}}{1 + \|u\|^2/e_0}(\tau u_{xx} + u) - \Gamma_{\text{loss}} u + \beta |u|^2 u.$$

A computation gives

$$g(\omega) = \frac{\sqrt{\omega}}{3(\sqrt{\omega} + e_0)} \left[\beta \omega^{\frac{3}{2}} + e_0(\beta - \tau \Gamma_{\text{gain}})\omega - 3\Gamma_{\text{loss}}\sqrt{\omega} + 3e_0(\Gamma_{\text{gain}} - \Gamma_{\text{loss}}) \right].$$

Since we are interested only in roots of g for $\omega > 0$ for which the first factor is always positive, it suffices to find roots of

$$\check{g}(\omega) = \beta \omega^{\frac{3}{2}} + e_0(\beta - \tau \Gamma_{\text{gain}})\omega - 3\Gamma_{\text{loss}}\sqrt{\omega} + 3e_0(\Gamma_{\text{gain}} - \Gamma_{\text{loss}}),$$

which is a cubic function in the variable $\sqrt{\omega}$. We fix Γ_{gain}, Γ_{loss}, τ and e_0. If $\Gamma_{\text{loss}} > \Gamma_{\text{gain}}$, then $g(\omega)$ has a unique positive root ω for each positive $\beta > 0$; see Figure 3(b). If, on the other other hand, $\Gamma_{\text{gain}} > \Gamma_{\text{loss}}$, then there is a number $\beta^* > 0$ that depends upon the aforementioned fixed parameters so that the following is true; see Figure 3(a): For $\beta \leq 0$, $g(\omega)$ has a unique positive root, while there are two positive roots for $0 < \beta < \beta^*$. Lastly, for $\beta > \beta^*$, there are no positive roots.

For $\epsilon > 0$, the soliton is given by

$$\phi(x) = \sqrt{\frac{\omega}{2}} \operatorname{sech}(\sqrt{\omega}x) \left[1 + \mathrm{i}\epsilon A_1 \ln(\operatorname{sech}(\sqrt{\omega}x)) \right] + \mathrm{O}(\epsilon^2) =: \phi_0(x) + \epsilon \phi_1(x) + \mathrm{O}(\epsilon^2) \tag{5.4}$$

where

$$A_1 = \frac{1}{2\omega(e_0 + \sqrt{\omega})} \left[\tau \Gamma_{\text{gain}} e_0 \omega - \Gamma_{\text{loss}}\sqrt{\omega} + e_0(\Gamma_{\text{gain}} - \Gamma_{\text{loss}}) \right].$$

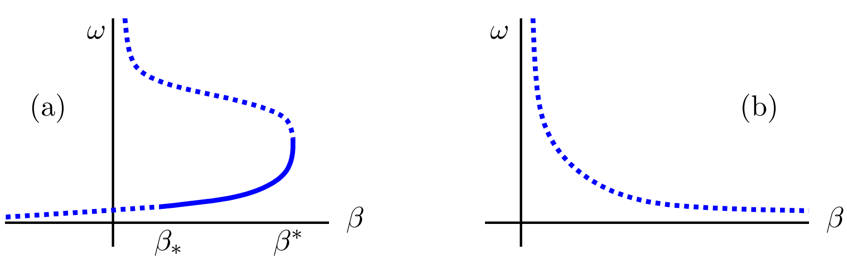

Figure 3: *The existence and stability properties of pulses of (5.1) are illustrated in (a) for $\Gamma_{\text{gain}} > \Gamma_{\text{loss}}$ and in (b) for $\Gamma_{\text{loss}} > \Gamma_{\text{gain}}$. In (a), stable pulses exist on the lower branch for $\beta \in (\beta_*, \beta^*)$ where the essential spectrum destabilizes for $\beta = \beta_*$ given by (5.7). In (b), the background state $u = 0$ is the only stable solution.*

5.2 Stability of the persisting solitary waves

Linearizing (5.1) about a solitary wave ϕ yields the eigenvalue problem

$$\begin{aligned}&\mathrm{i}\lambda v + v_{xx} + 4|\phi|^2(2v-\bar{v}) - \omega v \qquad (5.5)\\ &= \mathrm{i}\epsilon\left[\frac{\Gamma_{\text{gain}}}{1+\|\phi\|^2/e_0}(\tau v_{xx}+v) - \frac{2\Gamma_{\text{gain}}e_0}{(e_0+\|\phi\|^2)^2}(\tau\phi_{xx}+\phi)\,\mathrm{Re}\langle\phi,v\rangle_{L^2(\mathbb{R})} - \Gamma_{\text{loss}}v + \beta|\phi|^2(2v-\bar{v})\right].\end{aligned}$$

It follows from (5.5) that the real part of the rightmost point of the essential spectrum is given by

$$\mathrm{Re}\,\lambda = \epsilon\left(\frac{\Gamma_{\text{gain}}}{1+\|\phi\|^2/e_0} - \Gamma_{\text{loss}}\right).$$

Note that the background state $\phi = 0$ destabilizes prior to any solitary wave ϕ with $\|\phi\| > 0$. To leading order, the solitary waves found in the last section satisfy $\|\phi\|^2 = \sqrt{\omega}$ and their essential spectrum is therefore in the left half-plane if

$$\Gamma_{\text{loss}} > \frac{\Gamma_{\text{gain}}}{1+\sqrt{\omega}/e_0}. \qquad (5.6)$$

In fact, it is not hard to see that the pulse (5.4) exists (so that $g(\omega) = 0$) and has marginal essential spectrum that touches the imaginary axis (so that we have equality in (5.6)) precisely when

$$\beta = \beta_* = \tau\Gamma_{\text{loss}} > 0, \qquad \sqrt{\omega} = \sqrt{\omega_*} = \frac{e_0(\Gamma_{\text{gain}} - \Gamma_{\text{loss}})}{\Gamma_{\text{loss}}}. \qquad (5.7)$$

Next, we discuss the point spectrum of the operator (5.5). The linearized operator (5.5) maps the space of even H^1-functions into itself and the space of odd H^1-functions into itself because the persisting waves are even. In particular, we can compute the point spectrum by restricting the linearized operator to each one of these spaces and compute the spectrum of the restriction. On each of these two spaces, the nonlinear Schrödinger equation (5.2) linearized about the solitons (5.3) has an eigenvalue at $\lambda = 0$ with geometric multiplicity one and algebraic multiplicity two [28]. There is no other point spectrum besides the eigenvalues at zero. Since (5.1) is invariant under translations and gauge rotations, $\lambda = 0$ will remain to be an eigenvalue of the linearization about the persisting solitary waves on either space. We need to compute the second eigenvalue, on each of the two spaces, which may move away upon making ϵ non-zero. To accomplish this, we simply apply the results in Section 4.3. The eigenfunctions $v(\epsilon)$ are given by ϕ_x and $\mathrm{i}\phi$ with ϕ as in (5.4), while the adjoint eigenfunctions at $\epsilon = 0$ are given by $\partial_\omega\phi_0(x)$ and $x\phi_0(x)$ with ϕ_0 as in (5.3). Since the actual computations are straightforward but tedious, we omit them and simply present the results. The two bifurcating eigenvalues are given by

$$\begin{aligned}\lambda_{\text{even}} &= \epsilon\frac{4e_0}{(e_0+\sqrt{\omega})^2}\left[\frac{1}{6}\tau\Gamma_{\text{gain}}\omega^{\frac{3}{2}} + \frac{\Gamma_{\text{loss}}}{e_0}\omega + \left(2\Gamma_{\text{loss}} - \frac{3\Gamma_{\text{gain}}}{2}\right)\sqrt{\omega} + e_0(\Gamma_{\text{loss}} - \Gamma_{\text{gain}})\right]\\ \lambda_{\text{odd}} &= -\epsilon\frac{4\tau\Gamma_{\text{gain}}e_0\omega}{3(e_0+\sqrt{\omega})} < 0.\end{aligned}$$

We see that λ_{even} is positive when $\Gamma_{\text{loss}} > \Gamma_{\text{gain}}$. Thus, a necessary criterion for stability is

$$\Gamma_{\text{gain}} > \Gamma_{\text{loss}} > \frac{\Gamma_{\text{gain}}}{1+\sqrt{\omega}/e_0}$$

where we also take (5.6) into account.

Lastly, we have to discuss eigenvalues that may move off the essential spectrum upon making ϵ non-zero. Such bifurcations have been studied in [11, 12] for (local) perturbations of nonlinear Schrödinger equations. The Evans function of (5.2) has a root of order four at $\lambda = 0$ and a root at $\lambda = \pm\mathrm{i}\omega$, that is, at the edge of the essential spectrum. Thus, we need to compute the roots near $\lambda = \pm\mathrm{i}\omega$ for $\epsilon > 0$ close to zero. The Evans function $E_{\mathrm{nls}}(\gamma)$ of the nonlinear Schrödinger equation (5.2) near the branch point $\lambda = \mathrm{i}\omega$ is given by

$$E_{\mathrm{nls}}(\gamma) = 4\sqrt{2\omega}\gamma + \mathrm{O}(\gamma^2)$$

where

$$\lambda = \mathrm{i}(\omega - \gamma^2);$$

see Figure 4 and [11]. The branch point of (5.5) near $\lambda = \mathrm{i}\omega$ is given by

$$\lambda_{\mathrm{bp}}(\epsilon) = \mathrm{i}\omega + \epsilon \left[\frac{\Gamma_{\mathrm{gain}}}{1 + \sqrt{\omega}/e_0} - \Gamma_{\mathrm{loss}} \right]$$

Using this expression, equation (3.9), and the expressions for φ^{c} and ψ given in [11, Section 4], we obtain that

$$\partial_\epsilon E_{\mathrm{bp}}(0, \epsilon)\Big|_{\epsilon=0} = \mathrm{i}\frac{4\sqrt{2}\beta\omega}{3}$$

so that the Taylor expansion of the Evans function $E_{\mathrm{bp}}(\gamma, \epsilon)$ is given by

$$E_{\mathrm{bp}}(\gamma, \epsilon) = 4\sqrt{2\omega}\gamma + \mathrm{i}\epsilon\frac{4\sqrt{2}\beta\omega}{3} + \mathrm{O}(\gamma^2 + \epsilon^2).$$

For $\epsilon > 0$, the root of the Evans function $E_{\mathrm{bp}}(\gamma, \epsilon)$ near $\gamma = 0$ is therefore given by

$$\gamma = -\mathrm{i}\epsilon\frac{\beta\sqrt{\omega}}{3} + \mathrm{O}(\epsilon^2) \tag{5.8}$$

which corresponds to

$$\lambda = \lambda_{\mathrm{bp}}(\epsilon) + \mathrm{i}\epsilon^2\,\frac{\beta^2\omega}{9} + \mathrm{O}(\epsilon^3).$$

This root λ of the Evans function is an eigenvalue if, and only if, the real part of the associated value γ as given in (5.8) satisfies $\mathrm{Re}\,\gamma > 0$; see Figure 4 and Theorem 3. We have $\mathrm{Re}\,\gamma = \mathrm{O}(\epsilon^2)$ which is inconclusive. Instead of computing the higher-order correction, we argue that, even if $\mathrm{Re}\,\gamma > 0$, it can only contribute an eigenvalue that destabilizes prior to the essential spectrum if $\beta < 0$. Indeed, if $\beta > 0$, then $\mathrm{Im}\,\gamma < 0$ for $\epsilon > 0$ since $\omega > 0$. If $\mathrm{Re}\,\gamma > 0$, so that the associated root λ corresponds to an eigenvalue, then this eigenvalue lies to the left of the essential spectrum since $\mathrm{Re}\,\gamma = 0$ corresponds to the essential spectrum Σ_{ess}. Thus, such an eigenvalue cannot move into the right half-plane until after part of the essential spectrum is already destabilized.

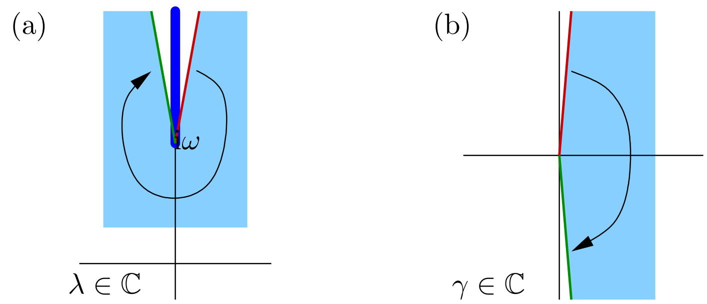

Figure 4: *The complex λ and γ planes are shown in (a) and (b), respectively, where λ and γ are related via $\lambda = \mathrm{i}(\omega - \gamma^2)$. In particular, $\mathrm{Re}\,\gamma = 0$ maps onto the essential spectrum Σ_{ess} shown as the thick solid line in (a). Roots of the Evans functions correspond to eigenvalues only if $\mathrm{Re}\,\gamma > 0$; otherwise, they correspond to resonance pole with "eigenfunctions" that grow exponentially.*

5.3 Summary

We obtained the following conditions:

$$\begin{aligned} \text{Existence:} \quad & -\beta\omega^{\frac{3}{2}} + e_0(\tau\Gamma_{\mathrm{gain}} - \beta)\omega + 3\Gamma_{\mathrm{loss}}\sqrt{\omega} + 3e_0(\Gamma_{\mathrm{loss}} - \Gamma_{\mathrm{gain}}) = 0 \\ \text{Stable essential spectrum:} \quad & \Gamma_{\mathrm{loss}} > \frac{\Gamma_{\mathrm{gain}}}{1 + \sqrt{\omega}/e_0} \\ \text{Stable point spectrum:} \quad & \frac{1}{6}\tau\Gamma_{\mathrm{gain}}\omega^{\frac{3}{2}} + \frac{\Gamma_{\mathrm{loss}}}{e_0}\omega + \left[2\Gamma_{\mathrm{loss}} - \frac{3\Gamma_{\mathrm{gain}}}{2}\right]\sqrt{\omega} + e_0(\Gamma_{\mathrm{loss}} - \Gamma_{\mathrm{gain}}) < 0 \end{aligned} \tag{5.9}$$

A necessary criterion for stability of the point spectrum near $\lambda = 0$ is $\Gamma_{\mathrm{gain}} > \Gamma_{\mathrm{loss}}$. If this inequality is satisfied, then (5.7) shows that the pulses destabilize in an essential instability at $\beta = \beta_* > 0$. In this case, the results of the previous section imply that the edge eigenvalue, if it exists, is still in the left half-plane. This indicates therefore that stable time-periodic waves, if they bifurcate at all, are indeed created by an essential instability and not by a Hopf bifurcation. We refer to [23, 24, 27] for bifurcation results near essential instabilities. Note, however, that the theory developed there does not apply to (5.1) due to the nonlocal terms present.

5.4 Numerical simulations

Lastly, we present numerical simulations of the governing equation (5.1) that indicate that essential instabilities occur and are supercritical. The numerical procedure employed uses a fourth-order Runge-Kutta method in t and a filtered pseudo-spectral method in x applied to (5.1) on a bounded interval with periodic boundary conditions. Note that the results obtained in [25] show that the essential instability on $\mathbb{R}$ persists as an instability of many point eigenvalues on sufficiently large bounded intervals with periodic boundary conditions. We take the parameters in (5.1) to be

$$e_0 = 1.0, \quad \tau = 0.1, \quad \Gamma_{\mathrm{gain}} = 0.2, \quad \Gamma_{\mathrm{loss}} = 0.1\,.$$

The remaining free parameter β is the bifurcation parameter which determines the stability of the pulse solutions. Evaluating (5.9) numerically, we find that stable pulses exist for $\beta \in (0.01, 0.0348)$:

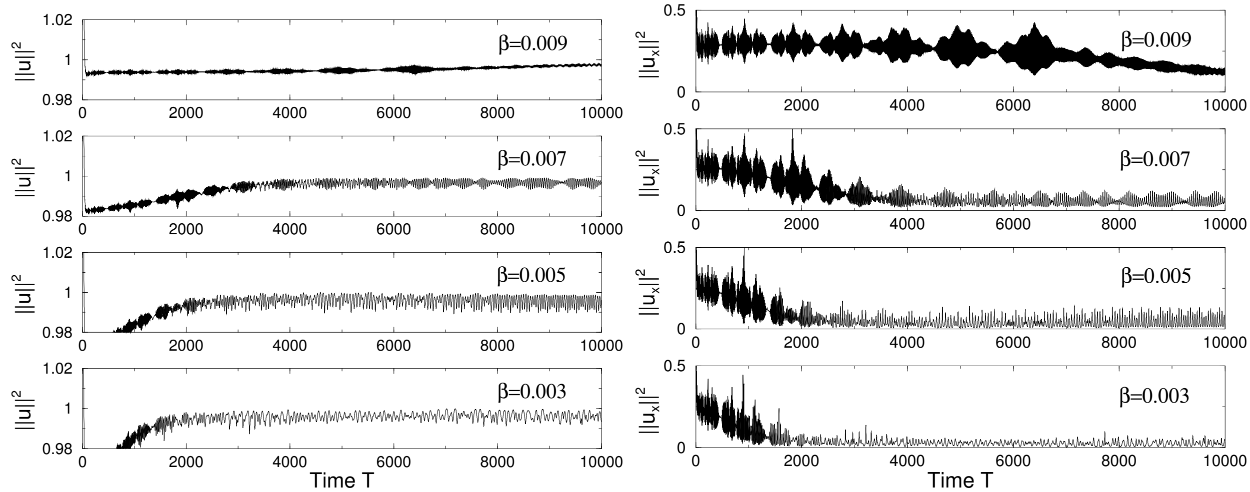

Figure 5: *The* L^2*-norms of* u *and* u_x *are shown for* $\beta = 0.003, 0.005, 0.007, 0.009$.

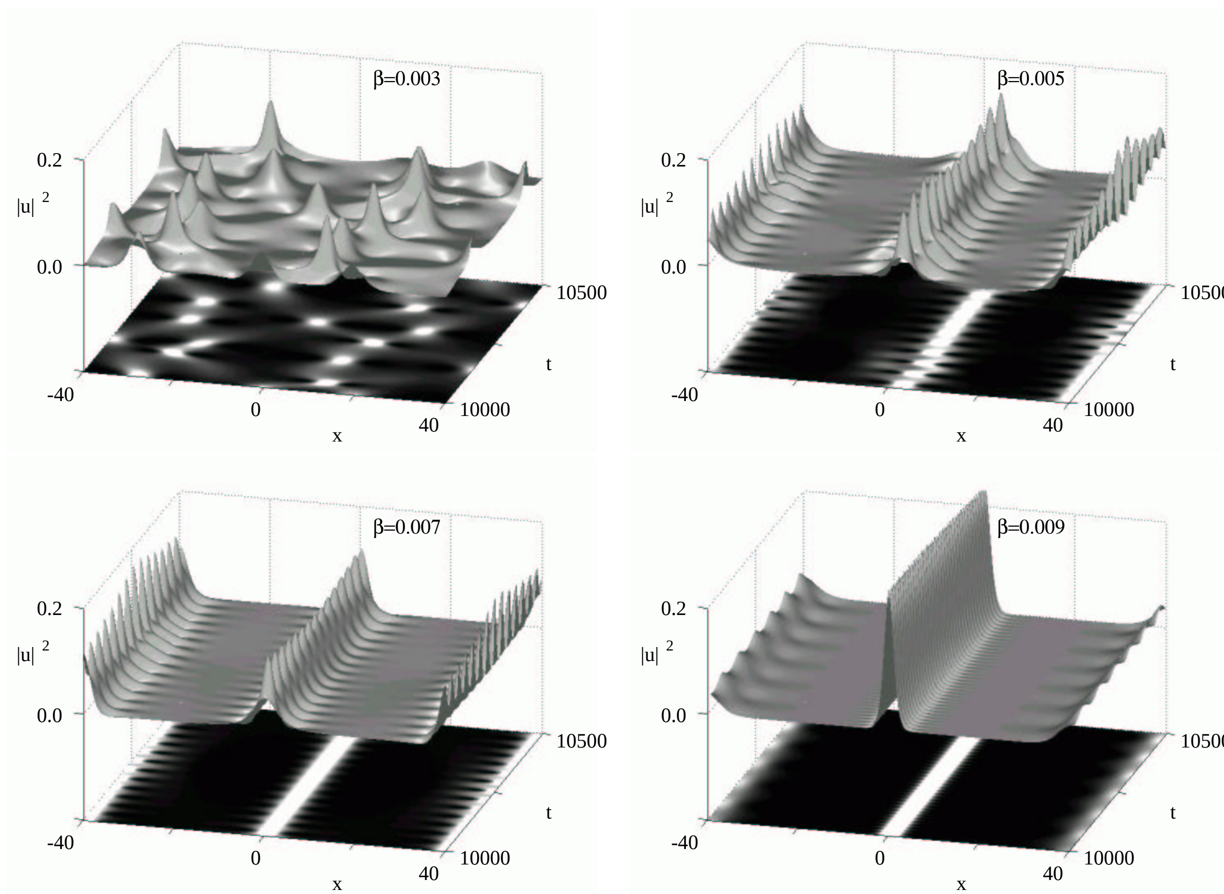

Figure 6: *Space-time plots of* $|u(x,t)|^2$ *are shown for* $\beta = 0.003, 0.005, 0.007, 0.009$.

the parameter value $\beta^* = 0.0348$ corresponds to a saddle-node bifurcation where λ_{even} crosses through zero, while (5.7) shows that the essential instability sets in at $\beta_* = 0.01$ with $\omega = 1$. Direct simulations indicate that quasi-periodic solutions with at least two frequencies appear for $\beta < \beta_*$.

We refer to Figure 5 for plots of the L^2-norms of the solution u and its spatial derivative u_x and to Figure 6 for space-time plots of $|u(x,t)|^2$ for different values of β.

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