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A new test for chaos in deterministic systems

By Georg A. Gottwald[1] and Ian Melbourne[2]

[1] *School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia* (gottwald@maths.usyd.edu.au)

[2] *Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey GU2 7XH, UK* (i.melbourne@surrey.ac.uk)

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We describe a new test for determining whether a given deterministic dynamical system is chaotic or non-chaotic. In contrast to the usual method of computing the maximal Lyapunov exponent, our method is applied directly to the time-series data and does not require phase-space reconstruction. Moreover, the dimension of the dynamical system and the form of the underlying equations are irrelevant. The input is the time-series data and the output is 0 or 1, depending on whether the dynamics is non-chaotic or chaotic. The test is universally applicable to any deterministic dynamical system, in particular to ordinary and partial differential equations, and to maps.

Our diagnostic is the real valued function

$$p(t) = \int_0^t \phi(\boldsymbol{x}(s)) \cos(\theta(s))\, \mathrm{d}s,$$

where ϕ is an observable on the underlying dynamics $\boldsymbol{x}(t)$ and

$$\theta(t) = ct + \int_0^t \phi(\boldsymbol{x}(s))\, \mathrm{d}s.$$

The constant $c > 0$ is fixed arbitrarily. We define the mean-square displacement $\boldsymbol{M}(t)$ for $p(t)$ and set $K = \lim_{t\to\infty} \log \boldsymbol{M}(t)/\log t$. Using recent developments in ergodic theory, we argue that, typically, $K = 0$, signifying non-chaotic dynamics, or $K = 1$, signifying chaotic dynamics.

Keywords: chaos; deterministic dynamical systems; Lyapunov exponents; mean-square displacement; Euclidean extension

1. Introduction

The usual test of whether a deterministic dynamical system is chaotic or non-chaotic is the calculation of the largest Lyapunov exponent λ. A positive largest Lyapunov exponent indicates chaos: if $\lambda > 0$, then nearby trajectories separate exponentially and if $\lambda < 0$, then nearby trajectories stay close to each other. This approach has been widely used for dynamical systems, whose equations are known (Abarbanel *et al*. 1993; Eckmann *et al*. 1986; Parker & Chua 1989). If the equations are not known or one wishes to examine experimental data, this approach is not directly applicable.

However, Lyapunov exponents may be estimated (Wolf *et al*. 1985; Sano & Sawada 1985; Eckmann *et al*. 1986; Abarbanel *et al*. 1993) by using the embedding theory of Takens (1981) or by approximating the linearization of the evolution operator. Nevertheless, the computation of Lyapunov exponents is greatly facilitated if the underlying equations are known and are low-dimensional.

In this article, we propose a new 0–1 test for chaos, which does not rely on knowing the underlying equations, and for which the dimension of the equations is irrelevant. The input is the time-series data and the output is either a '0' or a '1', depending on whether the dynamics is non-chaotic or chaotic. Since our method is applied directly to the time-series data, the only difference in difficulty between analysing a system of partial differential equations (PDEs) or a low-dimensional system of ordinary differential equations is the effort required to generate sufficient data. (As with all approaches, our method is impracticable if there are extremely long transients or once the dimension of the attractor becomes too large.) With experimental data, there is the additional effect of noise to be taken into consideration. We briefly discuss this important issue at the end of this paper. However, our aim in this paper is to present our findings for the situation of noise-free deterministic data.

2. Description of the 0–1 test for chaos

To describe the new test for chaos, we concentrate on the continuous-time case and denote a solution of the underlying system by $\boldsymbol{x}(t)$. The discrete-time case is handled analogously with obvious modifications. Consider an observable $\phi(\boldsymbol{x})$ of the underlying dynamics. The method is essentially independent of the actual form of ϕ—almost any choice of ϕ will suffice. For example, if $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, then $\phi(\boldsymbol{x}) = x_1$ is a possible and simple choice. Choose $c > 0$ arbitrarily and define

$$\left.\begin{aligned} \theta(t) &= ct + \int_0^t \phi(\boldsymbol{x}(s))\,\mathrm{d}s, \\ p(t) &= \int_0^t \phi(\boldsymbol{x}(s))\cos(\theta(s))\,\mathrm{d}s. \end{aligned}\right\} \tag{2.1}$$

(Throughout the examples in §§ 3 and 4 we fix $c = 1.7$ once and for all.) We claim that

(i) $p(t)$ is bounded if the underlying dynamics is non-chaotic; and

(ii) $p(t)$ behaves asymptotically like a Brownian motion if the underlying dynamics is chaotic.

The definition of p in (2.1), which involves only the observable $\phi(\boldsymbol{x})$, highlights the universality of the test. The origin and nature of the data which are fed into the system (2.1) are irrelevant for the test, and so is the dimension of the underlying dynamics.

Later on, we briefly explain the justification behind claims (i) and (ii). For the moment, we suppose that the claims are correct and show how to proceed.

To characterize the growth of the function $p(t)$, defined in (2.1), it is natural to look at the mean-square displacement (MSD) of $p(t)$, defined to be

$$\boldsymbol{M}(t) = \lim_{T\to\infty} \frac{1}{T}\int_0^T (p(t+\tau) - p(\tau))^2\,\mathrm{d}\tau. \tag{2.2}$$

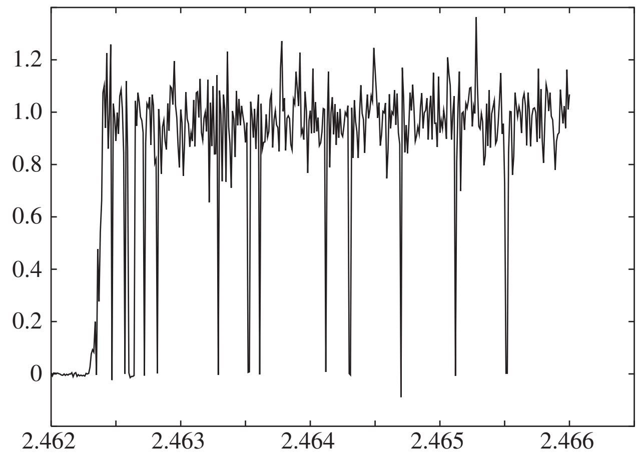

Figure 1. Asymptotic growth rate K of the mean-square displacement (2.2) versus ω for the van der Pol system (3.1) determined by a numerical simulation of the skew-product system (3.1) and (2.1) with $a = d = 5$, $c = 1.7$, $\phi(\boldsymbol{x}) = x_1 + x_2$ and ω varying from 2.462 to 2.466. The integration interval is $T = 2\,000\,000$ after an initial transient of 200 000 units of time.

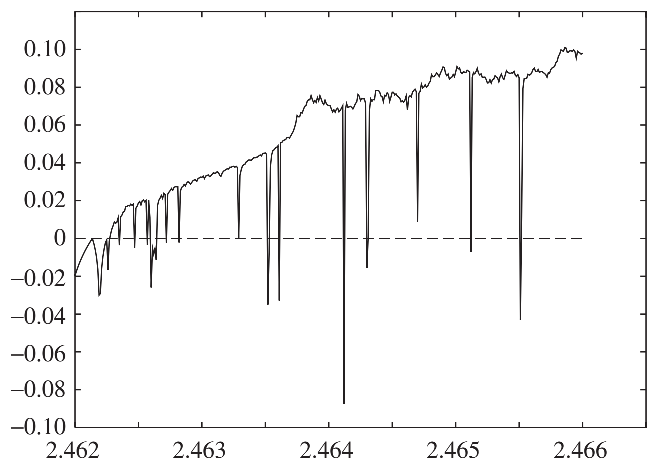

Figure 2. Largest Lyapunov exponent λ versus ω for the van der Pol system (3.1) with $a = d = 5$ and ω varying from 2.462 to 2.466 (cf. Parlitz & Lauterborn 1987). The integration interval is $T = 2\,000\,000$ after an initial transient of 200 000 units of time.

If the behaviour of $p(t)$ is Brownian, i.e. the underlying dynamics is chaotic, then $M(t)$ grows linearly in time; if the behaviour is bounded, i.e. the underlying dynamics is non-chaotic, then $M(t)$ is bounded. We define $K = \lim_{t\to\infty} \log \boldsymbol{M}(t)/\log t$ as the asymptotic growth rate of the MSD. The growth rate K can be numerically determined by means of linear regression of $\log \boldsymbol{M}(t)$ versus $\log t$ (Press *et al.* 1992). (To avoid negative logarithms we actually calculate $\lim_{t\to\infty} \log(\boldsymbol{M}(t) + 1)/\log t$, which obviously does not change the slope K.) This allows for a clear distinction of a non-chaotic and a chaotic system, as K may only take values $K = 0$ or $K = 1$. We have lost the possibility of quantifying the chaos by the magnitude of the largest Lyapunov exponent λ though.

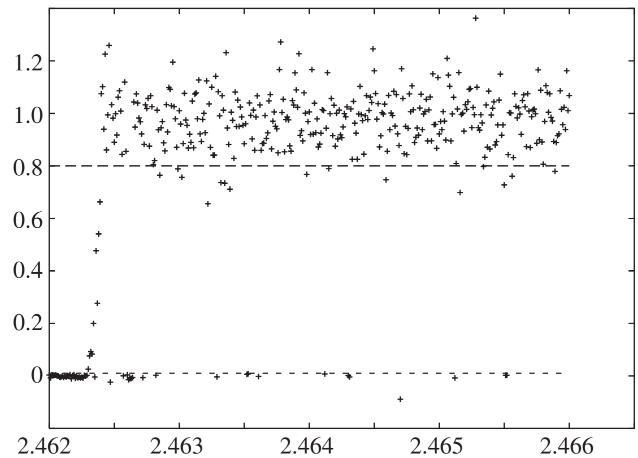

Figure 3. Asymptotic growth rate K versus ω for the van der Pol system (3.1), as in figure 1 with $T = 2\,000\,000$. The horizontal lines represent $K = 0.01$ and $K = 0.8$.

Numerically, one has to make sure that initial transients have died out so that the trajectories are on (or close to) the attractor at time zero, and that the integration time T is long enough to ensure $T \gg t$.

3. An example: the forced van der Pol oscillator

We illustrate the 0–1 test for chaos with the help of a concrete example, the forced van der Pol system,

$$\left.\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -d(x_1^2 - 1)x_2 - x_1 + a\cos\omega t, \end{aligned}\right\} \tag{3.1}$$

which has been widely studied in nonlinear dynamics (van der Pol 1927; Guckenheimer & Holmes 1990). For fixed a and d, the dynamics may be chaotic or non-chaotic depending on the parameter ω. Following Parlitz & Lauterborn (1987), we take $a = d = 5$ and let ω vary from 2.457 to 2.466 in increments of 0.000 01. Choose $\phi(\boldsymbol{x}) = x_1 + x_2$ and $c = 1.7$. We stress that the results are independent of these choices for all practical purposes. As described in § 5, almost all choices will work. (Deliberately poor choices, such as $c = 0$, or $\phi = 7$ or $\phi = t$ obviously fail; sensible choices that fail are virtually impossible to find.)

In figure 1 we show a plot of K versus ω. The periodic windows are clearly seen. As a comparison we show in figure 2 the largest Lyapunov exponent λ versus ω. Since the onset of chaos does not occur until after $\omega = 2.462$, we display the results only for the range $2.462 < \omega < 2.466$ in figures 1 and 2. (Both methods easily indicate regular dynamics for $2.457 < \omega < 2.462$.)

Naturally, we do not obtain the values $K = 0$ and $K = 1$ exactly; the mathematical results that underpin our method predict these values in the limit of infinite integration time. (The same caveat applies equally to the Lyapunov exponent method.) In producing the data for figures 1 and 2, we allowed for a transient of 200 000 units of time and then integrated up to time $T = 2\,000\,000$. In figure 1, data points are connected by straight lines to accentuate the periodic windows. In figure 3, the

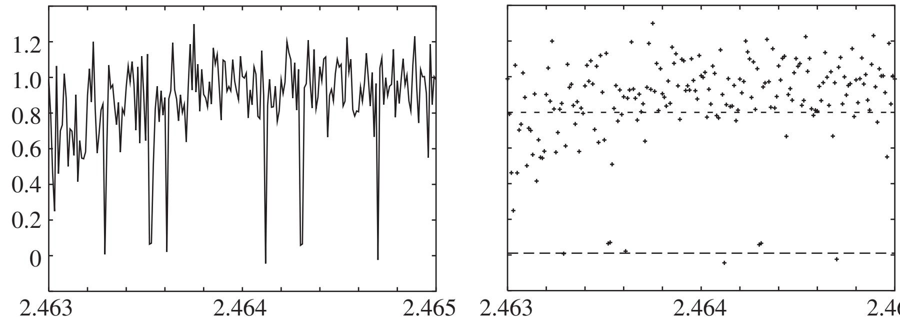

Figure 4. Asymptotic growth rate K versus ω varying from 2.463 to 2.465 for the van der Pol system as in figures 1 and 3 but with integration interval $T = 50\,000$ after an initial transient of 200 000 units of time. The horizontal lines represent $K = 0.01$ and $K = 0.8$.

results are reproduced but without the connecting lines, making it easier to discern the numerically computed values of K. As can be seen in figure 3, for most of the 400 data points in the range of ω, we obtain $K > 0.8$ or $K < 0.01$.

Next, we carry out the 0–1 test for the forced van der Pol system, when there is a more limited quantity of data. The results are shown in figure 4 for $2.463 < \omega < 2.465$. We again allow for a transient time 200 000 but then integrate only for $T = 50\,000$. The transitions between chaotic dynamics and periodic windows are almost as clear with $T = 50\,000$ as they are with $T = 2\,000\,000$ even though the convergence of K to 0 or 1 is better with $T = 2\,000\,000$.

Although it is easy to artificially round off the answer to one of the theoretically predicted possible values of 0 or 1, we have chosen not to do so. Moreover, the closeness of the numerically computed value of K to 0 or 1 provides a natural measure of the reliability of the test.

4. Further examples

To test the method on high-dimensional systems we investigated the driven and damped Kortweg–de Vries (KdV) equation (Kawahara & Toh 1988)

$$u_t + uu_x + \beta u_{xxx} + \alpha u_{xx} + \nu u_{xxxx} = 0 \tag{4.1}$$

on the interval $[0, 40]$ with periodic boundary conditions. This partial differential equation has non-chaotic solutions if the dispersion β is large and exhibits spatio-temporal chaos for sufficiently small β. Note that (4.1) reduces to the KdV equation when $\alpha = \nu = 0$, and reduces to the Kuramoto–Sivashinksy equation when $\beta = 0$.

We fix $\alpha = 2$, $\nu = 0.1$ and vary β. For $\beta = 0$, it is expected that the dynamics of the Kuramoto–Sivashinksy equation is chaotic for these parameter values. As an observable we used $\phi(u(x, t)) = u(x_0, t)$, where x_0 is an arbitrarily fixed position, and we iterated until time $T = 35\,000$. The 0–1 test confirms that the dynamics is chaotic at $\beta = 0$ (with $K = 0.939$). Also, we obtain $K = 0.989$ at $\beta = 0.1$ and $K = 0.034$ at $\beta = 4$, indicating chaotic and regular dynamics, respectively, at these two parameter values.

Finally, for discrete dynamical systems, we tried out the test with an ecological model whose chaotic component is coupled with strong periodicity (Cazelles &

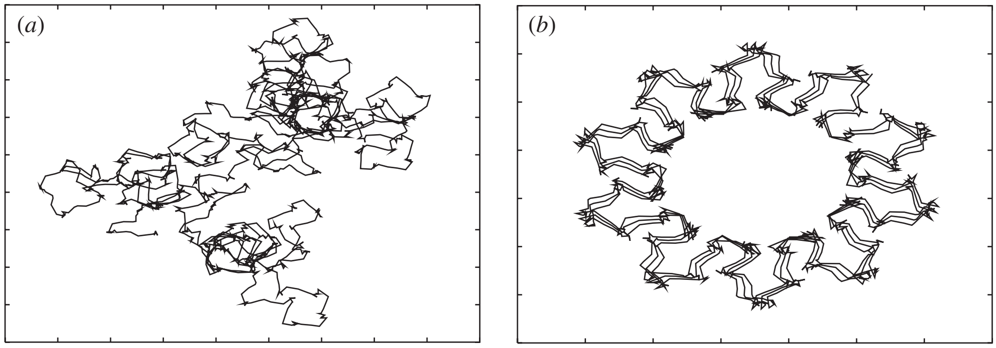

Figure 5. The dynamics of the translation components (p, q) of the $\boldsymbol{E}(2)$-extension (5.1) for the van der Pol system (3.1) with $a = d = 5$, $c = 1.7$, $\phi(\boldsymbol{x}) = x_1 + x_2$. These plots were obtained by integrating for $T \sim 1400$ (with time-step 0.01). (*a*) An unbounded trajectory is shown corresponding to chaotic dynamics at $\omega = 2.465\,50$. (*b*) A bounded trajectory is shown corresponding to regular dynamics at $\omega = 2.465\,51$.

Ferriere 1992; Brahona & Poon 1996). The model

$$x_{k+1} = 118 y_k \exp(-0.001(x_k + y_k)),$$
$$y_{k+1} = 0.2 x_k \exp(-0.07(x_k + y_k)) + 0.8 y_k \exp(-0.05(0.5 x_k + y_k))$$

has a non-connected chaotic attractor consisting of seven connected components. Our test yields $K = 1.023$ with only 10 000 data points and clearly shows that the dynamics is chaotic.

5. Justification of the 0–1 test for chaos

The function $p(t)$ can be viewed as a component of the solution to the skew-product system

$$\left.\begin{aligned} \dot{\theta} &= c + \phi(\boldsymbol{x}(t)), \\ \dot{p} &= \phi(\boldsymbol{x}(t)) \cos\theta, \\ \dot{q} &= \phi(\boldsymbol{x}(t)) \sin\theta, \end{aligned}\right\} \tag{5.1}$$

driven by the dynamics of the observable $\phi(\boldsymbol{x}(t))$. Here (θ, p, q) represent coordinates on the Euclidean group $\boldsymbol{E}(2)$ of rotations θ and translations (p, q) in the plane.

We note that inspection of the dynamics of the (p, q)-trajectories of the group extension provides very quickly (for small T) a simple visual test of whether the underlying dynamics is chaotic or non-chaotic, as can be seen from figure 5.

In Nicol *et al*. (2001) it has been shown that, typically, the dynamics on the group extension is sublinear and furthermore that the dynamics is bounded if the underlying dynamics is non-chaotic and unbounded (but sublinear) if the underlying dynamics is chaotic. Moreover, the p and q components each behave like a Brownian motion on the line if the chaotic attractor is uniformly hyperbolic (Field *et al*. 2003). A non-degeneracy result of Nicol *et al*. (2001) ensures that the variance of the Brownian motion is non-zero for almost all choices of observable ϕ. Recent work by Melbourne & Nicol (2003) indicates that these statements remain valid for large classes of non-uniformly hyperbolic systems, such as Hénon-like attractors.

There is a weaker condition on the 'chaoticity' of X that guarantees the desired growth rate $K = 1$ for the mean-square displacement (2.2): namely that the autocorrelation function of $\phi(\boldsymbol{x})\cos(\theta)$ decays at a rate that is better than quadratic. More precisely, let $\boldsymbol{x}(t)$ and $\theta(t)$ denote solutions to the skew-product equations (5.1) with initial conditions $\boldsymbol{x}_0$ and θ_0, respectively. If there are constants $C > 0$ and $\alpha > 2$ such that

$$\left| \int \phi(\boldsymbol{x}(t))\cos(\theta(t))\phi(\boldsymbol{x}_0)\cos\theta_0 \,\mathrm{d}\boldsymbol{x}_0 \,\mathrm{d}\theta_0 \right| \leqslant Ct^{-\alpha},$$

for all $t > 0$, then $K = 1$ as desired (Biktashev & Holden 1998; Ashwin *et al*. 2001; Field *et al*. 2003). (Again, results of Nicol *et al*. (2001), Field *et al*. (2003) and Melbourne & Nicol (2003) ensure that the appropriate non-degeneracy condition holds for almost all choices of ϕ.) There is a vast literature on proving decay of correlations (Baladi 1999) and this has been generalized to the equivariant setting for discrete time by Field *et al*. (2003) and Melbourne & Nicol (2003) and for continuous time by Melbourne & Török (2002). It follows from these references that $K = 1$, for large classes of chaotic dynamical systems.

One might ask why it is not better to work, instead of the $\boldsymbol{E}(2)$-extension, with the simpler $\mathbb{R}$-extension

$$\dot{p} = \phi(\boldsymbol{x}(t)).$$

In principle, $p(t)$ can again be used as a detector for chaos. However, by the ergodic theorem $p(t)$ will typically grow linearly with rate equal to the space average of ϕ. This would lead to $K = 2$ irrespective of whether the dynamics is regular or chaotic. Hence, it is necessary to subtract the linear term of $p(t)$ in order to observe the bounded/diffusive growth that distinguishes between regular/chaotic dynamics. Subtracting this linear term is a highly non-trivial numerical obstruction. The inclusion of the θ variable in the definition (2.1) of $p(t)$ and in the skew product (5.1) kills off the linear term.

6. Discussion

We have established a simple, inexpensive and novel 0–1 test for chaos in deterministic systems. (It is not the purpose of this paper to study stochastic systems.) The computational effort is of low cost, both in terms of programming efforts and in terms of actual computation time. The test is a binary test in the sense that it can only distinguish between non-chaotic and chaotic dynamics. This distinction is extremely clear by means of the diagnostic variable K, which has values either close to 0 or close to 1. The most powerful aspect of our method is that it is independent of the nature of the vector field (or data) under consideration. In particular the equations of the underlying dynamical system do not need to be known, and there is no practical restriction on the dimension of the underlying vector field. In addition, our method applies to the observable $\phi(\boldsymbol{x}(t))$ rather than the full trajectory $\boldsymbol{x}(t)$.

Related ideas (though not with the aim of detecting chaos) have been used for PDEs in the context of demonstrating hypermeander of spirals in excitable media (Biktashev & Holden 1998), where the spiral tip appears to undergo a planar Brownian motion (see also Ashwin *et al*. 2001). We note also the work of Coullet & Emilsson (1996), who studied the dynamics of Ising walls on a line, where the motion is the superposition of a linear drift and Brownian motion. (This is an example of an

$\mathbb{R}$-extension, which we mentioned briefly in § 5. As we pointed out then, the linear drift is an obstruction to using an $\mathbb{R}$-extension to detect chaos.)

From a purely computational point of view, the method presented here has a number of advantages over the conventional methods using Lyapunov exponents. At a more technical level, we note that the computation of Lyapunov exponents can be thought of abstractly as the study of the $\boldsymbol{GL}(n)$-extension

$$\dot{\boldsymbol{A}} = (d\boldsymbol{f})_{\boldsymbol{x}(t)}\boldsymbol{A},$$

where $\boldsymbol{GL}(n)$ is the space of $n \times n$ matrices, $\boldsymbol{A} \in \boldsymbol{GL}(n)$ and n is the size of the system. Note that the extension involves n^2 additional equations and is defined using the linearization of the dynamical system. To compute the dominant exponent, it is still necessary to add n additional equations, again governed by the linearized system. In contrast, our method requires the addition of two equations.

In this paper, we have concentrated primarily on the idealized situation, where there is an, in principle, unlimited amount of noise-free data. However, in § 3 we also demonstrated the effectiveness of our method for limited datasets. An issue which will be pursued in further work is the effect of noise, which is inevitably present in all experimental time-series. Preliminary results show that our test can cope with small amounts of noise without difficulty. A careful study of this capability, and comparison with other methods, is presently in progress.

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